Cosmological Inflation and Primordial Perturbation

(Notes for Lecture 1)

William Giarè

School of Mathematics & Statistics, The University of sheffield
Hicks Building · Hounsfield Road · Sheffield S3 7RH
w.giare@sheffield.ac.uk | www.williamgiare.com

– Abstract –

In these brief notes, I review the basic motivations underlying cosmological inflation as well as the main features of the single-field slow-roll paradigm. I pay some attention to the primordial inflationary fluctuations and the resulting primordial scalar and tensor spectra predicted by inflation. I discuss the relation between the primordial spectra and the CMB angular power spectrum.

1 Introduction

As often happens in cosmology and, more broadly, in science, observations challenge theories. As our experimental precision has increased over the years, many observational facts have come to light. Many of them are extremely difficult to explain in the context of the Hot Big Bang Theory.

For instance, observations of the Cosmic Microwave Background (CMB) teach us that the Early Universe was very homogeneous, and all CMB photons share the same temperature within small fluctuations of order $\delta T/T \approx 10^{-5}$. This is very hard to explain in the theoretical framework described by the Hot Big Bang Theory because CMB photons are separated by a distance greater than the particle horizon, and they have never communicated. So, according to the Hot Big Bang Theory, the last scattering surface should consist of many causally disconnected regions, and there is no dynamical reason why such regions (that have never "talked") could share similar physical conditions. We are forced to suppose a fine-tuning of thousands of initial conditions to explain homogeneity in the Early Universe.

Another observational fact difficult to explain within the Hot Big Bang Theory is that the Universe is spatially flat (or at the very least nearly flat). Information about the spatial geometry of the Universe is typically expressed in terms of the curvature density parameter $\Omega_k(t)$, releted to the total energy density of the universe

$$\Omega(t) \doteq \frac{\rho(t)}{\rho_c(t)}, \quad \rho_c(t) \doteq \frac{3H^2}{8\pi G}. \tag{1}$$

as $\Omega(t) = 1 - \Omega_k(t)$. Notice that $\Omega_k(t_0) = 0$ corresponds to a spatially flat Universe today while $\Omega_k(t_0) < 0$ and $\Omega_k(t_0) > 0$ correspond to a present-day spatially closed and open Universe, respectively. Using the Freemdan Equations

$$H^2 \equiv \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\kappa}{a^2}, \tag{2}$$

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + 3p). \tag{3}$$

the time evolution of $\Omega_k(t)$ be easily linked in terms of $H(t)$:

$$\Omega_k(t) \propto \frac{1}{(a\,H)^2} \tag{4}$$

Since the Hubble radius $1/(aH)$ grows with time for each component of the cosmic fluid with an equation of state $P/\rho \equiv \omega > -\frac{1}{3}$ (where P is the pressure and ρ is the energy density) when the Universe was dominated by radiation or matter, $\Omega_k(t) \propto a(t)^2$ and $\Omega_k(t) \propto a(t)$, respectively. This means that in Hot Big Bang cosmology, flatness is not a tracking solution of the FRW dynamics, and observing a flat geometry today would require imposing extreme fine-tuning at early times. Using the fact that in an expanding Universe the temperature scale ad $T \propto 1/a$, we can roughly have an idea of what $\Omega(t)$ should be at the Planck time $t_p \doteq \frac{1}{M_{pl}} = 5.4 \times 10^{-44}\,\mathrm{s}$ (i.e., the time when the temperature of the Universe is at the Planck epoch):

$$\frac{|\Omega - 1|_{T=T_{\rm Planck}}}{|\Omega - 1|_{T=T_0}} \approx \left(\frac{a_{\rm Pl}^2}{a_0^2}\right) \approx \left(\frac{T_0^2}{T_{\rm Pl}^2}\right) \approx \mathcal{O}\left(10^{-64}\right) \tag{5}$$

where we remember that the present epoch temperature of the Universe is $T_0 \approx 10^{-13}$ GeV. This means that in order to have a spatially flat Universe today, we would need to require a precision in the initial conditions within 1 part over 10^{60} at the Planck time.

Cosmological Inflation is believed to be the physical mechanism able to fix all the required initial conditions without controversial assumptions. At its core, inflation is an early epoch of "fast" accelerated expansion of the Universe, $\ddot{a} > 0$. To figure out how a phase of repulsive gravity can drive the Universe towards homogeneity and flatness, we recall that an accelerated expansion requires $\omega < -\frac{1}{3}$. In this case, also the Hubble radius $(aH)^{-1}$ decreases over time, and this automatically solves the flatness problem:

$$\underbrace{|1 - \Omega(a)|}_{\text{Driven to flatness}} = \underbrace{|-\kappa(a\,H)^{-2}|}_{\Leftarrow \text{decrease}}. \tag{6}$$

In particular, in the so-called de Sitter limit, $\omega = -1$, the spacetime expansion becomes exponentially accelerated, as well as the Hubble sphere exponentially shrinks: $(aH)^{-1} \propto e^{-Ht}$. In terms of the particle horizon, $\eta = -(1/H)e^{-Ht}$, we see that the initial singularity is pushed back to $\eta_i \to -\infty$, with the hypersurface $\eta = 0$ corresponding to the end of inflation. So, not only is the curvature exponentially driven to flatness, but now there is an "infinite" amount of conformal time $(d\eta = dt/a(t))$ to let the past light-cones of CMB photons intersect in such a way that the homogeneity observed in the CMB radiation is simply explained in terms of thermal equilibrium.

Anyway it is also evident that in an exact de Sitter background the end of inflation is reached at the cosmological time $t = \infty$ which means that inflation would go on forever. This is clearly due to the isometries of the de Sitter background which, being a maximally symmetric solution, preserve invariance under time translations. Therefore to ensure the end of inflation, the de Sitter limit, although valid at early times, must be broken near the end of inflation. Therefore we need a dynamical process able to transit from an inflating phase to a radiation dominated era.

