An Introduction to Inflationary Cosmology

(from theory to data analysis)

CosmoVerseSchool @Corfu

Lecture 1 of 2

— Theory Part —

Image: Planck's View of BICEP2/Keck Array Field. Credit: Jet Propulsion Laboratory, NASA and Caltech

William Giarè

w.giare@sheffield.ac.uk www.williamgiare.com

Research Associate in Theoretical Cosmology

The University of Sheffield
School of Mathematics & Statistics

Lecture materials: Google Drive Folder

For any further materials/clarifications/questions/curiosities/
feedback, feel free to Contact Me

ΛCDM Cosmology

General Relativity
To describe gravitational interactions

Standard Model
To describe fundamental interactions

Well understood

Inflation
To explain spatial flatness, homogeneity on large scales and inhomogeneities on small scales.

Cold Dark Matter
To Facilitate structure formation and explain the observational evidence for a missing mass in the Universe

Dark Energy (cosmological constant Λ)
To explain the late-time accelerated expansion of the Universe

Not so well understood

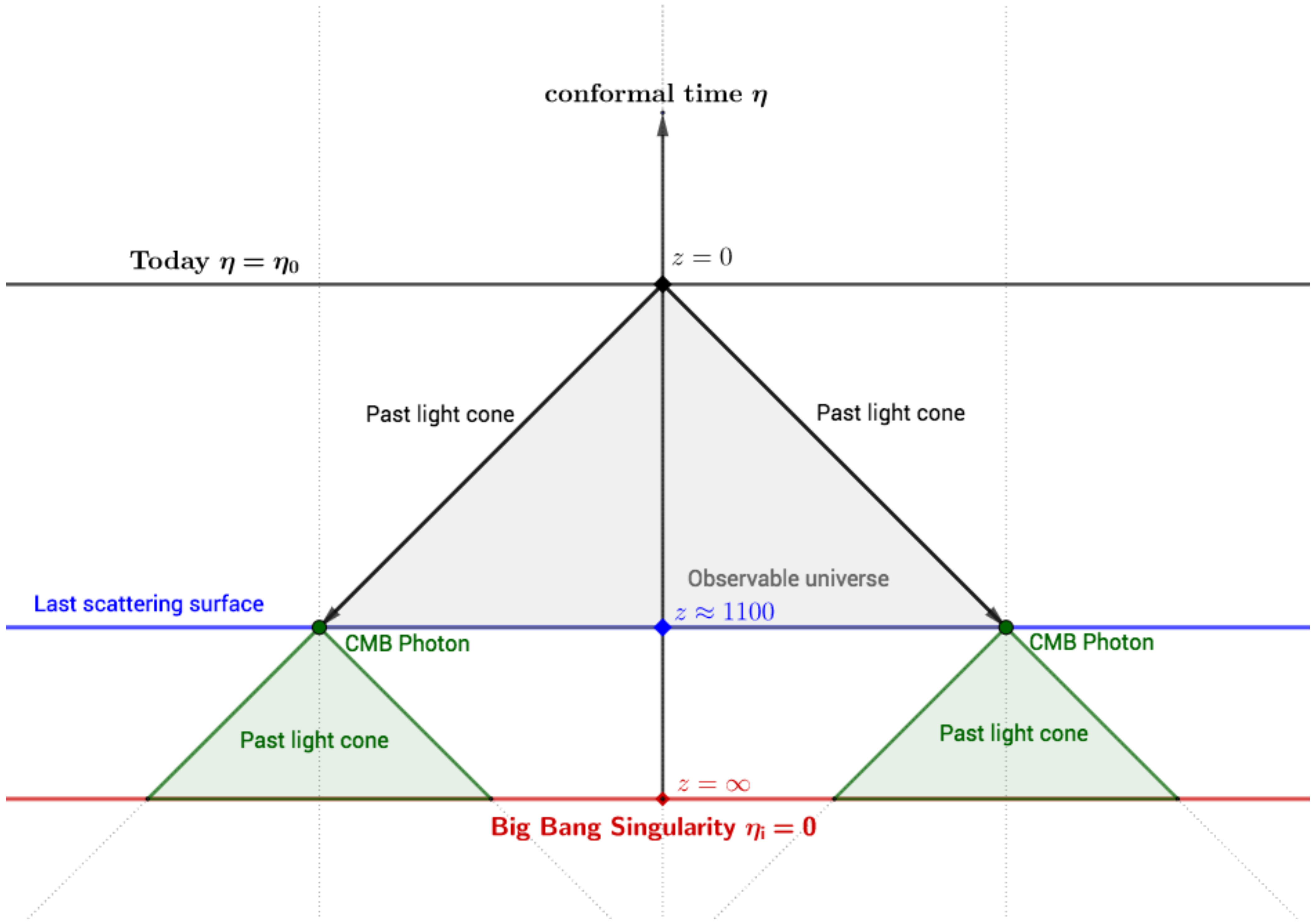
My pictorial representation of ΛCDM cosmology

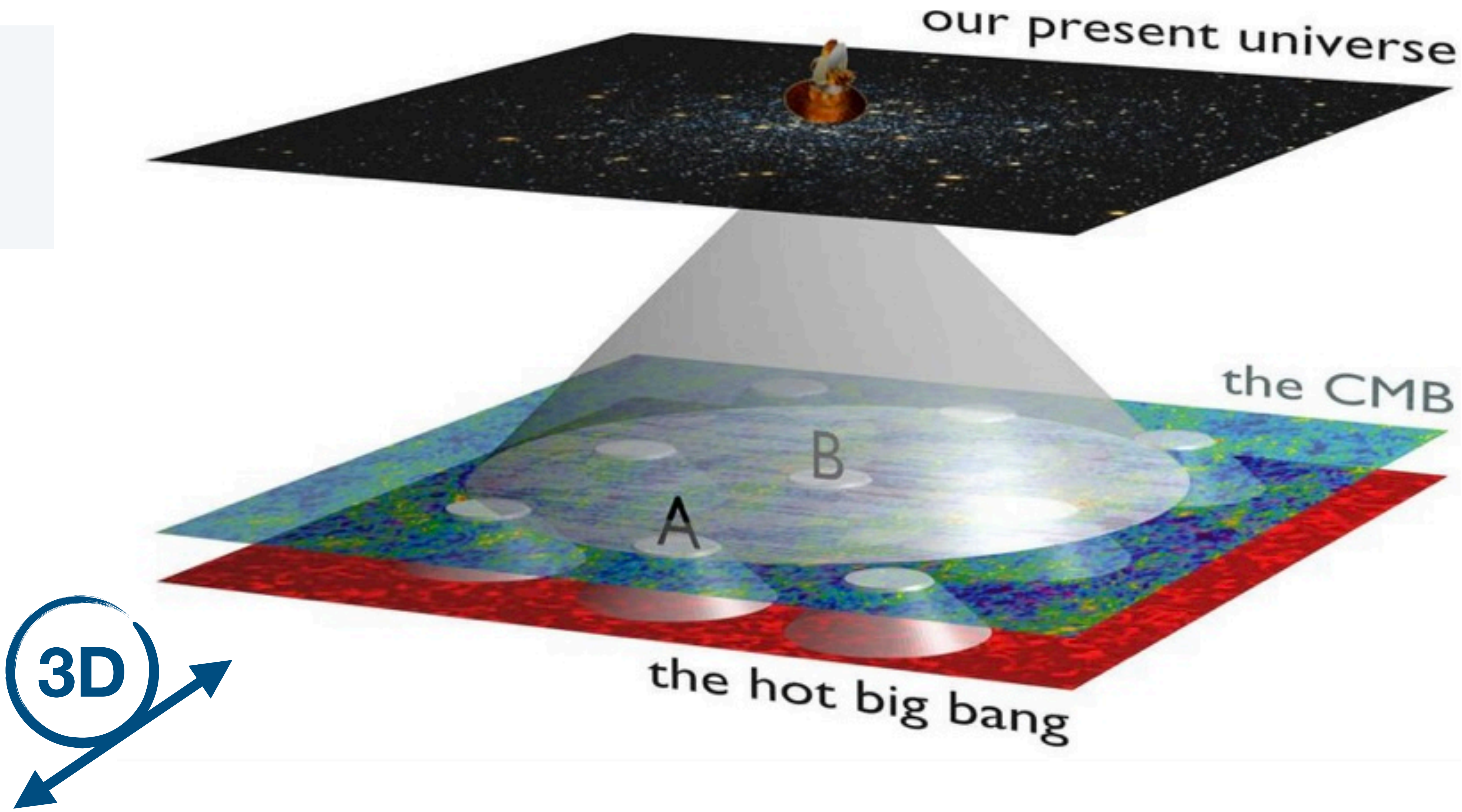
Why Inflation?

i.e. a phase of accelerated expansion $\ddot{a} > 0$

Horizon Problem:

All CMB photons share the same temperature within fluctuations $\delta T/T \simeq 10^{-5}$ but the last scattering surface consists of many causally disconnected regions





Why Inflation?

