Affine Inflation in Polynomial Affine Gravity in $3 + 1$ dimensions

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Abstract

The Polynomial Affine Gravity its a purely affine model that mediates gravitational interactions solely and exclusive through the affine connection instead of the metric tensor. In this paper we couple a scalar field through *inverse tensor densities* and its potential energy to the volume form. We formulte an effective action in the torsion-free sector couple with the scalar field and study the cosmological solutions.

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The Polynomial Affine Gravity model its a purely affine model on which we endowed the manifold only with an affine connection $(\mathcal{M}, \Gamma)$. This allow us to define the notion of parallelism by the covariant derivative ∇. Since we only have an affine connection Γ we can only deffine the following chain of geometric objects

$$\Gamma_{\mu}{}^{\sigma}{}_{\nu} \to \nabla_{\mu} \to \mathcal{R}_{\mu\sigma}{}^{\tau}{}_{\nu} \to \mathcal{R}_{\mu\nu} \tag{1}$$

Notice that in the absence of the metric tensor it is not possible to define the $\mathcal{R}$.

2.1 The action

In order to built the action of the Polynomial Affine Gravity we use the irreducible fields of the affine connection, by separating the connection into its symmetric and antisymmetric part

$$\hat{\Gamma}_{\mu}{}^{\sigma}{}_{\nu} = \Gamma_{\mu}{}^{\sigma}{}_{\nu} + \mathcal{B}_{\mu}{}^{\sigma}{}_{\nu} + \delta^{\sigma}_{[\mu}\mathcal{A}_{\nu]} \tag{2}$$

where $\Gamma_\mu{}^\sigma{}_\nu$ correspond to the symmetric part of the connection, $\mathcal{B}_\mu{}^\sigma{}_\nu$ its the traceless part of the torsion tensor and $\mathcal{A}_\mu$ its the vectorial part of the torsion tensor. Additionally, we need to define the volume form, which can be written using only the wedge product

$$\mathrm{d}V^{\alpha\beta\gamma\delta} = J(x)\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma \wedge \mathrm{d}x^\delta \tag{3}$$

However, since we want to couple a scalar field $\phi(x)$ to this model, we need to introduce the potential energy of the scalar field $\mathcal{V}(\phi)$. Inspired by the work of Hemza Azri in affine inflation, we couple the potential energy to the volume form in the following manner

$$\mathrm{d}V^{\alpha\beta\gamma\delta} = \mathrm{d}\hat{V}^{\alpha\beta\gamma\delta}\frac{1}{\mathcal{V}(\phi)} \tag{4}$$

The action must preserv the invariance under diffemorphism, which is why the symmetric part of the connection goes indirectly throught the covariant derivative. The fundamental fields to build the action are $\nabla, \mathcal{A}, \mathcal{B}, \mathrm{d}V$. Then we perform a sort of *dimensional structural analysis technique* studying everysingle possible non-trivial contribution to the action.