2 Single Field Slow-Roll Inflation

The simplest dynamical model of Inflation involves a scalar field ϕ, which from now on we call the inflaton, minimally coupled to gravity. The action reads

$$S = \int d^4x\,\sqrt{-g}\left[\frac{\bar{M}_p^2}{2}\,R + \frac{1}{2}\,g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right], \tag{7}$$

with $\bar{M}_p = 1/\sqrt{8\pi G}$ the *reduced* Planck Mass in the natural units $c = \hbar = 1$. This theory is said to be “minimal coupled to gravity” because there is not a direct coupling between the inflaton field and the metric tensor in the action.

2.1 Klein-Gordon Equation

The equation of motion can be obtained minimizing the action with respect to the field $\delta S_\phi/\delta\phi = 0$. A trivial computation gives:

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\partial^\mu\phi\right) + \frac{\delta V(\phi)}{\delta\phi} = 0. \tag{8}$$

In principle, to solve this equation in full generality we should use the FRW metric with a non-vanishing curvature since it is the inflation itself that drives the spacetime to be flat. Anyway, the observational predictions of inflation come out towards its ending phase when the spacetime is already stretched towards flatness. So, a consistent theory of initial conditions is not required for investigating the inflationary predictions, and we can simply use a flat FRW metric. In this way, from Eq. (8), we obtain

$$\ddot{\phi} + 3H\dot{\phi} - a^{-2}(t)\,\nabla^2\phi + \frac{\delta V(\phi)}{\delta\phi} = 0. \tag{9}$$

If we restrict our attention on homogeneous scalar fields, the gradient term vanishes $\nabla^2\phi = 0$ and the functional derivative $\delta V(\phi)/\delta\phi$ reduces to the ordinary one $dV(\phi)/d\phi \equiv V'(\phi)$. The equation of motion eventually becomes

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0. \tag{10}$$

2.2 Friedmann Equations

Minimizing the action with respect to the metric $\delta S/\delta g^{\mu\nu} = 0$ we can find the relation for Stress-Energy tensor

$$T^\phi_{\mu\nu} = g_{\mu\nu}\mathcal{L}_\phi - 2\frac{\delta\mathcal{L}_\phi}{\delta g^{\mu\nu}} \tag{11}$$

$$= -\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}\left(\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi - V(\phi)\right) \tag{12}$$

and get the relation for the energy-density and pressure in a inflaton-dominated Universe, namely:

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \tag{13}$$

$$P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \tag{14}$$

The equation of state ω is now a function of the scalar field

$$\omega_\phi \equiv \frac{P_\phi}{\rho_\phi} = \frac{\frac{1}{2}\dot{\phi}^2 - V(\phi)}{\frac{1}{2}\dot{\phi}^2 + V(\phi)}. \tag{15}$$

and we clearly see that if $V \gg \dot{\phi}^2$ a phase of repulsive gravity $\omega_\phi \approx -1 < -1/3$ is obtained and the Universe starts inflating. The Friedmann equations (2) and (3) for a Universe dominated by the homogeneous scalar field ϕ read as follows:

$$3\bar{M}_p^2\, H^2 = \rho_\phi = \left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right), \tag{16}$$

$$\bar{M}_p^2 \left(\frac{\ddot{a}}{a}\right) = -\frac{1}{6}\left(\rho_\phi + 3p_\phi\right) = -\frac{1}{3}\left(\dot{\phi}^2 - V(\phi)\right). \tag{17}$$

2.3 Slow-Roll conditions

This minimal scenario is commonly called slow roll inflation. The price to pay to achieve inflation is to put restrictions on the scalar field ϕ and above all on its potential $V(\phi)$. First of all, taking the derivative of the equation (16) and using the equation of motion (10) we have:

$$2\,\bar{M}_p^2 \dot{H} = -\dot{\phi}^2 \tag{18}$$

During the slow roll phase we want an almost exponentially expansion and so we have to require $\omega_\phi \approx -1$ that implies $\dot{\phi}^2 \ll V(\phi)$. So from equation (16)

$$V(\phi) \approx 3\,\bar{M}_p^2\, H^2, \tag{19}$$

and the slow roll condition is equivalent to require that

$$\epsilon_H \equiv -\frac{\dot{H}}{H^2} \ll 1. \tag{20}$$

On the other hand, taking the derivative of Eq.(19) and using the slow roll condition we get also

$$V_\phi(\phi) \approx -3\,H\dot{\phi}, \tag{21}$$

where $V_{\phi\ldots\phi}$ indicates the derivatives of the potential with respect to the filed. This implies $|\ddot{\phi}| \ll 3\,H|\dot{\phi}|$.

The slow-roll conditions can be expressed in terms of the slow-roll parameters

$$\epsilon_V \doteq \bar{M}_{\rm p}^2 \frac{1}{2}\left(\frac{V_\phi^2}{V^2}\right) \simeq \epsilon_H, \tag{22}$$

$$\eta_V \doteq \bar{M}_{\rm p}^2 \left(\frac{V_{\phi\phi}}{V}\right), \tag{23}$$

such that $\epsilon \ll 1$ and $|\eta_V| \ll 1$ Notice that the limit $\epsilon_V \to 0$ corresponds to an exactly de Sitter expansion.

2.3.1 Inflationary potential and Reheating

Inflation can be easily achieved when the potential looks like that shown in Figure 1. Along the flat plateau the kinetic energy of the scalar field $\dot{\phi}$ becomes negligible with respect to the potential energy $V(\phi)$ which is instead approximately constant. In this way, $\omega_\phi \approx -1$ and we have an almost de Sitter phase. On the other hand, when this condition breaks down, inflation ends and the scalar field typically falls into a potential well starting oscillating. This phase of oscillation around the vacuum state is called reheating and is required to restore particles in the Universe. Indeed during the slow period, the Universe is exponentially driven towards flatness and homogeneity but all its pre-inflationary contents are exponentially diluted as well. This means that at the end of inflation the Universe appears nearly empty and dominated by a scalar field in a state of coherent oscillation about the vacuum state. This looks far away from the Hot Big Bang picture: there is no radiation or particles but only an enormous amount of energy. Indeed, as by definition the energy-density during the inflationary expansion remains constant, the total energy $E_i = \rho V_i$ exponentially expands with the volume of the Universe. Such an exponential amount of energy can easily decay into radiation and particles