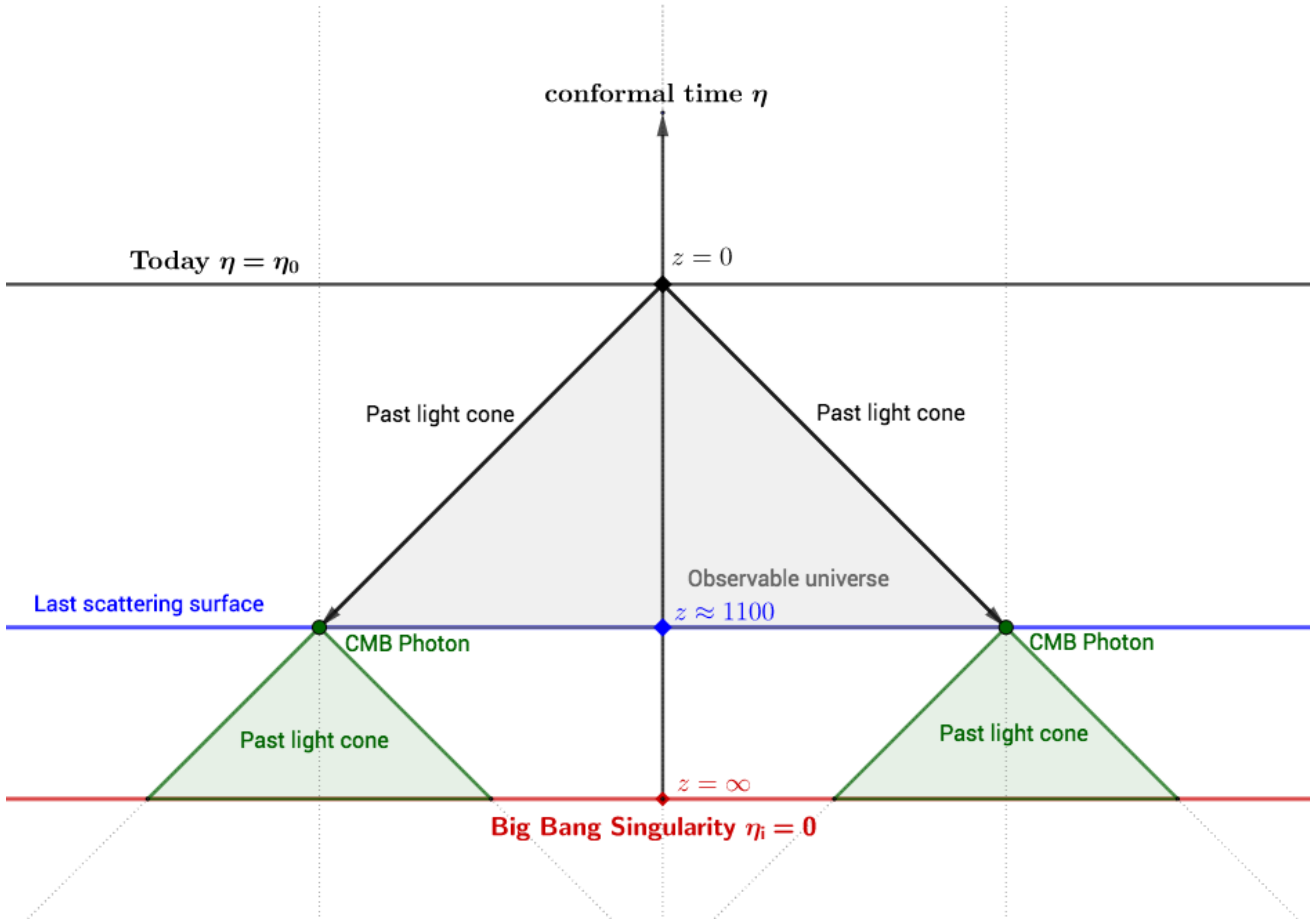
i.e. a phase of accelerated expansion $\ddot{a} > 0$

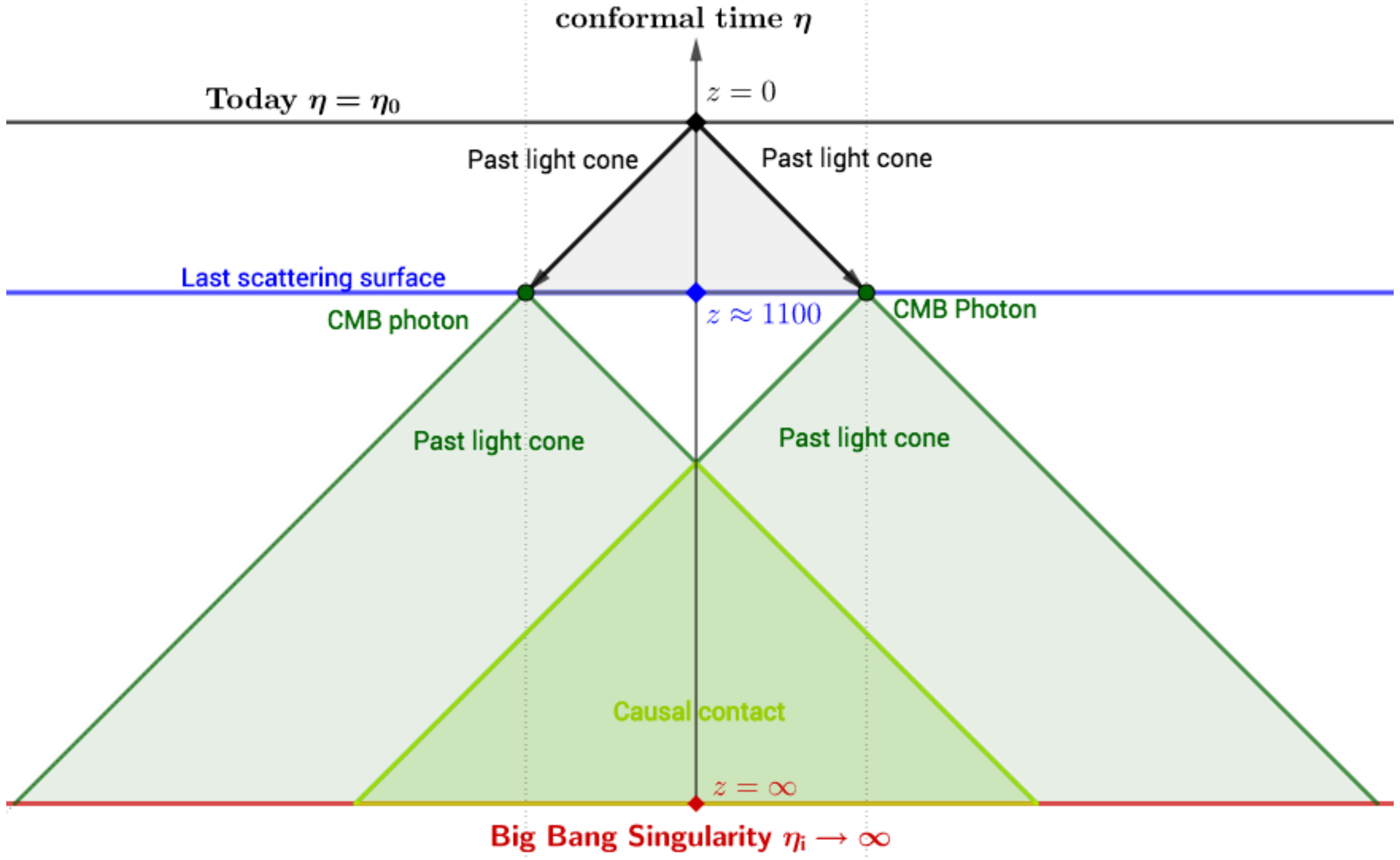
Horizon Problem:

