Affine Inflation in Polynomial Affine Gravity in $3 + 1$ dimensions

Jose Perdiguero Garate

20/12/2022

Abstract

The Polynomial Affine Gravity its a purely affine model that mediates gravitational interactions solely and exclusive through the affine connection instead of the metric tensor. In this paper we couple a scalar field through *inverse tensor densities* and its potential energy to the volume form. We formulte an effective action in the torsion-free sector couple with the scalar field and study the cosmological solutions.

Contents

1 Introduction

General Relativity, being so far the most successful model of gravitational interactions, is the ground basis of the standard model of cosmology, also known as ΛCDM. However, the later rely in a number of additional ingredients to describe consistently the evolution of the cosmos in a way that is compatible with the observations. It should be mentioned that the observations come from different ages of the Universe: cosmic microwave background, big bang nucleo-synthesis and baryon acoustic oscillations: and there are also additional observations coming from gravitational lenses and gravitational waves.

Nonetheless, at any early stage of the Universe it is need an inflationary epoch with the purpose of solving the flatness and horizon problems. The inflation is generally induced by a scalar field which couples to the gravitiational sector and possesses a self-interacting term that determines the prices conditions of the model.

2 Polynomial Affine Gravity

The Polynomial Affine Gravity model its a purely affine model on which we endowed the manifold only with an affine connection $(\mathcal{M}, \Gamma)$. This allow us to define the notion of parallelism by the covariant derivative ∇. Since we only have an affine connection Γ we can only deffine the following chain of geometric objects

$$\Gamma_{\mu}{}^{\sigma}{}_{\nu} \to \nabla_{\mu} \to \mathcal{R}_{\mu\sigma}{}^{\tau}{}_{\nu} \to \mathcal{R}_{\mu\nu} \tag{1}$$

Notice that in the absence of the metric tensor it is not possible to define the $\mathcal{R}$.

2.1 The action

In order to built the action of the Polynomial Affine Gravity we use the irreducible fields of the affine connection, by separating the connection into its symmetric and antisymmetric part

$$\hat{\Gamma}_{\mu}{}^{\sigma}{}_{\nu} = \Gamma_{\mu}{}^{\sigma}{}_{\nu} + \mathcal{B}_{\mu}{}^{\sigma}{}_{\nu} + \delta^{\sigma}_{[\mu}\mathcal{A}_{\nu]} \tag{2}$$

where $\Gamma_{\mu}{}^{\sigma}{}_{\nu}$ correspond to the symmetric part of the connection, $\mathcal{B}_{\mu}{}^{\sigma}{}_{\nu}$ its the traceless part of the torsion tensor and $\mathcal{A}_{\mu}$ its the vectorial part of the torsion tensor. Additionally, we need to define the volume form, which can be written using only the wedge product

$$\mathrm{d}V^{\alpha\beta\gamma\delta} = J(x)\mathrm{d}x^{\alpha} \wedge \mathrm{d}x^{\beta} \wedge \mathrm{d}x^{\gamma} \wedge \mathrm{d}x^{\delta} \tag{3}$$

However, since we want to couple a scalar field $\phi(x)$ to this model, we need to introduce the potential energy of the scalar field $\mathcal{V}(\phi)$. Inspired by the work of Hemza Azri in affine inflation, we couple the potential energy to the volume form in the following manner

$$\mathrm{d}V^{\alpha\beta\gamma\delta} = \mathrm{d}\hat{V}^{\alpha\beta\gamma\delta}\frac{1}{\mathcal{V}(\phi)} \tag{4}$$

The action must preserv the invariance under diffemorphism, which is why the symmetric part of the connection goes indirectly throught the covariant derivative. The fundamental fields to build the action are $\nabla, \mathcal{A}, \mathcal{B}, \mathrm{d}V$. Then we perform a sort of *dimensional structural analysis technique* studying everysingle possible non-trivial contribution to the action.