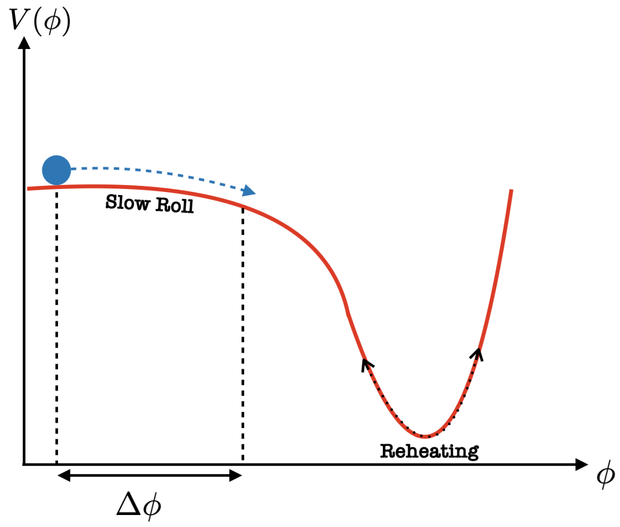

Figure 1: The typical shape of a good Inflationary Potential.

when the field starts oscillating during the reheating phase. The details of reheating clearly depend on the specific shape of the potential, nevertheless this is typically a very rapid process. When the field is oscillating around the minimum, we can approximate the potential as $V(\phi) \approx \frac{\lambda^2}{2}\phi^2$ so that:

$$\rho_\phi = \frac{1}{2}\left(\dot{\phi}^2 + \lambda^2\phi^2\right), \tag{24}$$

taking the time derivative, $\dot{\rho}_\phi = \dot{\phi}\,\ddot{\phi} + \lambda^2\phi\,\dot{\phi}$, and using Eq. (10) we get:

$$\dot{\rho}_\phi + 3H\rho_\phi = \frac{3H}{2}\underbrace{\left(\lambda^2\phi^2 - \dot{\phi}^2\right)}_{\text{oscillates}}. \tag{25}$$

The oscillating factor on the right hand side averages out to zero over one oscillation period and the long-time behavior of the energy density eventually reads:

$$\dot{\bar{\rho}}_\phi + 3H\bar{\rho}_\phi = 0. \tag{26}$$

Note that the inflaton field is doing small oscillations around the potential minimum and the energy density can decay into particles. If the decay is slow the inflaton energy density follows the equation:

$$\dot{\bar{\rho}}_\phi + (3H + \Gamma)\bar{\rho}_\phi = 0, \tag{27}$$

where Γ represents the inflation decay rate and so $-\Gamma\rho_\phi$ is the energy transferred to other particles. Whether the inflaton decays into bosons, the process may be very rapid and violent and it is known as pre-heating. Anyway the particles produced in this stage will eventually interact, creating other particles unless the thermal equilibrium will be restored at some temperature so that the standard Hot Big Bang evolution can start.

2.4 Primordial Fluctuations

The Cosmic Microwave Background radiation appears highly uniform, but we do observe temperature and polarization anisotropies. For instance, we measure differences in temperature of CMB photons of the order of $\delta T/T \sim 10^{-5}$. At first glance, one might assume these fluctuations simply exist, but what is striking is that they are correlated across scales well beyond the horizon at the time of decoupling. This poses the question: how can fluctuations that seemingly never interacted be correlated? Once again, inflation theory offers an intriguing solution to this problem.

The basic idea underlying the origin of the primordial perturbations is that, during inflation, the inflaton field ϕ evolving on the potential $V(\phi)$ will not have a completely classical dynamics, but it will also have some small quantum fluctuations around its classical trajectory. Quantum fluctuations of the inflaton field are so blown up on superhorizon scales by inflation itself becoming classical perturbations. Therefore inflation, combined with quantum mechanics, provides an elegant mechanism for generating the initial seeds of all structures in the Universe.

To describe a rigorous picture of quantum fluctuations, in general, we should consider perturbations in the metric, too. Nevertheless the Einstein equations relate perturbations in the metric to perturbations in the fields and so there is essentially only one physical degree of freedom. If we choose to work in the so called spatially flat Gauge, namely the Gauge in which the curvature of space-like hypersurfaces is zero and the spatial part of the metric is unperturbed, we can quantify this degree of freedom as the field fluctuations $\delta\phi$, leaving the metric unperturbed. At the same level of accuracy, we can also drop the contribution that arises from the inflationary potential $V(\phi)$, considering the field to be free. In this way, we can just focus on $\delta\phi$.

2.4.1 Perturbations in a free scalar Field

We split field and the fluctuations $\phi \to \phi + \delta\phi$. The perturbations in general will not be homogeneous, $\delta\phi = \delta\phi(t, x)$, so in their equation of motion we must consider also the spatial dependence and, in light of Eq. (9), we write

$$\ddot{\delta\phi} + 3H\dot{\delta\phi} - a^{-2}(t)\,\nabla^2\delta\phi = 0. \tag{28}$$

It is useful to use the conformal coordinates $\partial_t = (1/a)\partial_\eta$ in such a way that

$$\delta\phi'' + 2\left(\frac{a'}{a}\right)\delta\phi' - \nabla^2\delta\phi = 0 \tag{29}$$

where we used the notation $(\dots)\prime \equiv \partial_\eta(\dots)$. We have to quantize the field in a FRW spacetime. We proceed analogously to the canonical quantization process, expanding the field $\delta\phi$ into its Fourier components

$$\delta\phi(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \left[\delta\phi_{\mathbf{k}}(\eta)\, b_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + \delta\phi^*_{\mathbf{k}}(\eta)\, b^*_{\mathbf{k}}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\right] \tag{30}$$

and promoting the field $\delta\phi$ to be an operator $\delta\phi \to \hat{\delta\phi}$ and $(b_{\mathbf{k}}, b^*_{\mathbf{k}}) \to (\hat{b}_{\mathbf{k}}, \hat{b}^\dagger_{\mathbf{k}})$. We interpret $(\hat{b}_{\mathbf{k}}, \hat{b}^\dagger_{\mathbf{k}})$ as the common creation and annihilation operators. We also impose the canonical quantization conditions:

$$\left[\hat{b}_{\mathbf{k}}\,,\,\hat{b}^\dagger_{\mathbf{k'}}\right] = \delta^3(\mathbf{k} - \mathbf{k'}), \tag{31}$$

$$\left[\hat{b}_{\mathbf{k}}\,,\,\hat{b}_{\mathbf{k'}}\right] = \left[\hat{b}^\dagger_{\mathbf{k}}\,,\,\hat{b}^\dagger_{\mathbf{k'}}\right] = 0. \tag{32}$$