All CMB photons share the same temperature within fluctuations $\delta T/T \simeq 10^{-5}$ but the last scattering surface consists of many causally disconnected regions

Inflationary solution:

Inflation pushes backward in (conformal) time the Big Bang singularity explaining why all CMB photons share the same temperature





Why Inflation?

i.e. a phase of accelerated expansion $\ddot{a} > 0$

Flatness Problem:

Observations point towards a (nearly?) flat Universe today

$$\Omega_k(t) = -\frac{k}{a^2(t)H^2(t)}$$

Radiation $\Omega_k(t) \propto a^2$

Matter $\Omega_k(t) \propto a$

$\Omega_k = 0$ **today implies** $\Omega_k(t_{\rm pl}) \sim 10^{-64}$

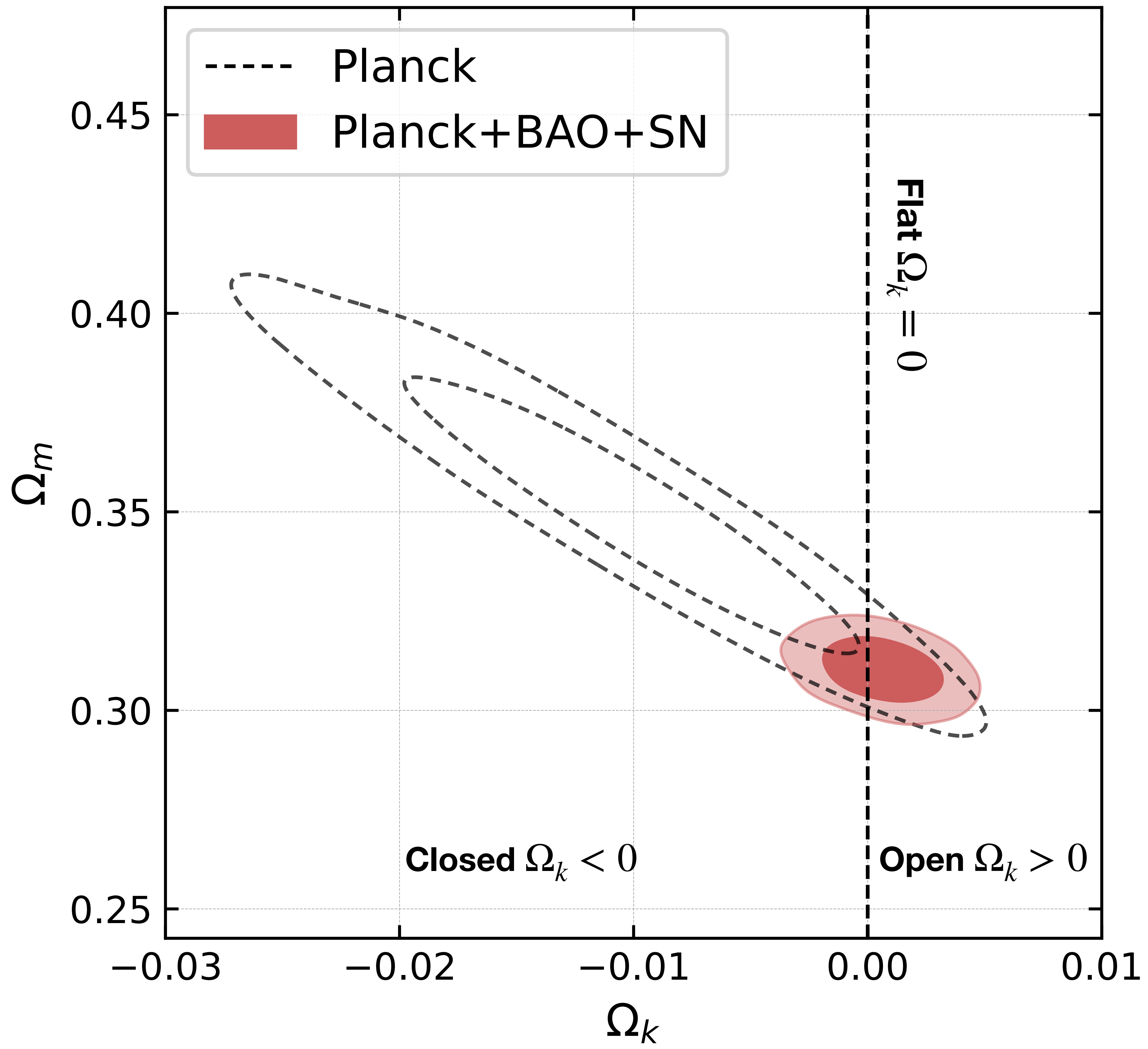
Inflationary solution:

$$\Omega_k(t) = -\frac{k}{a^2(t)H^2(t)}$$

Inflation $\Omega_k(t) \propto 1/a^2 \to 0$

Radiation $\Omega_k(t) \propto a^2$

Matter $\Omega_k(t) \propto a$



... You will be able to do this plot after Lecture 2 ;)

Single Field Slow-Roll Inflation

Action for a single scalar field minimally coupled to gravity

$$S = \int d^4x\sqrt{-g}\left[\frac{M_p^2}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]$$

Einstein-Hilbert action (GR)

$$S_{\rm EH} = \int d^4x\sqrt{-g}\left[\frac{M_p^2}{2}R\right]$$

Scalar field action $S_\phi = \int d^4x\sqrt{-g}\left[\mathscr{L}_\phi\right]$

with $\mathscr{L}_\phi = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)$

Single Field Slow-Roll Inflation

Action for a single scalar field minimally coupled to gravity

$$S = \int d^4x\sqrt{-g}\left[\frac{M_p^2}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]$$

Klein-Gordon Equation

$$\frac{\delta S_\phi}{\delta\phi} = 0 \longrightarrow \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\partial^\mu\phi\right) + \frac{\delta V(\phi)}{\delta\phi} = 0$$

$$\downarrow$$

$$\ddot{\phi} + 3H\dot{\phi} - a^{-2}(t)\,\nabla^2\phi + \frac{\delta V(\phi)}{\delta\phi} = 0$$

Step-by-step derivation

$$\begin{aligned}
\frac{\delta S_\phi}{\delta\phi} = 0 &= \int d^4x\,\sqrt{-g}\left[\frac{1}{2}\,\partial_\mu\delta\phi\partial^\mu\phi + \frac{1}{2}\,\partial_\mu\phi\partial^\mu\delta\phi - \frac{\delta V(\phi)}{\delta\phi}\delta\phi\right] = \\
&= \int d^4x\,\sqrt{-g}\left[\partial_\mu\delta\phi\partial^\mu\phi - \frac{\delta V(\phi)}{\delta\phi}\delta\phi\right] = \\
&= \int d^4x\,\underbrace{\partial_\mu\left[\sqrt{-g}\left(\partial^\mu\phi\,\delta\phi\right)\right]}_{\text{Surface term }=0} - \partial_\mu\left(\sqrt{-g}\partial^\mu\phi\right)\delta\phi - \sqrt{-g}\frac{\delta V(\phi)}{\delta\phi}\delta\phi = \\
&= \int d^4x\sqrt{-g}\left[-\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\partial^\mu\phi\right) - \frac{\delta V(\phi)}{\delta\phi}\right]\delta\phi.
\end{aligned}$$

$$\downarrow$$

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\partial^\mu\phi\right) + \frac{\delta V(\phi)}{\delta\phi} = 0.$$

Single Field Slow-Roll Inflation

Action for a single scalar field minimally coupled to gravity

$$S = \int d^4x\sqrt{-g}\left[\frac{M_p^2}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]$$