THen, the most general action in $3+1$ dimension up to boundary terms is

$$\begin{aligned} S = \int \mathrm{d}V^{\alpha\beta\gamma\delta}\Big[& B_1\mathcal{R}_{\mu\nu}{}^\mu{}_\rho\mathcal{B}_\alpha{}^\nu{}_\beta\mathcal{B}_\gamma{}^\rho{}_\delta + B_2\mathcal{R}_{\alpha\beta}{}^\mu{}_\rho\mathcal{B}_\gamma{}^\nu{}_\delta\mathcal{B}_\mu{}^\rho{}_\nu + B_3\mathcal{R}_{\mu\nu}{}^\mu{}_\alpha\mathcal{B}_\beta{}^\nu{}_\gamma\mathcal{A}_\delta + B_4\mathcal{R}_{\alpha\beta}{}^\sigma{}_\rho\mathcal{B}_\gamma{}^\rho{}_\delta\mathcal{A}_\sigma \\ & + B_5\mathcal{R}_{\alpha\beta}{}^\rho{}_\rho\mathcal{B}_\gamma{}^\sigma{}_\delta\mathcal{A}_\sigma + C_1\mathcal{R}_{\mu\alpha}{}^\mu{}_\nu\nabla_\beta\mathcal{B}_\gamma{}^\nu{}_\delta + C_2\mathcal{R}_{\alpha\beta}{}^\rho{}_\rho\nabla_\sigma\mathcal{B}_\gamma{}^\sigma{}_\delta + D_1\mathcal{B}_\nu{}^\mu{}_\lambda\mathcal{B}_\mu{}^\nu{}_\alpha\nabla_\beta\mathcal{R}_\gamma{}^\lambda{}_\delta \\ & + D_2\mathcal{B}_\alpha{}^\mu{}_\beta\mathcal{B}_\mu{}^\lambda{}_\nu\nabla_\lambda\mathcal{B}_\gamma{}^\nu{}_\delta + D_3\mathcal{B}_\alpha{}^\mu{}_\nu\mathcal{B}_\beta{}^\lambda{}_\gamma\nabla_\lambda\mathcal{B}_\mu{}^\nu{}_\delta + D_4\mathcal{B}_\alpha{}^\lambda{}_\beta\mathcal{B}_\gamma{}^\sigma{}_\delta\nabla_\lambda\mathcal{A}_\sigma + D_5\mathcal{B}_\alpha{}^\lambda{}_\beta\mathcal{A}_\sigma\nabla_\lambda\mathcal{B}_\gamma{}^\sigma{}_\delta \\ & + D_6\mathcal{B}_\alpha{}^\lambda{}_\beta\mathcal{A}_\gamma\nabla_\lambda A_\delta + D_7\mathcal{B}_\alpha{}^\lambda{}_\beta\mathcal{A}_\lambda\nabla_\gamma A_\delta + E_1\nabla_\rho\mathcal{B}_\alpha{}^\rho{}_\beta\nabla_\sigma\mathcal{B}_\gamma{}^\sigma{}_\delta + E_2\nabla_\rho\mathcal{B}_\alpha{}^\rho{}_\beta\nabla_\gamma\mathcal{A}_\delta \\ & + F_1\mathcal{B}_\alpha{}^\mu{}_\beta\mathcal{B}_\gamma{}^\sigma{}_\delta\mathcal{B}_\mu{}^\lambda{}_\rho\mathcal{B}_\sigma{}^\rho{}_\lambda + F_2\mathcal{B}_\alpha{}^\mu{}_\beta\mathcal{B}_\gamma{}^\nu{}_\lambda\mathcal{B}_\delta{}^\lambda{}_\rho\mathcal{B}_\mu{}^\rho{}_\nu + F_3\mathcal{B}_\nu{}^\mu{}_\lambda\mathcal{B}_\mu{}^\nu{}_\alpha\mathcal{B}_\beta{}^\lambda{}_\gamma\mathcal{A}_\delta + F_4\mathcal{B}_\alpha{}^\mu{}_\beta\mathcal{B}_\gamma{}^\nu{}_\delta\mathcal{A}_\mu\mathcal{A}_\nu\Big]. \end{aligned}$$

To couple the affine action to a scalar field, we need to introduce a kinetic term in the absence of the metric tensor. In order to do so, we build *inverse symmetric tensor densities*, by using the *dimensional analysis structure technique*

$$\mathrm{g}^{\mu\nu} = (\alpha\nabla_\lambda\mathcal{B}_\rho{}^\mu{}_\sigma + \beta\mathcal{A}_\lambda\mathcal{B}_\rho{}^\mu{}_\sigma)\,\mathrm{d}V^{\nu\lambda\rho\sigma} + \gamma\mathcal{B}_\kappa{}^\mu{}_\lambda\mathcal{B}_\rho{}^\nu{}_\sigma\mathrm{d}V^{\kappa\lambda\rho\sigma} \tag{5}$$

Using the above expression we can define the kinetic term

$$S_\phi = -\int \mathrm{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \tag{6}$$