Using Eq. (29), we easily find the equation for the Fourier components

$$\delta\phi''_k + 2\left(\frac{a'}{a}\right)\delta\phi'_k + k^2\delta\phi_k = 0 \tag{33}$$

where with k^2 we are intending the spatial Euclidean amplitude $k^2 = |\mathbf{k}|^2$. At this point it is useful to introduce the following field redefinition:

$$u_k \equiv a(\eta)\,\delta\phi_k(\eta), \tag{34}$$

The equation for the Fourier modes (33) in terms of the new field u_k reads

$$u''_k + \left[k^2 - \frac{a''}{a}\right]u_k = 0. \tag{35}$$

This is known as *Mukhanov equation* and can be considered the generalization of the Klein-Gordon equation in an expanding Universe. In the so called *Ultraviolet limit* $k \gg \frac{a''}{a}$, Eq. (35) simplifies to

$$u''_k + k^2\,u_k = 0. \tag{36}$$

whose solution is given by

$$u_k(\eta) = \frac{1}{\sqrt{2k}}\left(A_k\,e^{-ik\eta} + B_k\,e^{ik\eta}\right) \tag{37}$$

with A_k and B_k to be fixed by choosing an appropriate vacuum state (we will do this soon). On the other hand, in the so called *Infrared limit* $k \ll \frac{a''}{a}$ the equation (35) reads

$$a\,u''_k - a''\,u_k = 0, \tag{38}$$

with the easy solution

$$u_k \propto a(\eta) \Rightarrow \delta\phi_k = \text{const}. \tag{39}$$

proving a very interesting feature: the Fourier mode $\delta\phi_k$ does not evolve on the super-horizon scales (i.e. $k \ll aH$). This phenomenon is called *mode freezing*.

We now come back to the issue of the vacuum state. The mode amplitude depends on the constant A_k and B_k and all of their physics boils down the boundary condition for the field perturbations in the ultraviolet limit. This problem is strictly related to the vacuum selection in the canonical quantization process. Indeed with some efforts, one can show that the canonical quantization condition for the operators $(\hat{b}_k, \hat{b}_k^\dagger)$ translates into a boundary condition for the u_k and u_k^* modes that is nothing else but the *Wronskian condition*

$$W(u_k\,,\,u_k^*) \equiv u_k\,(u_k^*)' - (u_k)' u_k^* = i. \tag{40}$$

Using the solution (37) it is easy to see that this implies $|A_k|^2 - |B_k|^2 = 1$, the same condition that one would obtain in a Minkowski spacetime. Anyway, it is not enough to complete the solution and, in fact, a second relation arises from the vacuum selection in our FRW spacetime. We define the vacuum state for the FRW spacetime as the state where all the comoving observers see no particles which is to require that in the ultraviolet limit the FRW spacetime is asymptotically Minkowskian, *i.e.*, $A_k = 1 \quad$ and $\quad B_k = 0$. This is known as the *Bunch-Davies vacuum*. With this choice the solution in the ultraviolet limit eventually becomes

$$u_k(\eta) = \frac{1}{\sqrt{2k}} e^{-ik\eta}, \tag{41}$$

2.4.2 Power spectrum in a de Sitter spacetime

Notice that equation (35) depends on the spacetime background by a term $\frac{a''}{a}$ and so it is quite difficult to find a generic solution for this equation. However, here, we are interested in the quantum inflationary fluctuations and, since during inflation our spacetime is approximately de Sitter, it is worth finding an exact solution in this limit. By noting that in a de Sitter spacetime $\eta = -1/(a\,H)$, and $a''/a = 2/\eta^2$, the Mukhanov equation becomes:

$$u_k'' + \left(k^2 - \frac{2}{\eta^2}\right) u_k = 0. \tag{42}$$

By a direct substitution one can check that an exact solution is:

$$u_k = A_k\,\frac{e^{-ik\eta}}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right) + B_k\,\frac{e^{ik\eta}}{\sqrt{2k}}\left(1 + \frac{i}{k\eta}\right). \tag{43}$$

which fixing the Bunch-Davies vacuum ($A_k = 1$ and $B_k = 0$) eventually becomes

$$u_k = \frac{e^{-ik\eta}}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right). \tag{44}$$

So we have the complete expression of the field operator $\hat{\delta\phi}$ in the de Sitter spacetime

$$\hat{\delta\phi}(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \left[\left(\frac{u_{\mathbf{k}}}{a}\right) \hat{b}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + \left(\frac{u_{\mathbf{k}}^*}{a}\right) \hat{b}_{\mathbf{k}}^\dagger\, e^{-i\mathbf{k}\cdot\mathbf{x}}\right]. \tag{45}$$