Friedmann Equations

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0 \longmapsto T^{\phi}_{\mu\nu} = g_{\mu\nu}\mathscr{L}_\phi - 2\frac{\delta \mathscr{L}_\phi}{\delta g^{\mu\nu}}$$

$$\updownarrow$$

Energy-density

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$

Pressure

$$P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

Step-by-step derivation

$$\begin{aligned}\frac{\delta S_{\mathrm{E.H}}}{\delta g^{\mu\nu}} &= \int d^4x \left(\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\right)\frac{M_p^2}{2}R + \frac{\sqrt{-g}}{2}M_p^2\left(\frac{\delta R}{\delta g^{\mu\nu}}\right) = \\ &= \int d^4x \; -\frac{\sqrt{-g}}{2}g_{\mu\nu}\,\delta g^{\mu\nu}\left(\frac{M_p^2}{2}R\right) + \frac{\sqrt{-g}}{2}M_p^2\left(R_{\mu\nu}\delta g^{\mu\nu}\right) = \\ &= \int d^4x\,\frac{\sqrt{-g}}{2}\left[M_p^2\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\right]\delta g^{\mu\nu}\end{aligned}$$

$$\begin{aligned}\frac{\delta S_\phi}{\delta g_{\mu\nu}} &= \int d^4x\,\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\mathcal{L}_\phi + \sqrt{-g}\frac{\delta\mathcal{L}_\phi}{\delta g^{\mu\nu}}\delta g^{\mu\nu} = \\ &= \int d^4x \; -\frac{\sqrt{-g}}{2}g_{\mu\nu}\mathcal{L}_\phi\,\delta g^{\mu\nu} + \sqrt{-g}\frac{\delta\mathcal{L}_\phi}{\delta g^{\mu\nu}}\delta g^{\mu\nu} = \\ &= \int d^4x\,\frac{\sqrt{-g}}{2}\left[2\frac{\delta\mathcal{L}_\phi}{\delta g^{\mu\nu}} - g_{\mu\nu}\mathcal{L}_\phi\right]\delta g^{\mu\nu}\end{aligned}$$

$$\downarrow$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{M_p^2}\left(g_{\mu\nu}\mathcal{L}_\phi - 2\frac{\delta\mathcal{L}_\phi}{\delta g^{\mu\nu}}\right)$$

Single Field Slow-Roll Inflation

Action for a single scalar field minimally coupled to gravity

$$S = \int d^4x\sqrt{-g}\left[\frac{M_p^2}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]$$

Friedmann Equations

$$3M_p^2 H^2 = \rho_\phi = \left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)$$

$$M_p^2\left(\frac{\ddot{a}}{a}\right) = -\frac{1}{6}\left(\rho_\phi + 3P_\phi\right) = -\frac{1}{3}\left(\dot{\phi}^2 - V(\phi)\right)$$

Accelerated expansion $\ddot{a} > 0 \Longleftrightarrow P_\phi < -\rho_\phi/3$

Step-by-step derivation

$$\begin{aligned}\frac{\delta S_{\mathrm{E.H}}}{\delta g^{\mu\nu}} &= \int d^4x\left(\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\right)\frac{M_p^2}{2}R + \frac{\sqrt{-g}}{2}M_p^2\left(\frac{\delta R}{\delta g^{\mu\nu}}\right) = \\ &= \int d^4x\, -\frac{\sqrt{-g}}{2}g_{\mu\nu}\,\delta g^{\mu\nu}\left(\frac{M_p^2}{2}R\right) + \frac{\sqrt{-g}}{2}M_p^2\left(R_{\mu\nu}\delta g^{\mu\nu}\right) = \\ &= \int d^4x\,\frac{\sqrt{-g}}{2}\left[M_p^2\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\right]\delta g^{\mu\nu}\end{aligned}$$

$$\begin{aligned}\frac{\delta S_\phi}{\delta g_{\mu\nu}} &= \int d^4x\,\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\mathcal{L}_\phi + \sqrt{-g}\frac{\delta\mathcal{L}_\phi}{\delta g^{\mu\nu}}\delta g^{\mu\nu} = \\ &= \int d^4x\, -\frac{\sqrt{-g}}{2}g_{\mu\nu}\mathcal{L}_\phi\,\delta g^{\mu\nu} + \sqrt{-g}\frac{\delta\mathcal{L}_\phi}{\delta g^{\mu\nu}}\delta g^{\mu\nu} = \\ &= \int d^4x\,\frac{\sqrt{-g}}{2}\left[2\frac{\delta\mathcal{L}_\phi}{\delta g^{\mu\nu}} - g_{\mu\nu}\mathcal{L}_\phi\right]\delta g^{\mu\nu}\end{aligned}$$

$$\Downarrow$$

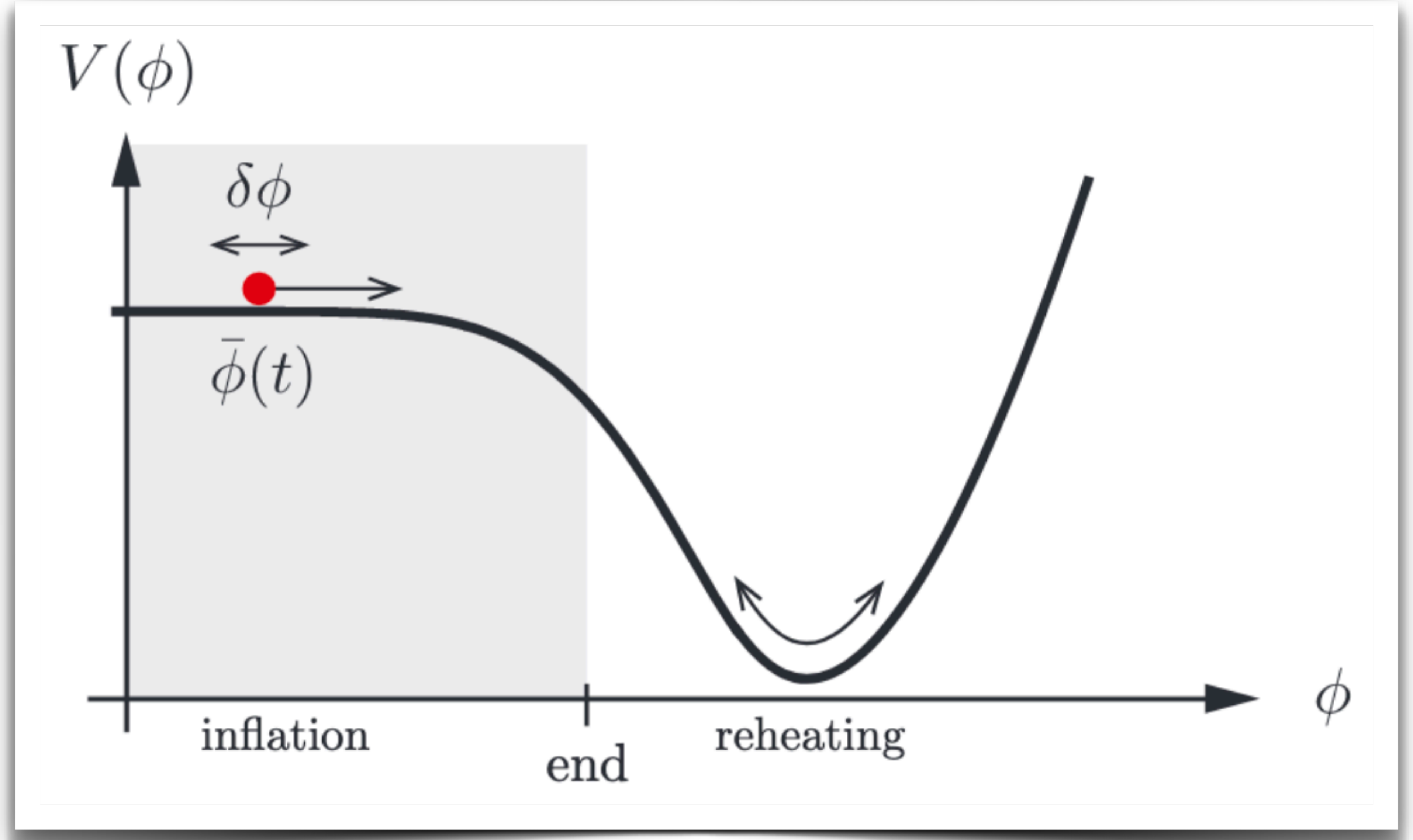
$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{1}{M_p^2}\left(g_{\mu\nu}\mathcal{L}_\phi - 2\frac{\delta\mathcal{L}_\phi}{\delta g^{\mu\nu}}\right)$$