Since we want to work on the torsion-free sector, it is worth to notice that only the terms that are linear in the torsion will have a non-trivial contribution, C_1 and C_2. Additionally, since our connection its an *equi-affine* connection, the trace of the Riemman tensor will vanish completly. Applying the same idea the to the scalar field action, only the α term survive. Thus, the effective action coupled with a scalar field is

$$S_{ef} = \int \mathrm{d}V^{\alpha\beta\gamma\delta}\Big[C_1\mathcal{R}_{\mu\alpha}{}^\mu{}_\nu - \alpha\partial_\alpha\phi\partial_\nu\Big]\nabla_\beta\mathcal{B}_\gamma{}^\nu{}_\delta.$$

2.2 Cosmological ansatz

In order to solve the field equations, one need to build an ansatz, since we want to do cosmology, we need to build an ansazts compatible with the symmetries of the cosmological principle, which are rotation and translations. It is possible to build an ansatz for our fundamental geometric objects using the Lie derivative along the Killing vector fields. The most general ansatz for the symmetric part of the connection $\Gamma_\mu{}^\sigma{}_\nu$ is

$$\Gamma_t{}^t{}_t = f(t), \quad \Gamma_i{}^t{}_j = g(t)S_{ij} \tag{7}$$

$$\Gamma_t{}^i{}_j = h(t)\delta^i_j, \quad \Gamma_i{}^j{}_k = \gamma_i{}^j{}_k \tag{8}$$

the traceless part of the torsion tensor $\mathcal{B}_{\mu}{}^{\sigma}{}_{\nu}$ is completely define by only one time depending function

$$\mathcal{B}_{\theta}{}^{r}{}_{\varphi} = \psi(t) r^2 \sin\theta\sqrt{1-\kappa r^2} \qquad \mathcal{B}_{r}{}^{\theta}{}_{\varphi} = \frac{\psi(t)\sin\theta}{\sqrt{1-\kappa r^2}}$$

$$\mathcal{B}_{r}{}^{\varphi}{}_{\theta} = \frac{\psi(t)}{\sqrt{1-\kappa r^2}\sin\theta}$$

and finally, the vectorial torsion tensor $\mathcal{A}_{\mu}$ is given by

$$\mathcal{A}_t = \eta(t) \tag{9}$$

Since, we have the tensor $g_{\mu\nu}$ presented as a particular solution to the field equation. Thus, we need to provied an ansatz for this tensor compatible with the symmetries of the cosmological principle

$$g_{\mu\nu} = b(t)\mathrm{d}t^2 + a(t)\left(\frac{1}{1-\kappa r^2}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2\right) \tag{10}$$

Additionally, its required that the covariant derivative of the object $g_{\mu\nu}$ must vanishes completly, from which we found that

$$b(t) = -b_0 \qquad h(t) = \frac{\dot{a}(t)}{a(t)} \qquad g(t) = \frac{\dot{a}(t)a(t)}{b_0} \tag{11}$$

It is important to remark to the above definitions will only have an effect on the parallel field equation, which is where the $g_{\mu\nu}$ object exist.

2.3 The field equations

The field equations are obtained using Kiwosjki's formalism and taking into account the symmetries and properties of the fundamental fields, we vary the action with respect to the fundamental fields. In the torsion-free limit, the field equation is

$$\nabla_{\mu}\left[\frac{1}{\mathcal{V}(\phi)}\left(C\partial_{\alpha}\phi\partial_{\lambda}\phi - \mathcal{R}_{\alpha\lambda}\right)\mathrm{d}V^{\mu\nu\rho\alpha}\right] + \frac{2}{3}\nabla_{\mu}\left[\frac{1}{\mathcal{V}(\phi)}\mathcal{R}_{\alpha\theta}\delta^{[\nu}_{\lambda}\mathrm{d}V^{\rho]\alpha\mu\theta}\right] = 0 \tag{12}$$

By multiplying the left hand side of the field equation by $\epsilon_{\nu\rho\tau\beta}$, the second term vanishes completly and the field equation is reduced even further to

$$\nabla_{[\mu}\left(\mathcal{R}_{\nu]\gamma}\frac{1}{\mathcal{V}(\phi)}\right) - C\nabla_{[\mu}\left(\partial_{\nu]}\phi\partial_{\gamma}\phi\frac{1}{\mathcal{V}(\phi)}\right) = 0 \tag{13}$$