One can now compute the power spectrum of the fluctuations around the vacuum state in the de Sitter limit

$$\langle 0|\hat{\delta\phi}(\eta, \mathbf{x})\hat{\delta\phi}(\eta, \mathbf{x'})|0\rangle = \int \frac{d^3k\, d^3k'}{(2\pi)^3} \left(\frac{u_{\mathbf{k}} u_{\mathbf{k'}}^*}{a^2}\right) \langle 0|b_{\mathbf{k}} b_{\mathbf{k'}}^\dagger|0\rangle e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{k'}\cdot\mathbf{x'}} + ... \tag{46}$$

where all the other terms omitted from the integral vanish. The only non vanishing matrix element

$$\langle 0|b_{\mathbf{k}} b_{\mathbf{k'}}^\dagger|0\rangle = \langle 0|b_{\mathbf{k}} b_{\mathbf{k'}}^\dagger - \underbrace{b_{\mathbf{k'}}^\dagger b_{\mathbf{k}}|0\rangle}_{=0} = \langle 0|\left[b_{\mathbf{k}}\,,\, b_{\mathbf{k'}}^\dagger\right]|0\rangle = \delta^3(\mathbf{k} - \mathbf{k'}). \tag{47}$$

gives

$$\begin{aligned} \langle 0|\hat{\delta\phi}(\eta, \mathbf{x})\hat{\delta\phi}(\eta, \mathbf{x'})|0\rangle &= \int \frac{d^3k}{(2\pi)^3} \left(\frac{|u_{\mathbf{k}}|^2}{a^2}\right) e^{i\,\mathbf{k}\cdot(\mathbf{x}-\mathbf{x'})} \\ &\doteq \int \frac{d^3k}{(2\pi)^3}\, P_{\delta\phi}(k)\, e^{i\,\mathbf{k}\cdot(\mathbf{x}-\mathbf{x'})}, \end{aligned} \tag{48}$$

where in the last line we have defined the power spectrum

$$P_{\delta\phi}(k) \doteq \frac{|u_{\mathbf{k}}|^2}{a^2}. \tag{49}$$

We define the *dimensionless* power spectrum as

$$\mathcal{P}_{\delta\phi} \doteq \frac{k^3}{2\pi^2} P_{\delta\phi}(k) = \frac{k^3}{2\pi^2}\frac{|u_{\mathbf{k}}|^2}{a^2}. \tag{50}$$

Using equation (41) we obtain for $|u_{\mathbf{k}}|^2$

$$|u_{\mathbf{k}}|^2 = \frac{1}{2k}\left(1 + \frac{1}{k^2\eta^2}\right), \tag{51}$$

which gives

$$\mathcal{P}_{\delta\phi} = \left(\frac{H}{2\pi}\right)^2 \left[1 + \left(\frac{k}{aH}\right)^2\right]. \tag{52}$$

This is the expression for the dimensionless power spectrum of the inflaton fluctuations in an exact de Sitter spacetime. Note that well outside the Hubble horizon (*i.e.*, $k \ll (aH)$) it approaches to be a constant:

$$\mathcal{P}_{\delta\phi} = \left(\frac{H}{2\pi}\right)^2. \tag{53}$$

This result is consistent with the phenomenon of modes freezing that we discussed above.

2.5 Primordial Spectra from Inflation

Thanks to these efforts, we are now ready to compute the power spectrum of primordial scalar and tensor perturbations. Perturbations can be decomposed according to their spin with respect to a local rotation of the spatial coordinates on hypersurfaces of constant time. We have three different types of perturbations:

- **Scalar perturbation**: they have spin 0 and are invariant underr rotation.
- **Tensor perturbations** (or gravitational waves): they have spin 2 and are the true degrees of freedom of the gravitational fields (i.e., they can exist even in the vacuum).
- **Vector perturbations**: they have spin 1 and arise from rotational velocity fields.

Due to the Scalar-Vector-Tensor decomposition theorem, at linear order, we can consider scalar and tensor pretensions separately. Additionally, in the simplest models of inflation we have only two degrees of freedom, so that only scalar and tensor modes are quantum mechanically exited. In what follows we therefore focus on scalar and tensor modes.

2.5.1 Primordial Scalar Modes

To calculate the power spectrum predicted by inflation for primordial scalar modes, we must use some Gauge-invariant quantity. We use the primordial curvature perturbation ζ

$$\zeta \equiv -\Psi - \frac{H}{\dot{\rho}}\delta\rho. \tag{54}$$

where Ψ is a 3-scalar called *spatial curvature perturbation*[1]. Geometrically, ζ measures the spatial curvature of constant-density hypersurfaces. The fact that ζ is gauge-independent allows us to adopt a more convenient gauge while ensuring that the results are gauge-independent. The inflaton quantum fluctuations can be easily related to the primordial curvature perturbations ζ in the zero spatial curvature gauge, where the spatial component of the metric is unperturbed ($\Psi = 0$), and the spacelike hypersurfaces at constant time are flat. According to equation (54), in the spatially flat Gauge, ζ for an inflaton-dominated Universe reads

$$\zeta \approx -\left(\frac{H}{\dot{\phi}}\right)\delta\phi. \tag{55}$$

Therefore the calculation of the two-point correlation function $\langle 0|\hat{\zeta}\hat{\zeta}|0\rangle$ is now straightforward:

$$\langle 0|\hat{\zeta}\,\hat{\zeta}|0\rangle = \left(\frac{H}{\dot{\phi}}\right)^2 \langle 0|\hat{\delta\phi}\hat{\delta\phi}|0\rangle, \tag{56}$$

from which we can define the dimensionless spectrum as follows:

$$\mathcal{P}_s = \left(\frac{H}{\dot{\phi}}\right)^2 \mathcal{P}_{\delta\phi}. \tag{57}$$

[1]We recall that the most general line element is

$$ds^2 = -\left(1 + 2\Phi\right)dt^2 + 2\,a(t)B_i\,dx^i\,dt + a^2(t)\left[\left(1 - 2\Psi\right)\delta_{ij} + 2E_{ij}\right]dx^i\,dx^j,$$

where Φ is a 3-scalar called *Lapse*; Ψ is a 3-scalar called *spatial curvature perturbation*; B_i is a 3-vector called *shift*; E_{ij} is a spatial symmetric and traceless 3-tensor called *shear*.

Using Eq. (19) and Eq. (21) – or equivalently Eq. (22) and Eq. (23) – we eventually get:

$$\mathcal{P}_s = \left(\frac{1}{8\pi^2 M_{\rm pl}^2}\right)\left(\frac{H^2}{\epsilon_V}\right) = \left(\frac{1}{12\pi^2 M_{\rm pl}^6}\right)\left(\frac{V^3}{V_\phi^2}\right), \tag{58}$$

This is the dimensionless power spectrum for scalar perturbations predicted by inflation at the time of horizon crossing. Note that since ζ is a gauge independent quantity (or more rigorously speaking a gauge fixed quantity), this result is gauge independent as well.