THen, the most general action in $3+1$ dimension up to boundary terms is

$$\begin{aligned} S = \int \mathrm{d}V^{\alpha\beta\gamma\delta}\Big[& B_1\mathcal{R}_{\mu\nu}{}^{\mu}{}_{\rho}\mathcal{B}_{\alpha}{}^{\nu}{}_{\beta}\mathcal{B}_{\gamma}{}^{\rho}{}_{\delta} + B_2\mathcal{R}_{\alpha\beta}{}^{\mu}{}_{\rho}\mathcal{B}_{\gamma}{}^{\nu}{}_{\delta}\mathcal{B}_{\mu}{}^{\rho}{}_{\nu} + B_3\mathcal{R}_{\mu\nu}{}^{\mu}{}_{\alpha}\mathcal{B}_{\beta}{}^{\nu}{}_{\gamma}\mathcal{A}_{\delta} + B_4\mathcal{R}_{\alpha\beta}{}^{\sigma}{}_{\rho}\mathcal{B}_{\gamma}{}^{\rho}{}_{\delta}\mathcal{A}_{\sigma} \\ & + B_5\mathcal{R}_{\alpha\beta}{}^{\rho}{}_{\rho}\mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta}\mathcal{A}_{\sigma} + C_1\mathcal{R}_{\mu\alpha}{}^{\mu}{}_{\nu}\nabla_{\beta}\mathcal{B}_{\gamma}{}^{\nu}{}_{\delta} + C_2\mathcal{R}_{\alpha\beta}{}^{\rho}{}_{\rho}\nabla_{\sigma}\mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta} + D_1\mathcal{B}_{\nu}{}^{\mu}{}_{\lambda}\mathcal{B}_{\mu}{}^{\nu}{}_{\alpha}\nabla_{\beta}\mathcal{R}_{\gamma}{}^{\lambda}{}_{\delta} \\ & + D_2\mathcal{B}_{\alpha}{}^{\mu}{}_{\beta}\mathcal{B}_{\mu}{}^{\lambda}{}_{\nu}\nabla_{\lambda}\mathcal{B}_{\gamma}{}^{\nu}{}_{\delta} + D_3\mathcal{B}_{\alpha}{}^{\mu}{}_{\nu}\mathcal{B}_{\beta}{}^{\lambda}{}_{\gamma}\nabla_{\lambda}\mathcal{B}_{\mu}{}^{\nu}{}_{\delta} + D_4\mathcal{B}_{\alpha}{}^{\lambda}{}_{\beta}\mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta}\nabla_{\lambda}\mathcal{A}_{\sigma} + D_5\mathcal{B}_{\alpha}{}^{\lambda}{}_{\beta}\mathcal{A}_{\sigma}\nabla_{\lambda}\mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta} \\ & + D_6\mathcal{B}_{\alpha}{}^{\lambda}{}_{\beta}\mathcal{A}_{\gamma}\nabla_{\lambda}A_{\delta} + D_7\mathcal{B}_{\alpha}{}^{\lambda}{}_{\beta}\mathcal{A}_{\lambda}\nabla_{\gamma}A_{\delta} + E_1\nabla_{\rho}\mathcal{B}_{\alpha}{}^{\rho}{}_{\beta}\nabla_{\sigma}\mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta} + E_2\nabla_{\rho}\mathcal{B}_{\alpha}{}^{\rho}{}_{\beta}\nabla_{\gamma}\mathcal{A}_{\delta} \\ & + F_1\mathcal{B}_{\alpha}{}^{\mu}{}_{\beta}\mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta}\mathcal{B}_{\mu}{}^{\lambda}{}_{\rho}\mathcal{B}_{\sigma}{}^{\rho}{}_{\lambda} + F_2\mathcal{B}_{\alpha}{}^{\mu}{}_{\beta}\mathcal{B}_{\gamma}{}^{\nu}{}_{\lambda}\mathcal{B}_{\delta}{}^{\lambda}{}_{\rho}\mathcal{B}_{\mu}{}^{\rho}{}_{\nu} + F_3\mathcal{B}_{\nu}{}^{\mu}{}_{\lambda}\mathcal{B}_{\mu}{}^{\nu}{}_{\alpha}\mathcal{B}_{\beta}{}^{\lambda}{}_{\gamma}\mathcal{A}_{\delta} + F_4\mathcal{B}_{\alpha}{}^{\mu}{}_{\beta}\mathcal{B}_{\gamma}{}^{\nu}{}_{\delta}\mathcal{A}_{\mu}\mathcal{A}_{\nu}\Big]. \end{aligned}$$

In previous works we have mentioned some of the features of the action: (i) Its rigidity, since contains all possible combinations of the fields and their derivatives; (ii) All the coupling constants are dimensionless, which might be a sign of conformal invariance, and also ensure that the model is power-counting renormalisable; (iii) The field equations are second order differential equations, and the Einstein spaces are a subset of their solu- tions; (iv) The supporting symmetry group is the group of diffeomorphisms, desirable for the background independence of the model; (v) Even though there is no fundamental metric, it is possible to obtain emergent (connection- descendent) metric tensors; (vi) The cosmological constant appears in the solutions as an integration constant, changing the paradigm concerning its interpretation; (vii) The model can be extended to be coupled with a scalar field, and the field equations are equivalent to those of General Relativity interacting with a massless scalar field.

To couple the affine action to a scalar field, we need to introduce a kinetic term in the absence of the metric tensor. In order to do so, we build *inverse symmetric tensor densities*, by using the *dimensional analysis structure technique*

$$\mathrm{g}^{\mu\nu} = (\alpha \nabla_\lambda \mathcal{B}_\rho{}^\mu{}_\sigma + \beta \mathcal{A}_\lambda \mathcal{B}_\rho{}^\mu{}_\sigma)\, \mathrm{d}V^{\nu\lambda\rho\sigma} + \gamma \mathcal{B}_\kappa{}^\mu{}_\lambda \mathcal{B}_\rho{}^\nu{}_\sigma \mathrm{d}V^{\kappa\lambda\rho\sigma} \tag{5}$$

Using the above expression we can define the kinetic term

$$S_\phi = -\int \mathrm{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \tag{6}$$

