Numerical solutions for boson stars in $f(\mathcal{R})$ gravity using PINNs

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1 Introduction

2 Dynamics in $f(\mathcal{R})$

In Palatini's formalisms, the manifold is endowed with two fundamental and independent fields: the metric tensor $g_{\mu\nu}$ and the affine connection $\Gamma^{\alpha}{}_{\beta\gamma}$, the former allow us to define the notion of distance, whereas the latter define the notion of parallelism. The action is written as follow

$$S[g,\Gamma] = \frac{1}{2k}\int \mathrm{d}^4x\sqrt{-g}f(\mathcal{R}) - \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\mathcal{P}(X,\Phi), \tag{1}$$

where the first term correspond to a generalization of the Einstein-Hilbert action, by replacing the functional with an arbitrary function of the Ricci scalar, and the second term stands for the matter sector defined as

$$\mathcal{P} = X - 2V(\Phi), \tag{2}$$

where $X = g^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi$ and $V(\Phi) = -\frac{1}{2}\mu^2\Phi\Phi$, with μ as the mass of the complex scalar field. The field equations are obtained through varying the action with respect the fundamental fields

$$f_{\mathcal{R}}\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f_{\mathcal{R}} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_{\mathcal{R}} = k\mathcal{T}_{\mu\nu}, \tag{3}$$

$$\nabla_\lambda\left(\sqrt{-g}f_{\mathcal{R}}g^{\mu\nu}\right) = 0 \tag{4}$$

where Eqs.(3) and (4) are the field equation coming from varying the action with respect to the inverse metric tensor $g^{\mu\nu}$ and the affine connection $\Gamma^{\alpha}{}_{\beta\gamma}$ respectively. Additionally, the energy-momentum tensor is defined as

$$\mathcal{T}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{P}(X,\Phi)\right)}{\delta g^{\mu\nu}}. \tag{5}$$

An exact solution to Eq.(4) can be found by introducing a conformal transformation of the metric tensor

$$q_{\mu\nu} = f_{\mathcal{R}}g_{\mu\nu}. \tag{6}$$

where the conformal factor is written as $f_{\mathcal{R}}$. The conformal transformation, allow us to write an explicit relation between the affine connection with the conformal metric

$$\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2} q^{\lambda\rho} \left(\partial_\mu q_{\rho\nu} + \partial_\nu q_{\rho\mu} - \partial_\rho q_{\mu\nu} \right), \tag{7}$$

this are the Christof fel symbols, and we can use it in Eq.(3) to find an equation that only involves the metric tensor and the matter.

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} q_{\mu\nu}\mathcal{R} = \frac{k}{f_{\mathcal{R}}}\mathcal{T}_{\mu\nu} - \frac{\mathcal{R} f_{\mathcal{R}} - f}{2 f_{\mathcal{R}}} q_{\mu\nu} - \frac{3}{2 f_{\mathcal{R}}^2}\left(\partial_\mu f_{\mathcal{R}} \partial_\nu f_{\mathcal{R}} - \frac{1}{2} q_{\mu\nu} (\partial f_{\mathcal{R}})^2 \right) + \frac{1}{f_{\mathcal{R}}}\left(\nabla_\mu \nabla_\nu f_{\mathcal{R}} - q_{\mu\nu} \Box f_{\mathcal{R}} \right) \tag{8}$$

using

$$f(\mathcal{R}) = \mathcal{R} + \xi \mathcal{R}^2 \tag{9}$$

Working in the Einstein-frame, the action is written as

$$S = \frac{1}{2k} \int \mathrm{d}^4 x \sqrt{-q}\mathcal{R} - \frac{1}{2} \int \mathrm{d}^4 x \sqrt{-q} \mathcal{K}(Z, \Phi), \tag{10}$$

where $\mathcal{K}(Z, \Phi)$ is defined as

$$\mathcal{K}(Z, \Phi) = \frac{Z - \xi k Z^2}{1 - 8\xi k V} - \frac{2V}{1 - 8\xi k V}. \tag{11}$$

The first field equation, is obtained by varying the action with respect to the inverse metric $g^{\mu\nu}$

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R} g_{\mu\nu} = k\mathcal{T}_{\mu\nu}, \tag{12}$$

where the energy-momentum tensor $\mathcal{T}_{\mu\nu}$ is defined as

$$\mathcal{T}_{\mu\nu} = \frac{\left(\partial_\mu \Phi \partial_\nu \Phi + \partial_\mu \Phi \partial_\nu \Phi\right)(1 - 2\xi k Z) - q_{\mu\nu}\left(Z\left(1 - \xi k Z\right) + \mu^2 |\Phi|^2 \right)}{2 + 8\xi k \mu^2 |\Phi|^2}. \tag{13}$$

The second field equation is obtained with the variation with respect to the complex scalar field

$$\mathrm{d}s^2 = -\alpha^2(x)\mathrm{d}t^2 + \beta^2(x)\mathrm{d}x^2 + x^2\mathrm{d}\theta^2 + x^2\sin^2\theta\mathrm{d}\varphi^2 \tag{14}$$

3 Numerical analysis

4 Final remarks