Single Field Slow-Roll Inflation

Action for a single scalar field minimally coupled to gravity

$$S = \int d^4x\sqrt{-g}\left[\frac{M_p^2}{2}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]$$

Typical Inflationary potential



Getting Inflation:

$$w(\phi) = \frac{P(\phi)}{\rho(\phi)} = \frac{\frac{\dot{\phi}^2}{2} - V(\phi)}{\frac{\dot{\phi}^2}{2} + V(\phi)} < -\frac{1}{3}$$

↓

Slow-Roll Conditions

$$V(\phi) \gg \dot{\phi} \qquad \frac{V_\phi^2}{V} \ll H^2 \qquad |V_{\phi\phi}| \ll H^2$$

→

Slow-Roll Parameters

$$\epsilon \equiv -\frac{\dot{H}}{H^2} \simeq \frac{M_\mathrm{p}^2}{2}\left(\frac{V_\phi^2}{V^2}\right) \ll 1$$

$$|\eta| \equiv \left|M_\mathrm{p}^2\left(\frac{V_{\phi\phi}}{V}\right)\right| \ll 1$$

Primordial Perturbations

Planck 2018

We observe **temperature and polarisation anisotropies in the CMB**

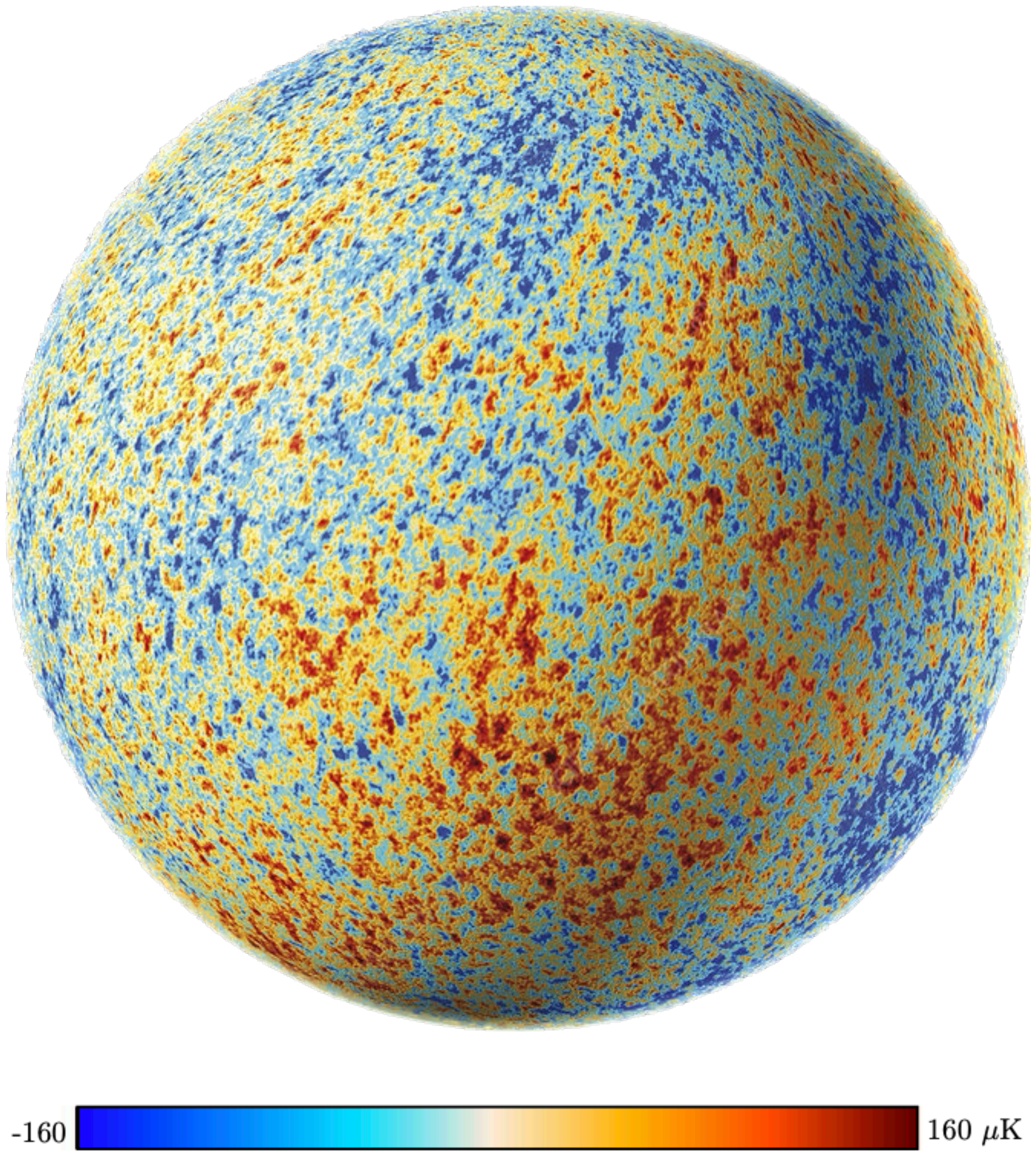
We believe these irregularities **originated during inflation:**

1) The Inflaton field will undergo **quantum fluctuations**

$$\phi \to \bar{\phi} + \delta\phi$$

2) Fluctuations will be stretched on super-horizon scales

3) They will re-enter the causal horizon later, sourcing primordial density fluctuations



PRIMORDIAL PERTURBATIONS

Planck 2018

We observe **temperature and polarisation anisotropies in the CMB**

Strategy:

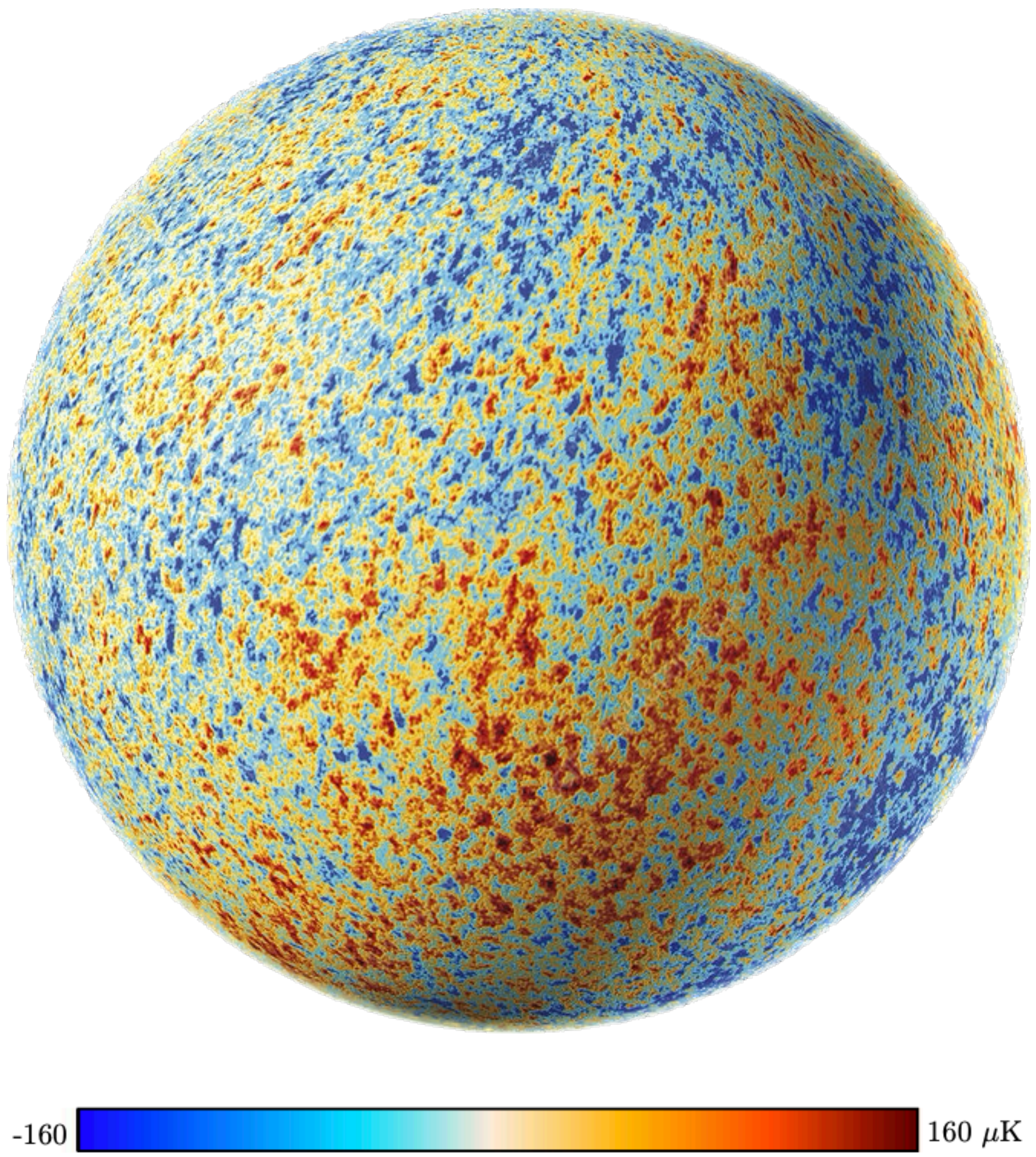
1) **Quantum fluctuations of a (free) scalar field in de Sitter**