Since we want to work on the torsion-free sector, it is worth to notice that only the terms that are linear in the torsion will have a non-trivial contribution, C_1 and C_2. Additionally, since our connection its an *equi-affine* connection, the trace of the Riemman tensor will vanish completly. Applying the same idea the to the scalar field action, only the α term survive. Thus, the effective action coupled with a scalar field is

$$S_{ef} = \int \mathrm{d}V^{\alpha\beta\gamma\delta} \left[C_1 \mathcal{R}_{\mu\alpha}{}^\mu{}_\nu - \alpha \partial_\alpha \phi \partial_\nu \right] \nabla_\beta \mathcal{B}_\gamma{}^\nu{}_\delta.$$

2.2 Cosmological ansatz

In order to solve the field equations, one need to build an ansatz, since we want to do cosmology, we need to build an ansazts compatible with the symmetries of the cosmological principle, which are rotation and translations. It is possible to build an ansatz for our fundamental geometric objects using the Lie derivative along the Killing vector fields. The most general ansatz for the symmetric part of the connection $\Gamma_\mu{}^\sigma{}_\nu$ is

$$\Gamma_t{}^t{}_t = f(t), \quad \Gamma_i{}^t{}_j = g(t) S_{ij} \tag{7}$$

$$\Gamma_t{}^i{}_j = h(t)\delta^i_j, \quad \Gamma_i{}^j{}_k = \gamma_i{}^j{}_k \tag{8}$$

Since we are going to be working in the torsio-free sector, in principle there is no neccesity to compute the ansatz for the torsion tensor, however, just for the sake of completness, we are going to include them. The traceless part of the torsion tensor $\mathcal{B}_\mu{}^\sigma{}_\nu$ is completely define by only one time depending function

$$\mathcal{B}_\theta{}^r{}_\varphi = \psi(t) r^2 \sin\theta \sqrt{1-\kappa r^2} \qquad \mathcal{B}_r{}^\theta{}_\varphi = \frac{\psi(t)\sin\theta}{\sqrt{1-\kappa r^2}}$$

$$\mathcal{B}_r{}^\varphi{}_\theta = \frac{\psi(t)}{\sqrt{1-\kappa r^2}\sin\theta}$$

and finally, the vectorial torsion tensor $\mathcal{A}_\mu$ is given by

$$\mathcal{A}_t = \eta(t) \tag{9}$$

2.3 The field equations

The field equations are obtained using Kiwosjki's formalism and taking into account the symmetries and properties of the fundamental fields, we vary the action with respect to the fundamental fields. In the torsion-free limit, the field equation is

$$\nabla_\mu \left[\frac{1}{\mathcal{V}(\phi)} \left(C\partial_\alpha \phi \partial_\lambda \phi - \mathcal{R}_{\alpha\lambda} \right) \mathrm{d}V^{\mu\nu\rho\alpha} \right] + \frac{2}{3} \nabla_\mu \left[\frac{1}{\mathcal{V}(\phi)} \mathcal{R}_{\alpha\theta} \delta^{[\nu}_\lambda \mathrm{d}V^{\rho]\alpha\mu\theta} \right] = 0 \tag{10}$$

By multiplying the left hand side of the field equation by $\epsilon_{\nu\rho\tau\beta}$, the second term vanishes completly and the field equation is reduced even further to

$$\nabla_{[\mu} \left(\mathcal{R}_{\nu]\gamma} \frac{1}{\mathcal{V}(\phi)} \right) - C \nabla_{[\mu} \left(\partial_{\nu]} \phi \partial_\gamma \phi \frac{1}{\mathcal{V}(\phi)} \right) = 0 \tag{11}$$

This is the field equation in Polynomial Affine Graivity coupled with a scalar field in the torsion-free sector.

3 Cosmolpgical solutions

4 Cosmological Solutions with the scalar field ϕ

The field equation in the torsion-free sector coupled with a scalar field is given by

$$\nabla_{[\mu}\left(\mathcal{R}_{\nu]\gamma}\frac{1}{\mathcal{V}(\phi)}\right) - C\nabla_{[\mu}\left(\partial_{\nu]}\phi\partial_\gamma\phi\frac{1}{\mathcal{V}(\phi)}\right) = 0 \tag{12}$$

which can be rewritten as

$$\nabla_{[\mu}\mathcal{S}_{\nu]\gamma} = 0 \tag{13}$$

where the $\mathcal{S}_{\mu\nu}$ tensor is a symmetric tensor defined by

$$\mathcal{S}_{\mu\nu} = \left(\mathcal{R}_{\mu\nu} - C\partial_\mu\phi\partial_\nu\phi\right)\frac{1}{\mathcal{V}(\phi)} \tag{14}$$

From here, we distinguish three types of solutions

$$\mathcal{S}_{\mu\nu} = 0 \qquad \nabla_\eta\mathcal{S}_{\mu\nu} = 0 \qquad \nabla_{[\eta}\mathcal{S}_{\mu]\nu} = 0 \tag{15}$$