A particular solution to the above equation is

$$\mathcal{R}_{\mu\nu} - C\partial_{\mu}\phi\partial_{\nu}\phi = \Lambda\mathcal{V}(\phi)g_{\mu\nu} \tag{14}$$

which can be written as

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} + \Lambda\mathcal{V}(\phi)g_{\mu\nu} = C\left(\partial_{\mu}\phi\partial_{\nu}\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2\right) \tag{15}$$

Taking the divergence ∇^{μ} of the above equation leads to

$$C\nabla^{\mu}\nabla_{\mu}\phi - \Lambda\mathcal{V}'(\phi) = 0 \tag{16}$$

This is the Klein-Gordon field equation which requires the existence of the $g_{\mu\nu}$ object, to define the d'Alembert operator. Without this object, it is not possible to obtain a Klein-Gordon field equation, and additionally we require that the integration constant $\Lambda \neq 0$.

From the field equation obtained by varying the action, we distinguish two families of solutions, the first one being the *Reduced field equation* couple with the Klein-Gordon field equation

$$\frac{\mathcal{R}_{\mu\nu} - C\partial_{\mu}\phi\partial_{\nu}\phi}{\mathcal{V}(\phi)} = \Lambda g_{\mu\nu} \qquad C\nabla^{\mu}\nabla_{\mu}\phi - \Lambda\mathcal{V}'(\phi) = 0 \tag{17}$$

where the Klein-Gordon field equation emerge naturally from the parallelism constraint. The second type of solution its given by *Harmonic field equation* which is coupled with a kinetic term and potential energy

$$\nabla_{[\mu}\left(\mathcal{R}_{\nu]\gamma}\frac{1}{\mathcal{V}(\phi)}\right)-C\nabla_{[\mu}\left(\partial_{\nu]}\phi\partial_{\gamma}\phi\frac{1}{\mathcal{V}(\phi)}\right)=0 \tag{18}$$

3 Cosmological Solutions

Here we study all the possible solutions to the field equations under the simmetries of the cosmological principle. Notice that, in the harmonic field equation the tensor $g_{\mu\nu}$ does not exist, therefore, the connection's coefficients are written just as the ones obtained by building the ansatz.

The Ricci tensor can be use as an emergent metric tensor only when its well defined and not degenerate, meaning its temporal and spatial part are not trivial. Under the cosmological ansatz the Ricci tensor is

$$\mathcal{R}_{tt}=-3\left(\dot{h}(t)+h^2(t)\right) \qquad \mathcal{R}_{rr}=\frac{\dot{g}(t)+g(t)h(t)+2\kappa}{1-\kappa r^2} \tag{19}$$

Notice that the expression $\dot{g}(t)+g(t)h(t)+2\kappa$ its the affine analogue to the well-known scale factor $a(t)$ in the FRW universe. As one does in *General Relativity*, demanding that its covariant derivative must vanishes completly, therefore, we are able to completely define the affine functions of the symmetric part of the connection

$$h(t)=\sqrt{A_1}\tanh\left(t\sqrt{A_1}\right) \qquad g(t)=\frac{\kappa\sinh\left(2t\sqrt{A_1}\right)}{2\sqrt{A_1}} \tag{20}$$

The above definitions ensures that $\nabla_\alpha\mathcal{R}_{\beta\gamma}=0$, this asservs the metricity condition. Then, the Ricci tensor is written as

$$\mathcal{R}_{tt}=-3A_1 \qquad \mathcal{R}_{rr}=\frac{3\kappa\cosh^2(t\sqrt{A_1})}{1-\kappa r^2} \tag{21}$$

Here we have an *affine scale factor* given by

$$a_f(t)=\left(3\kappa\cosh^2(t\sqrt{A_1})\right)^{1/2} \tag{22}$$

3.1 Reduced equations

Under the cosmological ansatz the field equation for the geometric part is written as follow

$$0=3\ddot{a}(t)+Ca(t)\dot{\phi}^2(t)-\Lambda b_0 a(t)\mathcal{V}(\phi) \tag{23}$$

$$0=\ddot{a}(t)a(t)+2\dot{a}^2(t)+2b_0\kappa-\Lambda b_0 a^2(t)\mathcal{V}(\phi) \tag{24}$$