Notice that in an exact de Sitter spacetime the spectrum of inflaton fluctuations is exactly scale independent. Therefore in an almost de Sitter epoch, we expect the scale dependence to be very small. This is why the primordial scalar spectrum is commonly parametrized with a power

$$\mathcal{P}_s(k) = A_s\left(\frac{k}{k_*}\right)^{n_s-1} \tag{59}$$

which includes only an amplitude $A_s = \mathcal{P}(k_*)$ (evaluated at the pivot scale k_*) and a scalar spectral index (or scalar tilt)

$$n_{\rm s} - 1 \equiv \frac{d\log\mathcal{P}_{\rm s}}{d\log k}. \tag{60}$$

Therefore, the value of the scalar tilt is determined by (the derivative of) the inflationary potential and can be expressed in terms of the slow-roll parameters – Eq. (22) and Eq. (23) – as:

$$n_s - 1 = 2\eta_V - 6\epsilon_V. \tag{61}$$

2.5.2 Primordial Gravitational Waves

Along with scalar modes, during inflation, the tensor degrees of freedom (i.e., the metric) will be quantum mechanically excited. Therefore, from inflation, we expect a background of Primordial Gravitational Waves. A gravitational wave may be viewed as a ripple of space-time in the FRW background metric. In general the linear tensor perturbations may be written as

$$ds^2 = a^2(\eta)\left[-d\eta^2 + (\delta_{ij} + h_{ij})\,dx^i\,dx^j\right] \tag{62}$$

Tensor modes (i.e., metric perturbations) are intrinsically gauge invariant at linear order and can described in terms of the transverse and traceless part of the symmetric 3×3 matrix h_{ij}. To characterize the contribution of each wavenumber k we move to the Fourier space. Focusing on one particular polarisation state, and assuming isotropy, the gravitational wave field h_k satisfies the following equation:

$$h_k'' + 2\frac{a'}{a}h_k' + k^2 h_k = 0, \tag{63}$$

where $(..)'$ denotes the derivative with respect to conformal time. It is more convenient to use a new variable $u_k(\eta) \equiv a(\eta)h_k(\eta)$ satisfying

$$u'' + \left[k^2 - \frac{a''}{a}\right]u = 0. \tag{64}$$

For each polarization ($\times$ and $+$) this is nothing but the Mukhanov equation (35). Therefore we can use the results already derived to obtain the (dimensionless) tensor power spectrum. Taking into account all the factors corresponding to the two different polarization states, it reads:

$$\mathcal{P}_{\rm t} = \frac{8}{\bar{M}_p^2}\left(\frac{H}{2\pi}\right)^2 = \left(\frac{2}{3\pi^2 M_{\rm pl}^4}\right)V. \tag{65}$$

Once again the symmetries of the almost de Sitter background constrain the scale dependence to be very small and also in this case the tensor spectrum is commonly parametrized with a power

$$\mathcal{P}_{\rm t}(k) = rA_s\left(\frac{k}{k_*}\right)^{n_{\rm t}} \tag{66}$$

which includes only an amplitude $A_{\rm t} \equiv rA_{\rm s} \equiv \mathcal{P}_{\rm t}(k_*)$ and a tensor spectral index (or tensor tilt)

$$n_{\rm t} \doteq d\log\mathcal{P}_{\rm T}/d\log k. \tag{67}$$

The tensor tilt and the tensor amplitude can be both related to the inflationary parameter ϵ as

$$n_{\rm t} = -2\epsilon_V = -r/8 \tag{68}$$

leading to the well-known slow-roll consistency relation for single-field inflation. This relation is a *unique* prediction of single-field inflation and an experimental confirmation would provide direct evidence for inflation.

3 Imprintings in the CMB

After recombination the Universe becomes transparent to photons and today the Universe is embedded into this fossil electromagnetic radiation that dates back to 380.000 years after the Big Bang singularity.

Despite Cosmic Microwave Background radiation appears to be very homogeneous and isotropic; we observe small intrinsic temperature *anisotropies* and *polarization* that are crucial in our understanding of the underlying physics of the Early Universe. In what follows we point out their connection with inflation and how we can acutally test inflation using CMB data

3.1 Temperature Anisotropies

The physics of CMB anisotropies is well understood and described in terms of linear perturbation theory. The angular variations in temperature that we observe today are a snapshot of the *local* properties of relic photons at redshift $z \sim 1100$ that must be related to primordial perturbations. Therefore anisotropies encode information on the primordial perturbation itself. Here we first introduce the formalism used to describe CMB anisotropies and then we review the main physical processes that sourced them.

It is useful to define the so-called *brightness function*

$$\Theta(\eta, x, \hat{n}) \equiv \frac{\delta T(\eta, x, \hat{n})}{T(\eta)}, \tag{69}$$

where $\hat{n} = \mathbf{p}/p$ is the unitary vector which defines the direction of the photon momentum; x is a given point of the space and η is the conformal time. It is somehow useful to define also the direction in which the photon is seen $\hat{e} = -\hat{n}$. Indeed the brightness function depends equivalently on $\hat{n}$ or $\hat{e}$. Since the photons we observe today were emitted on a 3-sphere given by the intersection between the last scattering surface and our past light cone, it is natural to expand the bright function into spherical harmonics. The multipoles expansion reads

$$\Theta(\eta, x, \hat{n}) = \sum_{\ell \geq 1} \sum_{m} (-1)^{\ell} \Theta_{\ell m}(\eta, x) Y_{\ell m}(\hat{n}). \tag{70}$$

where

$$Y_{\ell m} = \left[\frac{2\ell + 1}{4\pi} \frac{(\ell - m)!}{(\ell + m)!} \right]^{\frac{1}{2}} P_{\ell}^{m}(\cos\theta)\, e^{i\, m\phi} \tag{71}$$

are the spherical harmonics while θ and ϕ represent the usual spherical coordinates that identify the direction $\hat{n}$. $P_{\ell}^{m}(\cos\theta)$ are the associated Legendre functions

$$P_{\ell}^{m}(x) = (-1)^{\ell} \frac{(1 - x^2)^{\frac{m}{2}}}{2^{\ell} \ell!} \frac{d^{\ell+m}}{dx^{\ell+m}} (1 - x^2)^{\ell} \tag{72}$$