4.1 $\mathcal{S}_{\mu\nu} = 0$

In the case where the symmetric tensor $\mathcal{S}_{\mu\nu}$ vanishes completely we are left with two differential equations

$$\dot{h} + h^2 + C\dot{\phi}^2 = 0 \qquad \dot{g} + gh + 2\kappa = 0 \tag{16}$$

where the potential energy $\mathcal{V}(\phi)$ does not play any role. Here we have two unknown functions of time and the scalar field, the best we can do is to solve the scalar field $\phi(t)$ in terms of the time derivative of the $h(t)$ function , as well as the $g(t)$ in terms of $h(t)$

$$\phi(t) = \phi_0 \pm \int \mathrm{d}t\sqrt{\frac{\dot{h}+h^2}{-C}} \qquad g(t) = e^{-\int \mathrm{d}t\, h}\left(g_0 - 2\kappa\int \mathrm{d}t\, e^{\int \mathrm{d}t\, h}\right) \tag{17}$$

4.2 $\nabla_\eta\mathcal{S}_{\mu\nu} = 0$

The second case its the affine parallel scenario, where the covariant derivative of the $\mathcal{S}_{\mu\nu}$ vanishes completely

$$0 = C\mathcal{V}'\dot{\phi}^3 - 2C\mathcal{V}\dot{\phi}\ddot{\phi} - 3\mathcal{V}\left(2h\dot{h} + \ddot{h}\right) + 3\mathcal{V}'\dot{\phi}\left(h^2 + \dot{h}\right) \tag{18}$$

$$0 = Cg\dot{\phi}^2 + 2gh^2 - 2\kappa h - h\dot{g} + 3g\dot{h} \tag{19}$$

$$0 = \left(2gh^2 + 4\kappa h + h\dot{g} - g\dot{h} - \ddot{g}\right)\mathcal{V} + \left(gh + 2\kappa + \dot{g}\right)\mathcal{V}'\dot{\phi} \tag{20}$$

It is usual in cosmology to be working in a flat space-time meaning that $\kappa = 0$ and additionally, to impose the *slow-roll* conditions, which are given by

$$\mathcal{V}(\phi) \gg \dot{\phi}^2(t) \qquad \mathcal{V}'(\phi) \gg \ddot{\phi}(t) \tag{21}$$

Under the above assumptions, the system of differential equations is reduced to

$$0 = C\mathcal{V}'\dot{\phi}^3 - 3\mathcal{V}\left(2h\dot{h} + \ddot{h}\right) + 3\mathcal{V}'\dot{\phi}\left(h^2 + \dot{h}\right) \tag{22}$$

$$0 = Cg\dot{\phi}^2 + 2gh^2 - h\dot{g} + 3g\dot{h} \tag{23}$$

$$0 = \left(2gh^2 + h\dot{g} - g\dot{h} - \ddot{g}\right)\mathcal{V} + \left(gh + \dot{g}\right)\mathcal{V}'\dot{\phi} \tag{24}$$

In cosmology, you give a potential energy term like *Power-Law* or *Starobinsky* and solve the field equations for the scale factor $a(t)$ and the scalar field $\phi(t)$. Here, we are going the give the potential energy term and solve the field equations for the affine functions $h(t)$ and $g(t)$ and the scalar field $\phi(t)$.

First we are going to work with the *Power-Law* potential $\mathcal{V}(\phi) = \beta\phi^j(t)$, then the equations are

$$0 = -6h\dot{h}\phi + 3jh^2\dot{\phi} + 3j\dot{h}\dot{\phi} + jC\dot{\phi}^3 - 3\phi\ddot{h} \tag{25}$$

$$0 = -h\dot{g} + g\left(2h^2 + 3\dot{h} + C\dot{\phi}^2\right) \tag{26}$$

$$0 = h\left(2gh + \dot{g} - g\dot{h}\right) + \frac{j(gh + \dot{g})\dot{\phi}}{\phi} - \ddot{g} \tag{27}$$

Notice that for our symmetric connection, comparing our connection coefficients with the ones already known in the FRW universe, we can deduced that $h(t) = \frac{\dot{a}}{a}$ and $g(t) = \dot{a}a$. Reeplacing those definitions in the system of differential equations, leads to

$$0 = 3\phi\dot{a}\ddot{a} + a\left(jaC\dot{\phi}^3 + 3j\dot{\phi}\ddot{a} - 3\phi\dddot{a}\right) \tag{28}$$

$$0 = -\frac{2\dot{a}^2}{a} + aC\dot{\phi}^2 + 2\ddot{a} \tag{29}$$

$$0 = ja\dot{\phi}\left(2\dot{a}^2 + a\ddot{a}\right) + \phi\left(4\dot{a}^3 - 3a\dot{a}\ddot{a} - a^2\dddot{a}\right) \tag{30}$$

It is convenient to introduce the following variable change

$$\ddot{a} = \dot{u}(t) \qquad\qquad \dddot{a}(t) = \ddot{u}(t) \qquad\qquad \Phi(t) = \frac{\dot{\phi}}{\phi} \tag{31}$$