Combining the two equations we obtained

$$3H^2(t)=\Lambda b_0\mathcal{V}(\phi)-\frac{1}{2}C\dot{\phi}^2(t)-\frac{3b_0\kappa}{a^2(t)} \tag{25}$$

where the first function is the Hubble function $H(t)$. Notice that for $\kappa=0$ we recover the classical Friedmann equation. Additionally, the field equation for the scalar is given by

$$\ddot{\phi}(t)+3h(t)\dot{\phi}(t)+\Lambda b_0\mathcal{V}'(\phi)=0 \tag{26}$$

We are able to recover Einstein-Hillbert coupled with a scalar field, therefore, there is no new information in this type of solution.

3.2 Harmonic equations

The *Harmonic field equation* coupled with the scalar field is given by

$$\nabla_{[\mu}\left(\mathcal{R}_{\nu]\gamma}\frac{1}{\mathcal{V}(\phi)}\right)-C\nabla_{[\mu}\left(\partial_{\nu]}\phi\partial_{\gamma}\phi\frac{1}{\mathcal{V}(\phi)}\right)=0 \tag{27}$$

which can be rewritten as

$$\nabla_{[\mu}\mathcal{S}_{\nu]\gamma}=0 \tag{28}$$

where the $\mathcal{S}_{\mu\nu}$ tensor is a symmetric tensor defined by

$$\mathcal{S}_{\mu\nu}=\left(\mathcal{R}_{\mu\nu}-C\partial_{\mu}\phi\partial_{\nu}\phi\right)\frac{1}{\mathcal{V}(\phi)} \tag{29}$$

The *Harmonic field equation* under the symmetries of the cosmological ansatz can be written as follow

$$C\mathcal{V}(\phi)g(t)\dot{\phi}^2(t)+\mathcal{V}(\phi)\left(4g(t)h^2(t)+2\kappa h(t)+2g(t)\dot{h}(t)-\ddot{g}(t)\right)+\mathcal{V}'(\phi)\dot{\phi}(t)\left(g(t)h(t)+2\kappa+\dot{g}(t)\right)=0 \tag{30}$$

Since we have two unknown functions of time and the scalar field, it is not possible to solve the above equation. However, if the Ricci tensor it is not degenerate, then it can serve the function as a metric tensor. Therefore, demanding that is covariant derivative must be zero, the above equation is reduced to

$$\dot{\phi}(t)+\frac{3\sqrt{A_1}\cosh\left(t\sqrt{A_1}\right)}{C\sinh\left(t\sqrt{A_1}\right)}\left(\frac{\mathcal{V}'(\phi)}{\mathcal{V}(\phi)}\right)=0 \tag{31}$$

At this point we can proceed as one usually does in classical cosmology, given a potential you find the scale factor and the scalar field, whereas here given a potential, you only need to determined the scalar field. The most well known potentials are the *Power-Law potential* and *Starobinsky potential.*

Taking first the *Power-Law potential* meaning $\mathcal{V}(\phi)=\beta\phi^n(t)$, then the above equation leads as

$$\dot{\phi}(t)+\frac{1}{\phi(t)}\left(\frac{3\sqrt{A_1}\cosh(t\sqrt{A_1})}{C\sinh(t\sqrt{A_1})}\right)=0 \tag{32}$$

which can be solved analytically.

Then we take *Starobinsky potential* written as $\mathcal{V}(\phi)=\alpha\left(1-e^{-\beta\phi}\right)^2$, the equation takes the form of

$$\dot{\phi}(t)+e^{-\beta\phi}\left(\frac{6\beta\sqrt{A_1}\cosh(t\sqrt{A_1})}{C\sinh(t\sqrt{A_1})}\right)=0 \tag{33}$$

which can be solved analytically.

4 Conclusions