2) Spectrum of primordial **inflationary scalar modes**

3) Spectrum of primordial **inflationary gravitational waves**

4) Connecting **primordial spectra to CMB angular power spectra**

?



Primordial Perturbations

Fluctuations of a (free) scalar field

$$S_\phi = \int d^4x\sqrt{-g}\left[\mathscr{L}_\phi\right] \equiv \int d^4x\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right]$$

We split field and fluctuations $\phi \to \phi + \delta\phi$, where $\delta\phi = \delta\phi(t,x)$

The equation of motion can be expressed in terms of the new variable $u_k = a(\eta)\delta\phi_k(\eta)$:

Mukhanov-Sasaki Equation

$$u_k'' + \left[k^2 - \frac{a''}{a}\right]u_k = 0$$

Step-by-step derivation

- We start from the Klein-Gordon equation for $\delta\phi(t,x)$:

$$\ddot{\delta\phi} + 3H\dot{\delta\phi} - a^{-2}(t)\,\nabla^2\delta\phi = 0$$

- We switch to conformal time $\partial_t = (1/a)\partial_\eta$:

$$\delta\phi'' + 2\left(\frac{a'}{a}\right)\delta\phi' - \nabla^2\delta\phi = 0$$

- We Fourier expand $\delta\phi(\eta,x)$:

$$\delta\phi(\eta,\mathbf{x}) = \int\frac{d^3k}{(2\pi)^{3/2}}\left[\delta\phi_k(\eta)\,b_k e^{ik\cdot x} + \delta\phi_k^*(\eta)\,b_k^*\,e^{-ik\cdot x}\right]$$

- The Klein-Gordon equation for the Fourier modes $\delta\phi_k$ are:

$$\delta\phi_k'' + 2\left(\frac{a'}{a}\right)\delta\phi_k' + k^2\delta\phi_k = 0$$

- We define the new field $u_k \equiv a(\eta)\,\delta\phi_k(\eta)$:

$$u_k'' + \left[k^2 - \frac{a''}{a}\right]u_k = 0$$

Primordial Perturbations

Fluctuations of a (free) scalar field

$$u_k'' + \left[k^2 - \frac{a''}{a}\right] u_k = 0$$

- **Ultraviolet limit** $k \gg \frac{a''}{a}$ → $u_k(\eta) = \frac{1}{\sqrt{2k}}\left(A_k\, e^{-ik\eta} + B_k\, e^{ik\eta}\right)$

- **Infrared limit** $k \ll \frac{a''}{a}$ → $u_k \propto a(\eta) \Rightarrow \delta\phi_k \sim$ const.

Fourier modes $\delta\phi_k$ do not evolve in the infrared limit.

This phenomenon is called **mode freezing.**

Numerical Solution of the Mukhanov-Sasaki equation

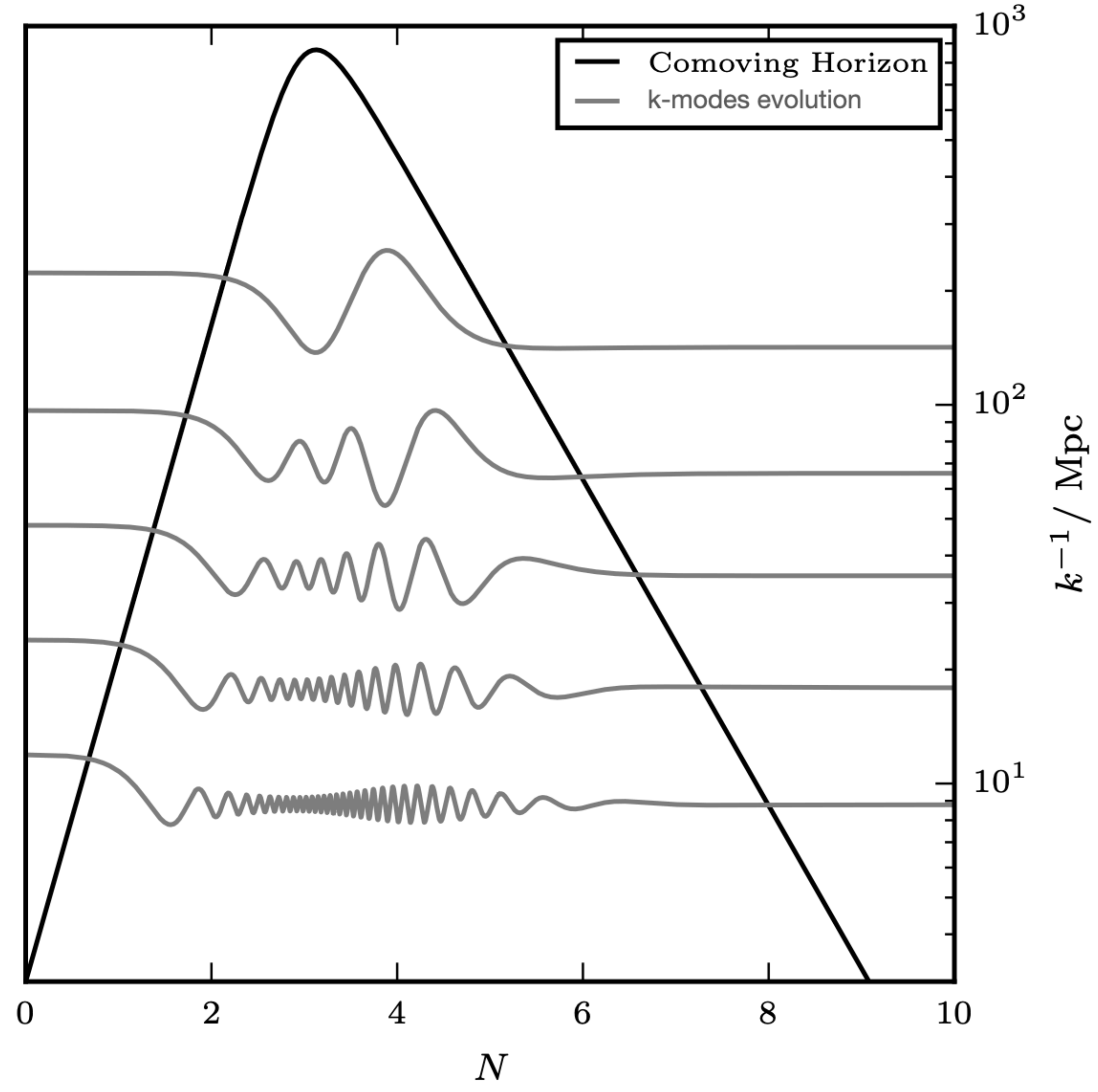


Image Taken from arXiv:1809.11095

Primordial Perturbations

Fluctuations of a (free) scalar field in a de Sitter spacetime

$$u_k'' + \left[k^2 - \frac{2}{\eta^2}\right] u_k = 0$$

an exact solution is:

$$u_k = A_k \frac{e^{-ik\eta}}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right) + B_k \frac{e^{ik\eta}}{\sqrt{2k}}\left(1 + \frac{i}{k\eta}\right)$$

Hereafter we set $A_k = 1$ and $B_k = 0$

(Corresponding to fixing the Bunch-Davies vacuum)