In the above notation, the equations are written as

$$0 = 3\ddot{u} + 3\dot{u}\left(-h - \Phi\right) - jaC\phi^2\Phi^3 \tag{32}$$

$$0 = \frac{\dot{u}}{a} - h^2 + \frac{1}{2}C\dot{\phi}^2 \tag{33}$$

$$0 = \ddot{u} + \dot{u}\left(3h - j\Phi\right) - 2ah^2\left(2h + j\Phi\right) \tag{34}$$

From the third equation, it is possible to find an expression for the scale factor

$$a = \frac{\dot{u}\left(3h - j\Phi\right) + \ddot{u}}{2h^2\left(2h + j\Phi\right)} \tag{35}$$

Reeplacing the above expresion for the scale factor in the first and second equation leads to

$$0 = \ddot{u} - \dot{u}\left(\frac{12h^4 + 18jh^3\Phi + 6j^2h^2\Phi^2 + 3jCh\phi^2\Phi^3 - j^2\phi^2\Phi^4C}{12h^3 + 6jh^2\Phi - j\phi^2\Phi^3C}\right) \tag{36}$$

$$0 = \ddot{u} - \dot{u}\left(\frac{2h^2\left(h + 3j\Phi\right) + C\left(3h - j\Phi\right)\dot{\phi}^2}{2h^2 - C\dot{\phi}^2}\right) \tag{37}$$

comparing the coefficients $\dot{u}$ yields to

$$\frac{12h^4 + 18jh^3\Phi + 6j^2h^2\Phi^2 + 3jCh\phi^2\Phi^3 - j^2\phi^2\Phi^4C}{12h^3 + 6jh^2\Phi - j\phi^2\Phi^3C} = \frac{2h^2\left(h + 3j\Phi\right) + C\left(3h - j\Phi\right)\dot{\phi}^2}{2h^2 - C\dot{\phi}^2} \tag{38}$$

which after some simplification, can be reduced to

$$h\left(2h + j\Phi\right)\left(\frac{6h}{-12h^3 - 6jh^2\Phi + jC\phi^2\Phi^3} + \frac{1}{2h^2 - C\dot{\phi}}\right) = 0 \tag{39}$$

notice that we have three types of solution, however the most interesting solutions its coming from solving the second differential equation by replacing the hubble function $h(t)$ and $\Phi(t)$

$$\frac{\dot{\phi}}{\phi} + \frac{2\dot{a}}{aj} = 0 \to \phi(t) = a(t)^{-2/j}\phi_0 \tag{40}$$

From here, reeplacing the scalar field $\phi(t)$ in terms of the scale factor $a(t)$ into the two differential equations leads

$$\ddot{u} + 5\frac{\dot{u}\dot{a}}{a} = 0 \tag{41}$$

which in terms of only the scale factors

$$\dddot{a} + 5\frac{\ddot{a}\dot{a}}{a} = 0 \tag{42}$$

Notice that the complexity of the differential equations system is reduced entirely by imposing the equality of the factors. The above differential equation for the scale factor can be solved analytically in terms of the inverse hypergeometric function. Additionally, it can be solved numerically.

Next, lets work the Starobinsky's potential $\mathcal{V}(\phi) = \alpha\left(1 - e^{-\beta\phi}\right)^2$, then the system of differential equations is written as

$$0 = 6h\dot{h} + 6\beta\left(h^2 + \dot{h}\right)\dot{\phi} + 2\beta C\dot{\phi}^3 + 3\ddot{h} - 3e^{\beta\phi}\left(2h\dot{h} + \ddot{h}\right) \tag{43}$$

$$0 = -h\dot{g} + g\left(2h^2 + 3\dot{h} + C\dot{\phi}^2\right) \tag{44}$$

$$0 = 2e{-}\beta\phi\left(gh + \dot{g}\right)\dot{\phi} + \left(1 - e^{-\beta\phi}\right)\left(h\left(2gh + \dot{g}\right) - g\dot{h} - \ddot{g}\right) \tag{45}$$

which can be rewritten just like the previous case in terms of the scale factor as

$$0 = 3\left(-1 + e^{\beta\phi}\right)\dot{a}\ddot{a} + a\left(2\beta aC\dot{\phi}^3 + 6\beta\dot{\phi}\ddot{a} - 3\left(-1 + e^{\beta\phi}\right)\right) \tag{46}$$

$$0 = -\frac{2\dot{a}^2}{a} + aC\dot{\phi}^2 + 2\ddot{a} \tag{47}$$

$$0 = 4\left(-1 + e^{\beta\phi}\right)\dot{a}^3 + 4\beta a\dot{a}^2\dot{\phi} - 3\left(-1 + e^{\beta\phi}\right)a\dot{a}\ddot{a} + a^2\left(2\beta\dot{\phi}\ddot{a} - \dddot{a}\left(-1 + e^{\beta\phi}\right)\right) \tag{48}$$

then, we introduce the new variables

$$\ddot{a}(t) = \dot{u}(t) \qquad \dddot{a}(t) = \ddot{u} \qquad \Phi(t) = \left(1 - e^{-\beta\phi}\right) \tag{49}$$

in the new variables, the system of differential equations is rewritten as

$$0 = 2\beta\dot{\phi}\left(3\dot{u} + aC\dot{\phi}^2\right) + 3e^{\beta\phi}\Phi\left(h\dot{u} - \ddot{u}\right) \tag{50}$$