It is well known that the Spherical harmonics are a complete orthonormal set of functions which means that

$$\int Y_{\ell m} Y^{*}_{\ell' m'}\, d\Omega = \delta_{\ell \ell'} \delta_{m m'} \tag{73}$$

Notice also that in the equation (70) we can absorb the factor $(-1)^{\ell}$ into spherical harmonics using $\hat{e}$ instead of $\hat{n}$:

$$\Theta(\eta, x, \hat{n}) = \sum_{\ell \geq 1} \sum_{m} \Theta_{\ell m}(\eta, x) Y_{\ell m}(\hat{e}) \tag{74}$$

In the sum we do not consider the monopole contribution $\ell = 0$. The reason is that such term carries information about the energy of relic photons at different positions but we cannot measure CMB photons at positions different than ours and so we cannot measure this effect which is proportional to the photons energy fluctuations. For this reason we start the spherical harmonics expansion considering the dipole contribution $\ell = 1$.

The dipole term, $\ell = 1$, in the sum is due to the doppler shift caused by the relative motion between the observer and the photons fluid. We stress that we are *not* comoving observers because we move by the Earth's motion. Therefore in our frame a dipole effect is expected and indeed observed. We can evaluate the Doppler shift to the first order in the relative photon velocity v_{γ}:

$$\sum_{m} \Theta_{1m}(\eta, x) Y_{1m}(\hat{e}) = -v_{\gamma} \cdot \hat{e}. \tag{75}$$

While the CMB dipole is the dominant effect, it does not give any appreciable information about the intrinsic primordial temperature fluctuations and consequently should be removed from the map of CMB anisotropies.

Multipoles with $\ell \geq 2$. In this case the effect of Earth motion on multipoles is proportional to $(v_{\gamma})^{\ell}$ and so it becomes small as $v_{\gamma} \ll 1$. Multipoles with $\ell \geq 2$ show a small magnitude of order 10^{-5}, that cannot be brought back to the Earth motion effect since this is expected to be at least of order 10^{-6} for $\ell = 2$. Therefore multipoles with $\ell \geq 2$, while small, are a snapshot of the intrinsic anisotropies in the CMB radiation that are related to its underlying physical production and evolution. From now on, we call such terms $a_{\ell m}$:

$$a_{\ell m} \doteq \Theta_{\ell m}(\eta_0, x_0 = 0), \quad \text{for } \ell \geq 2. \tag{76}$$

Here x_0 is our position chosen to be the origin of coordinates. We are interested in the stochastic properties of the CMB multipoles $a_{\ell m}$. We first note that invariance under rotations implies that $\langle a_{\ell m}\rangle = 0$ and that the two-point correlator therefore reads

$$\langle a_{\ell m} a^*_{\ell' m'}\rangle = C_\ell^{\mathrm{TT}}\, \delta_{\ell\, \ell'} \delta_{m\, m'}, \tag{77}$$

where $C_\ell^{\mathrm{TT}} \doteq \langle\, |a_{\ell m}|^2\rangle$ is the *angular power spectrum* of the CMB anisotropies and $\langle \dots \rangle$ means an average over an ensemble of realizations of Universes like ours. Clearly, we only have one Universe, so we cannot compute such a mean or variance over an ensemble of different Universes. Yet we can estimate the coefficients C_ℓ from observations approximately, by evaluating the average over m:

$$C_\ell = \frac{1}{2\ell + 1} \sum_{m=-\ell}^{m=\ell} |a_{\ell m}|^2 \tag{78}$$

3.2 Polarization

The temperature anisotropies originated from primordial fluctuations, are polarized by the Thomson scattering. Recombination was not an instantaneous process: while protons and electrons were combining into neutral hydrogen, the photons developed a quadrupole anisotropy that was converted into CMB polarization by the Thomson scattering.

To introduce the relevant Stokes parameters we can consider a simple plane wave coming from the positive direction of the z axis:

$$E(t) = \frac{1}{2}\left[\mathbf{E}\, e^{i\omega\, t} + \mathbf{E}^*\, e^{-i\omega\, t}\right], \tag{79}$$

the amplitude $\mathbf{E}$ can be decomposed into its (x, y) components as $E_\phi = E_x \cos\phi + E_y \sin\phi$. Defining the unpolarized intensity I

$$I \equiv \overline{|E_x|^2} + \overline{|E_y|^2}, \tag{80}$$

and the *Stokes parameters* $\{Q, U, V\}$

$$Q \equiv \overline{|E_x|^2} - \overline{|E_y|^2}, \tag{81}$$

$$U \equiv 2\,\mathrm{Re}\left[\overline{E_x^* E_y}\right], \tag{82}$$

$$V \equiv 2\,\mathrm{Im}\left[\overline{E_x^* E_y}\right] \tag{83}$$

one obtains

$$\overline{|E_\phi^2|} = I + Q\,\cos 2\phi + U\,\sin 2\phi. \tag{84}$$

We recall that Q and U are two Stokes parameters that specify the polarization plane, while V is a third Stokes parameter that measures the intensity of circular polarization. If we perform a rotation around the axis z of an angle φ so that $\phi \to \phi + \varphi$ clearly $\overline{|E_\phi^2|}$ must not change. This implies that under rotation the Stokes parameters (Q, U) change as

$$\begin{pmatrix} Q \\ U \end{pmatrix} \to \begin{pmatrix} \cos 2\varphi & -\sin 2\varphi \\ \sin 2\varphi & \cos 2\varphi \end{pmatrix} \begin{pmatrix} Q \\ U \end{pmatrix}. \tag{85}$$