Scale factor (derivatives) in a de Sitter spacetime

$$d\eta = \frac{1}{H\,a^2}\,da \Rightarrow \eta = -\frac{1}{H\,a}$$

$$a(\eta) = -\frac{1}{H\,\eta}$$

$$a'(\eta) = \frac{1}{H\,\eta^2}$$

$$a''(\eta) = -\frac{2}{H\,\eta^3}$$

$$\downarrow$$

$$\frac{a''}{a} = \frac{2}{\eta^2}$$

Primordial Perturbations

Canonical Quantization

We have the complete expression of the $\delta\phi$ in de Sitter

$$\delta\phi(\eta, x) = \int \frac{d^3k}{(2\pi)^{3/2}} \left[\left(\frac{u_k}{a}\right) b_k e^{ik\cdot x} + \left(\frac{u_k^*}{a}\right) b_k^* \, e^{-ik\cdot x} \right]$$

$$u_k = \frac{e^{-ik\eta}}{\sqrt{2k}} \left(1 - \frac{i}{k\eta}\right)$$

We promote $\delta\phi \to \hat{\delta\phi}$ and $b_k \to \hat{b}_k$ ($b_k^* \to \hat{b}_k^\dagger$) to operator

$$\left[\hat{b}_k, \hat{b}_{k'}^\dagger\right] = \delta^3(k - k') \qquad \left[\hat{b}_k, \hat{b}_{k'}\right] = \left[\hat{b}_k^\dagger, \hat{b}_{k'}^\dagger\right] = 0$$

Important Notes

- The operators $b_k^\dagger$ and b_k can be interpreted as creation and annihilation operators respectively. In particular this defines the vacuum state as the state that satisfies:
$$b_k^\dagger|0\rangle = |1\rangle$$
$$b_k^\dagger|0\rangle = 0$$
- We can build a Hilbert space for our quantum field theory defining a set of states $|n(k_1), ..., n(k_n)\rangle$ representing the number of particles with momenta $\{k_1, \ldots, k_n\}$ on which the creation and annihilation operators act as follows:
$$b_{\mathbf{k}}^\dagger \, |n(\mathbf{k})\rangle = \sqrt{n+1} \, |n(\mathbf{k}) + 1\rangle,$$
$$b_{\mathbf{k}} \, |n(\mathbf{k})\rangle = \sqrt{n} \, |n(\mathbf{k}) - 1\rangle.$$
- Notice that having fixed $A_k = 1$ and $B_k = 0$ (imposing the spacetime to be asymptotically Minkowski) implicitly selected our vacuum state.

Primordial Perturbations

Spectrum of zero-point quantum fluctuations

$$\langle 0 \,|\, \hat{\delta\phi}(\eta, x)\hat{\delta\phi}(\eta, x') \,|\, 0 \rangle = \int \frac{d^3k}{(2\pi)^3} P_{\delta\phi}(k)\, e^{ik\cdot(x-x')}$$

Where:

$$P_{\delta\phi}(k) \doteq \frac{|\,u_k\,|^2}{a^2} = \frac{H^2}{2k^3}\left[1 + \left(\frac{k}{aH}\right)^2\right] \longrightarrow \frac{H^2}{2k^3} \text{ for } k \ll aH$$

We define the dimensionless spectrum $\mathscr{P}_{\delta\phi}(k) \equiv \dfrac{k^3}{2\pi^2} P_{\delta\phi}(k)$

$$\mathscr{P}_{\delta\phi}(k) = \left(\frac{H}{2\pi}\right)^2 \text{ for } k \ll aH$$

Step-by-step derivation

- We start from the expression of the field operator $\hat{\delta\phi}$:

$$\hat{\delta\phi}(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \left[\left(\frac{u_{\mathbf{k}}}{a}\right) \hat{b}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + \left(\frac{u^*_{\mathbf{k}}}{a}\right) \hat{b}^\dagger_{\mathbf{k}}\, e^{-i\mathbf{k}\cdot\mathbf{x}}\right]$$

- The only non vanishing matrix element around the vacuum state is:

$$\langle 0|\hat{\delta\phi}(\eta, \mathbf{x})\hat{\delta\phi}(\eta, \mathbf{x'})|0\rangle = \int \frac{d^3k\, d^3k'}{(2\pi)^3} \left(\frac{u_{\mathbf{k}} u^*_{\mathbf{k'}}}{a^2}\right) \langle 0|b_{\mathbf{k}} b^\dagger_{\mathbf{k'}}|0\rangle e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\mathbf{k'}\cdot\mathbf{x'}}$$

- We can easily compute it with the following trick

$$\langle 0|b_{\mathbf{k}} b^\dagger_{\mathbf{k'}}|0\rangle = \langle 0|b_{\mathbf{k}} b^\dagger_{\mathbf{k'}} - \underbrace{b^\dagger_{\mathbf{k'}} b_{\mathbf{k}}|0\rangle}_{=0} = \langle 0|\left[b_{\mathbf{k}}\,,\, b^\dagger_{\mathbf{k'}}\right]|0\rangle = \delta^3(\mathbf{k} - \mathbf{k'})$$

- The integral becomes

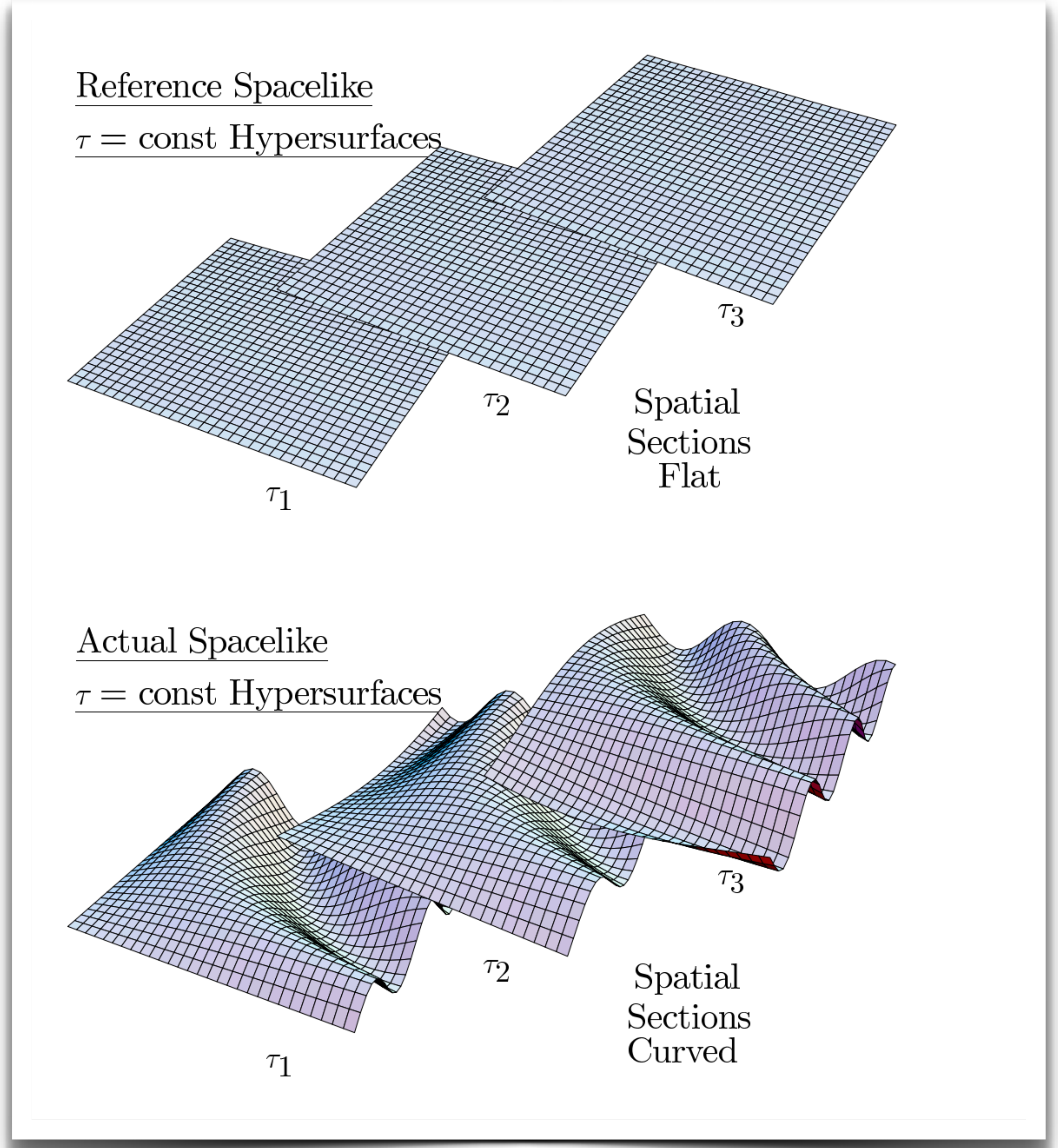
$$\begin{aligned}\langle 0|\hat{\delta\phi}(\eta, \mathbf{x})\hat{\delta\phi}(\eta, \mathbf{x'})|0\rangle &= \int \frac{d^3k}{(2\pi)^3} \left(\frac{|u_{\mathbf{k}}|^2}{a^2}\right) e^{i\,\mathbf{k}\cdot(\mathbf{x}-\mathbf{x'})} \\ &\equiv \int \frac{d^3k}{(2\pi)^3} P_{\delta\phi}(k)\, e^{i\,\mathbf{k}\cdot(\mathbf{x}-\mathbf{x'})}\end{aligned}$$