$$0 = 2\dot{u} + a\left(-2h^2 + C\dot{\phi}^2\right) \tag{51}$$

$$0 = 2e^{-\beta\phi}\left(2ah^2 + \dot{u}\right)\dot{\phi} + \Phi\left(4ah^3 - 3h\dot{u}\ddot{u}\right) \tag{52}$$

from the third equation we can find an expression for the scale factor

$$a = \frac{-2\beta\dot{u}\dot{\phi} + e^{\beta\phi}\left(3h\dot{u} + \ddot{u}\right)}{4h^2\left(e^{\beta\phi}h\Phi + \beta\dot{\phi}\right)} \tag{53}$$

reeplacing the above expression in the first and second equation leads to

$$0 = \ddot{u} - \dot{u}\left(h + \frac{2e^{-\beta\phi}\beta\dot{\phi}}{\Phi} + \frac{4\beta Ch\dot{\phi}^3}{6e^{\beta\phi}h^3\Phi + 6\beta h^2\dot{\phi} - \beta C\dot{\phi}^3}\right) \tag{54}$$

$$0 = \ddot{u} - \dot{u}\left(\frac{e^{\beta\phi}\left(-2\beta\dot{\phi}\left(-6h^2 + C\dot{\phi}^2\right) + e^{\beta\phi}h\Phi\left(2h^2 + 3C\dot{\phi}^2\right)\right)}{\Phi\left(2h^2 - C\dot{\phi}^2\right)}\right) \tag{55}$$

comparing both coefficients

$$h + \frac{2e^{-\beta\phi}\beta\dot{\phi}}{\Phi} + \frac{4\beta Ch\dot{\phi}^3}{6e^{\beta\phi}h^3\Phi + 6\beta h^2\dot{\phi} - \beta C\dot{\phi}^3} = \frac{e^{\beta\phi}\left(-2\beta\dot{\phi}\left(-6h^2 + C\dot{\phi}^2\right) + e^{\beta\phi}h\Phi\left(2h^2 + 3C\dot{\phi}^2\right)\right)}{\Phi\left(2h^2 - C\dot{\phi}^2\right)} \tag{56}$$

after some simplification yields one equation

$$0 = 2e^{-\beta\phi}h\left(e^{\beta\phi}h\Phi + \beta\dot{\phi}\right)\left(6\beta h^2 + 3e^{\beta\phi}Ch\Phi\dot{\phi} - \beta C\dot{\phi}^2\right) \tag{57}$$

notice that we can solve the first differential equation to find a relation between the scale factor and the scalar field

$$\dot{\phi} + \frac{\dot{a}\left(-1 + e^{\beta\phi}\right)}{a\beta} = 0 \to \phi(t) = \frac{1}{\beta}\log\left(\frac{a}{ae^{\beta\phi_0} + a}\right) \tag{58}$$

reeplacing the above the relation in the two equations leads to the same differential equations, which written in the original variables is

$$\dddot{a} + 5\frac{\ddot{a}\dot{a}}{a} = 0 \tag{59}$$

Notice that this is the same equation for the *Power-Law* potential.

4.3 $\nabla_{[\eta}\mathcal{S}_{\mu]\nu} = 0$

Under the symmetries of the cosmological ansatz we only have one equation

$$C\mathcal{V}(\phi)g\dot{\phi}^2 + \mathcal{V}(\phi)\left(4gh^2 + 2\kappa h + 2g\dot{h} - \ddot{g}\right) + \mathcal{V}'(\phi)\dot{\phi}\left(gh + 2\kappa + \dot{g}\right) = 0 \tag{60}$$

5 Conclusions

6 Appendix A: Dimensional Analysis

In this section we briefly show how to build the action and coupling the scalar field in the absence of the metric tensor, by using a sort of *dimensional analysis* technique.

6.1 Building the action

In order to build the most general action while preserving the invariance under diffeormophism, we perform a *dimensional analysis* technique. First, we define an operator $\mathcal{N}$ to count the number of free index and a second operator $\mathcal{W}$ to define the weight density of the object. Applying both operators to the fundamental fields leads to

$$\mathcal{N}(\mathcal{A}_\mu) = -1 \qquad \mathcal{N}(\mathcal{B}_\mu{}^\lambda{}_\nu) = -1 \qquad \mathcal{N}(\Gamma_\mu{}^\lambda{}_\nu) = -1 \qquad \mathrm{d}V^{\alpha\beta\gamma\delta} = 4 \tag{61}$$

$$\mathcal{W}(\mathcal{A}_\mu) = 0 \qquad \mathcal{W}(\mathcal{B}_\mu{}^\lambda{}_\nu) = 0 \qquad \mathcal{W}(\Gamma_\mu{}^\lambda{}_\nu) = 0 \qquad \mathrm{d}V^{\alpha\beta\gamma\delta} = 1 \tag{62}$$