Using the combination $Q_\pm \equiv Q \pm i\, U$ we find a more compact expression:

$$Q_\pm \to e^{\pm 2\, i\, \varphi}\, Q_\pm. \tag{86}$$

A similar description can be applied also to CMB radiation. However, for a black-body radiation $I \propto T^4$ and $\Theta = \frac{\delta T}{T} = \frac{\delta I}{4I}$. Therefore it is useful to redefine the Stokes parameters in a cosmological contest with the following normalization

$$Q_\pm \to \frac{Q_\pm}{4I} \tag{87}$$

According to the $Q_\pm$ transformation propriety under rotations (86), we clearly see that it is a spin-2 field that can be expanded in terms of spin-weighted spherical harmonics. The *spin-weighted spherical harmonics* can be defined in terms of rotations matrices a:

$${}_sY_{\ell m}(\theta, \phi) = \sqrt{\left(\frac{2\ell + 1}{4\pi}\right)} \mathcal{D}^\ell_{-s,m}(\phi, \theta, 0). \tag{88}$$

They reduce to ordinary spherical harmonics when $s = 0$. In this case $s = 2$ and defining ${}_2Y_{\ell m} \equiv Y^{\pm}_{\ell m}$, one can expand $Q_\pm$ as:

$$Q_\pm(\hat{e}) = \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} Q^{\pm}_{\ell m}\, Y^{\pm}_{\ell m}(\hat{e}). \tag{89}$$

At this point we can introduce the Polarization multipoles $E_{\ell m}$ and $B_{\ell m}$ defined as:

$$Q^{\pm}_{\ell m} \equiv E_{\ell m} \pm i\, B_{\ell m} \tag{90}$$

One can show that under parity transformation $\hat{e} \to -\hat{e}$ the E modes $E_{\ell m} \to (-1)^{\ell} E_{\ell m}$ while the B modes $B_{\ell m} \to (-1)^{\ell+1} B_{\ell m}$. Therefore the E-modes are *parity-even* while the B-modes are *parity-odd*. Roughly speaking, we can think of the E-modes as the gradient of a scalar and the B-modes as the curl of a vector. The stochastic properties under rotations and parity transformations allow us to define the following correlators among $a_{\ell m}$, $E_{\ell m}$ and $B_{\ell m}$

$$\langle a_{\ell m} a^*_{\ell' m'} \rangle = C^{TT}_{\ell}\, \delta_{\ell\, \ell'} \delta_{m\, m'} \tag{91}$$

$$\langle a^*_{\ell m}\, E_{\ell' m'} \rangle \equiv C^{TE}_{\ell} \delta_{\ell\ell'} \delta_{mm'} \tag{92}$$

$$\langle E^*_{\ell m}\, E_{\ell' m'} \rangle \equiv C^{EE}_{\ell} \delta_{\ell\ell'} \delta_{mm'} \tag{93}$$

$$\langle B^*_{\ell m}\, B_{\ell' m'} \rangle \equiv C^{BB}_{\ell} \delta_{\ell\ell'} \delta_{mm'}, \tag{94}$$

where for sake of completeness we wrote also the correlator (77). Notice that since B is parity-odd, while T and E are parity-even, in a parity-conserving theory we expect $C^{TB}_{\ell} = C^{EB}_{\ell} = 0$ and for each ℓ we can define the so called *covariance matrix*

$$\hat{C}_\ell \doteq \begin{pmatrix} C^{TT}_{\ell} & C^{TE}_{\ell} & 0 \\ C^{ET}_{\ell} & C^{EE}_{\ell} & 0 \\ 0 & 0 & C^{BB}_{\ell} \end{pmatrix}. \tag{95}$$

3.3 CMB Transfer Functions

The inflationary fluctuations are directly related to the small irregularities observed in the Cosmic Microwave Background. he mappings between the primordial spectra (i.e., $\mathcal{P}_{\rm s}$ and $\mathcal{P}_{\rm T}$) predicted by inflation and the CMB angular power spectra spectra (i.e. the different $C_{\ell s}$) are specified by the so-called *CMB transfer functions*, T^X_ℓ, defined as

$$\frac{\ell(\ell+1)}{2\pi} C^{X\,Y,\,\rm scalar}_{\ell} = \int_0^\infty d\ln k\; T^X_\ell(k)\, T^Y_\ell(k)\, \mathcal{P}_{\rm s}(k), \tag{96}$$

$$\frac{\ell(\ell+1)}{2\pi} C^{X\,Y,\,\rm tensor}_{\ell} = \int_0^\infty d\ln k\; \hat{T}^X_\ell(k)\, \hat{T}^Y_\ell(k)\, \mathcal{P}_{\rm T}(k), \tag{97}$$

where for scalar perturbations X and Y run over $X, Y = \{T, E\}$ while for tensor perturbations X and Y run over $X, Y = \{T, E, B\}$. Notice that $C^{\rm scalar}_{\ell}$ quantifies the contributions coming from scalar modes that is weighted by the (scalar) transfer functions $T^Y_\ell(k)$. Instead, $C^{\rm tensor}_{\ell}$ quantifies the contribution coming from tensor modes, weighted by the (tensor) transfer functions $\hat{T}^Y_\ell(k)$. The total contribution is given by

$$C^{\rm tot}_{\ell} = C^{\rm scalar}_{\ell} + C^{\rm tensor}_{\ell} \tag{98}$$

The transfer functions depend only on known physics: a set of coupled Einstein-Boltzmann equations at linear order. Roughly speaking, the form of the linear transformations encoded in the transfer functions probe the time evolution while the primordial power spectrum is determined by inflation. For analytical calculations, it is worth remembering that at each ℓ, the main contribution in C_ℓ is given by from the projection along the diagonal $\ell = kD(z_{\rm CMB})$, where $D(z_{\rm CMB}) \sim 14$ Gpc is the angular diameter distance to recombination. However the transfer functions can be obtained by numerically solving the linearized Einstein-Boltzmann equations. The two most popular Boltzmann solver codes are CAMB and CLASS.