A generic term will have the following form

$$\mathcal{O} = \mathcal{A}^m\mathcal{B}^n\Gamma^p\mathrm{d}V^q \tag{63}$$

Applying the operators defined above, yield the equations

$$\mathcal{N}(\mathcal{O}) = 4q - m - n - p \qquad \mathcal{W}(\mathcal{O}) = q \tag{64}$$

Notice that we are interested in building scalar densities, meaning that the number of free index must be zero and the weight density must be equal to the unity. Therefore, we have two constraints

$$m + n + p = 4 \qquad q = 1 \tag{65}$$

The terms contributing to the action are shown in the below table

From the above table, one use the symmetries of the tensor to see which terms will have non trivial contribution to the action. For example, the term with four $\mathcal{A}$ does not contribute to the action since its contraction with the volume element is identically zero. Whenever two covariant derivatives are contracted with the volume form they give a curvature tensor, and since the curvature is defined for the symmetric part of the connection, such curvature satisfy the torsion-free Bianchi identities, which relate some of the several possible contractions of the indices. An additional argument that helps to drop contraction of indices is that $\mathcal{B}$ is traceless.

$\mathcal{A}^m$	$\mathcal{B}^n$	Γ^p	Type of configuration	Action term
4	0	0	$\mathcal{AAAA}$	0
3	1	0	$\mathcal{AAAB}$	0
3	0	1	$\mathcal{AAA}\nabla$	0
2	2	0	$\mathcal{AABB}$	F_4
2	1	1	$\mathcal{AAB}\nabla$	D_6, D_7
2	0	2	$\mathcal{AA}\nabla\nabla$	0
1	3	0	$\mathcal{ABBB}$	F_3
1	2	1	$\mathcal{ABB}\nabla$	D_4, D_5
1	1	2	$\mathcal{AB}\nabla\nabla$	B_3, B_4, B_5, E_2
1	0	3	$\mathcal{A}\nabla\nabla\nabla$	0
0	4	0	$\mathcal{BBBB}$	F_1, F_2
0	3	1	$\mathcal{BBB}\nabla$	D_1, D_2, D_3
0	2	2	$\mathcal{BB}\nabla\nabla$	B_1, B_2, E_1
0	1	3	$\mathcal{B}\nabla\nabla\nabla$	C_1, C_2
0	0	4	$\nabla\nabla\nabla\nabla$	0

Table 1: Possible terms contributing to the action of Polynomial Affine gravity

6.2 Coupling the scalar field ϕ

In order to form the kinetic term without the metric tensor we proceed just like the subsection before, however, here we are interested in building tensor densities of the type $(0, 2)$, and additionally we require that it has to be symmetric. Then, the constraint equations are

$$m + n + p = 2 \qquad\qquad q = 1 \tag{66}$$

From here we present the table with all possible configuration The first type of term, can only

$\mathcal{A}^m$	$\mathcal{B}^n$	Γ^p	Type of configuration
2	0	0	$\mathcal{AA}$
1	1	0	$\mathcal{AB}$
1	0	1	$\mathcal{A}\nabla$
0	2	0	$\mathcal{BB}$
0	1	1	$\mathcal{B}\nabla$
0	0	2	$\nabla\nabla$

Table 2: Possible terms to contract the kinetic term of the scalar field ϕ

be configurate in one form where two vectorial torsion $\mathcal{A}$ must be contracted with the volumen element

$$\mathcal{A}_\alpha \mathcal{A}_\beta \mathrm{d}V^{\alpha\beta\mu\nu}$$

meaning that its contributio vanishes completely. The second type, produces only two possible terms

$$\mathcal{A}_\sigma \mathcal{B}_\alpha{}^\sigma{}_\beta \mathrm{d}V^{\alpha\beta\mu\nu} + \mathcal{A}_\alpha \mathcal{B}_\beta{}^\mu{}_\gamma \mathrm{d}V^{\alpha\beta\gamma\nu}$$

The first term can not be used because it is an antisymmetric object, therefore its coupling with the kinetic term of the scalar field wil vanishes completely, whereas the second produce non trivial contribution. The thir type term has only one contribution

$$\nabla_\alpha \mathcal{A}_\beta \mathrm{d}V^{\alpha\beta\mu\nu}$$

which is trivial. The fourth type has two possible configuration

$$\mathcal{B}_\alpha{}^\mu{}_\beta \mathcal{B}_\gamma{}^\nu{}_\delta \mathrm{d}V^{\alpha\beta\gamma\delta} + \mathcal{B}_\alpha{}^\sigma{}_\beta \mathcal{B}_\sigma{}^\mu{}_\gamma \mathrm{d}V^{\alpha\beta\gamma\nu}$$

Here both term have non trivial contribution, however, since we are going to be dealing with the torsion-free sector, both term will no have an effective action on the field equaitons. The fith configuration has two possible terms

$$\nabla_\alpha \mathcal{B}_\beta{}^\mu{}_\gamma \mathrm{d}V^{\alpha\beta\gamma\nu} + \nabla_\sigma \mathcal{B}_\alpha{}^\sigma{}_\beta \mathrm{d}V^{\alpha\beta\mu\nu}$$

Notice that second term will produce a trivial term since its antisymmetric in its two free indices, whereas the first term have a non trivial contribution. Finally, the las type

$$\mathcal{R}_{\alpha\beta}{}^\mu{}_\gamma \mathrm{d}V^{\alpha\beta\gamma\nu} + \mathcal{R}_{\alpha\beta}{}^\sigma{}_\sigma \mathrm{d}V^{\alpha\beta\mu\nu}$$

both terms are trivial through Bianchi's identity and the antisymmetry of the volumen element.

7 Appendix B: GR as a subspace of PAG

The field equation is given by

$$\nabla_{[\mu}\left(\mathcal{R}_{\nu]\gamma}\frac{1}{\mathcal{V}(\phi)}\right) - C\nabla_{[\mu}\left(\partial_{\nu]}\phi\partial_\gamma\phi\frac{1}{\mathcal{V}(\phi)}\right) = 0 \tag{67}$$

A particular solution of the above field equation is given by

$$\nabla_\mu\left(\mathcal{R}_{\nu\gamma}\frac{1}{\mathcal{V}(\phi)}\right) - C\nabla_\mu\left(\partial_\nu\phi\partial_\gamma\phi\frac{1}{\mathcal{V}(\phi)}\right) = \nabla_\mu\left(\Lambda g_{\nu\gamma}\right) \tag{68}$$

Notice that in order to be consistent, the object $g_{\mu\nu}$ must be symmetric, so taking the antysymmetrize covariant derivative it vanishes completely, allowing us to recover the field equation. This gives rise a metric tensor named $g_{\mu\nu}$. Next, a particular solution of the above equation is

$$\mathcal{R}_{\mu\nu} - C\partial_\mu\phi\partial_\nu\phi = \Lambda\mathcal{V}(\phi)g_{\mu\nu} \tag{69}$$

and by using the metric tensor, the above equation can be rewritten in a more well-known equation

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} + \Lambda\mathcal{V}(\phi)g_{\mu\nu} = C\left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2\right) \tag{70}$$

If we vary the action with respect to the scalar field and take the torsion-free limit, then we will have a trivial contribution, however, we can still have the field equation for the scalar field by proceeding in a different manner. Taking the divergence ∇^μ of the above equation leads to

$$\Lambda g_{\mu\nu}\nabla^\mu\mathcal{V}(\phi) = C\left(\nabla^\mu\partial_\mu\phi\partial_\nu\phi + \partial_\mu\phi\nabla^\mu\partial_\nu\phi - g_{\mu\nu}\partial\phi\nabla^\mu\partial\phi\right) \tag{71}$$

after some simplification, the above equation leads to

$$C\nabla^\mu\nabla_\mu\phi - \Lambda\mathcal{V}'(\phi) = 0 \tag{72}$$

which is the Klein-Gordon field equation. Notice that in order to obtain the above equation we require the existence of the $g_{\mu\nu}$ object, to define the d'Alembert operator. Without this object, it is not possible to obtain a Klein-Gordon field equation, additionally we require that the integration constant $\Lambda \neq 0$ and also we demand that the covariant derivative of $g_{\mu\nu}$ vanishes completely, to satisfy the metricity condition.

Since, we have the tensor $g_{\mu\nu}$ presented as a particular solution to the field equation. Thus, we need to provied an ansatz for this tensor compatible with the symmetries of the cosmological principle. This is donde by computing the Lie derivative of an arbitrary tensor type $(0,2)$ along the Killing vectors that generate the symmetries of rotation and translations. This compute leads to

$$g_{\mu\nu} = b(t)\mathrm{d}t^2 + a(t)\left(\frac{1}{1-\kappa r^2}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2\right) \tag{73}$$

Demanding that its covariant derivative vaniahes, we found restriction for the above functions in terms of the connection coefficients

$$b(t) = -b_0 \qquad h(t) = \frac{\dot{a}(t)}{a(t)} \qquad g(t) = \frac{\dot{a}(t)a(t)}{b_0} \tag{74}$$

Replacing the cosmological ansatz for g and Γ in the field equation yields to

$$0 = 3\ddot{a}(t) + Ca(t)\dot{\phi}^2(t) - \Lambda b_0 a(t)\mathcal{V}(\phi) \tag{75}$$

$$0 = \ddot{a}(t)a(t) + 2\dot{a}^2(t) + 2b_0\kappa - \Lambda b_0 a^2(t)\mathcal{V}(\phi) \tag{76}$$

Combining the two equations we obtained we get

$$3H^2(t) = \Lambda b_0 \mathcal{V}(\phi) - \frac{1}{2}C\dot{\phi}^2(t) - \frac{3b_0\kappa}{a^2(t)} \tag{77}$$

where the first function is the Hubble function $H(t)$, here we are able to recover the FRW field equation coupled with a scalar field. Additionally, the field equation for the scalar is given by the Klein-Gordon field equation, replacing the cosmological ansatz leads to

$$\ddot{\phi}(t) + 3H(t)\dot{\phi}(t) + \Lambda b_0 \mathcal{V}'(\phi) = 0 \tag{78}$$

Thus we are able to recover Einstein-Hillbert coupled with a scalar field.