- Using that in de Sitter $u_k = \frac{e^{-ik\eta}}{\sqrt{2k}}\left(1 - \frac{i}{k\eta}\right)$ and $a = -1/(H\eta)$ we get:

$$P_{\delta\phi}(k) \equiv \frac{|u_{\mathbf{k}}|^2}{a^2} = \frac{H^2}{2k^3}\left[1 + \left(\frac{k}{aH}\right)^2\right]$$

Primordial Perturbations During Inflation

Image Taken from arXiv:hep-ph/0210162



1) Scalar Perturbations: $\delta\phi \rightarrow \delta\rho \rightarrow$ **Primordial density perturbations**

In the isotropic and homogeneous Universe, $\rho = \rho(t)$. Requiring not to have gauge modes defines a privileged coordinate choice (the comoving coordinates).

In a perturbed Universe, $\rho = \rho(t, \bar{x})$. The split into homogeneous background and perturbation is not unique, but it depends on the coordinates.

Because of perturbations, there is not a privileged coordinate system, and we need to fix a gauge and work with gauge-independent quantities.

2) Tensor Perturbations: $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} + h_{\mu\nu} \rightarrow$ **Primordial Gravity Waves**

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -h_+ & h_\times & 0 \\ 0 & h_\times & h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Tensor perturbations are intrinsically gauge-invariant at linear order

Primordial Scalar Modes from Inflation

Gauge Freedom

When considering primordial scalar modes, we need to pay attention to the so-called Gauge freedom

To solve ambiguities between real and fake perturbations, we use the gauge-independent primordial curvature perturbation ζ

$$\zeta \equiv -\Psi - \frac{H}{\dot{\rho}}\delta\rho$$

The quantum fluctuations in the inflaton field can be easily related to the primordial curvature perturbations in the zero spatial curvature gauge ($\Psi = 0$)

$$\zeta \approx -\left(\frac{H}{\dot{\phi}}\right)\delta\phi$$

Important Note

We can classify perturbations using their helicity (or spin). Consider a rotation of the coordinate system around the wave-vector k by an angle θ a perturbation is said to have helicity m if

$$\delta Q(\eta, k) \rightarrow e^{im\theta}\delta Q(\eta, k)$$

We define:

- **Scalar perturbations** those with helicity $m = 0$
- **Vector perturbations** those with helicity $m = \pm 1$
- **Tensor perturbations** those with helicity $m = \pm 2$

Primordial Scalar Modes from Inflation

Primordial Scalar Spectrum

The spectrum of primordial scalar modes can be easily computed in terms of the (gauge-invariant) primordial curvature perturbation ζ

$$\langle 0|\hat{\zeta}\hat{\zeta}|0\rangle = \left(\frac{H}{\dot{\phi}}\right)^2 \langle 0|\hat{\delta\phi}\hat{\delta\phi}|0\rangle$$

From this we define the (dimensionless) spectrum $\mathscr{P}_s = \left(\frac{H}{\dot{\phi}}\right)^2 \mathscr{P}_{\delta\phi}$

$$\mathscr{P}_s = \left(\frac{1}{8\pi^2 M_{\rm pl}^2}\right)\left(\frac{H^2}{\epsilon}\right)$$

Step-by-step derivation

- We start from $\hat{\zeta}(\eta,\mathbf{x}) = -\frac{H}{\dot{\phi}}\hat{\delta\phi}(\eta,\mathbf{x})$:

$$\begin{aligned}
\langle 0|\hat{\zeta}(\eta,\mathbf{x})\hat{\zeta}(\eta,\mathbf{x}')|0\rangle &= \left(\frac{H}{\dot{\phi}}\right)^2 \langle 0|\hat{\delta\phi}(\eta,\mathbf{x})\hat{\delta\phi}(\eta,\mathbf{x}')|0\rangle \\
&= \int \frac{d^3k}{(2\pi)^3}\left(\frac{H}{\dot{\phi}}\right)^2 P_{\delta\phi} \\
&\equiv \int \frac{d^3k}{(2\pi)^3} P_{\rm s}
\end{aligned}$$

where $P_{\rm s} \equiv \left(\frac{H}{\dot{\phi}}\right)^2 P_{\delta\phi}$

- We define the dimensionless spectrum

$$\mathcal{P}_{\rm s} \equiv \left(\frac{H}{\dot{\phi}}\right)^2 \mathcal{P}_{\delta\phi}$$

- Using $\mathcal{P}_{\delta\phi} \simeq \frac{H^2}{(2\pi)^2}$

$$\mathcal{P}_{\rm s} = \left(\frac{H}{\dot{\phi}}\right)^2 \frac{H^2}{(2\pi)^2}$$

- Using the slow-roll conditions $\dot{\phi}^2 = -2M_{\rm pl}^2\dot{H}$ and $\epsilon \equiv -\dot{H}/H^2$

$$\mathcal{P}_{\rm s} = \left(\frac{1}{8\pi^2 M_{\rm pl}^2}\right)\left(\frac{H^2}{\epsilon}\right)$$

Primordial Gravitational Waves from Inflation

Primordial Tensor Spectrum

Tensor perturbations are are described in terms of the transverse and traceless part of the symmetric 3×3 matrix h_{ij}

In Fourier space, $u_k(\eta) \equiv a(\eta)h_k(\eta)$ satisfies:

$$u_k'' + \left[k^2 - \frac{a''}{a}\right] u_k = 0$$

Accounting for all polarization states $(+, \times)$ and normalization factors $(M_p/2)$, the (dimensionless) spectrum of primordial tensor modes is

$$\mathscr{P}_\mathrm{t}(k) = \frac{8}{M_p^2}\left(\frac{H}{2\pi}\right)^2$$

Step-by-step derivation

- We start from the linearized Einstein Equations $\delta G_{\mu\nu} = \delta T_{\mu\nu}$

$$\delta G^i_j = \frac{1}{2a^2}\left(h''_{ij} + 2\frac{a'}{a}h'_{ij} - \nabla^2 h_{ij}\right) \qquad \delta T^i_j = 0 \text{ (for perfect fluid)}$$

- This implies

$$h''_{ij} + 2\frac{a'}{a}h'_{ij} - \nabla^2 h_{ij} = 0$$

- Expanding in Fourier space, the equation for the Fourier modes h_k for each polarization state (+,×) is

$$h''_k + 2\frac{a'}{a}h'_k + k^2 h_k = 0$$

- we introduce the fields $u_k \equiv a(\eta)\, h_k$ satisfying the Mukhanov-Sasaki Equation:

$$u''_k + \left[k^2 - \frac{a''}{a}\right] u_k = 0$$

- In order to use the formalism developed from a scalar field, we have to be sure that h and ϕ have the same physical unit. We introduce the canonical normalized fields $\psi_k = \frac{M_p}{2} u_k$ (satisfying the same Mukhanov-Sasaki Equation)

- For each polarization state, the spectrum of ψ can be easily computed as

$$\mathcal{P}_\psi = \frac{H^2}{(2\pi)^2}$$

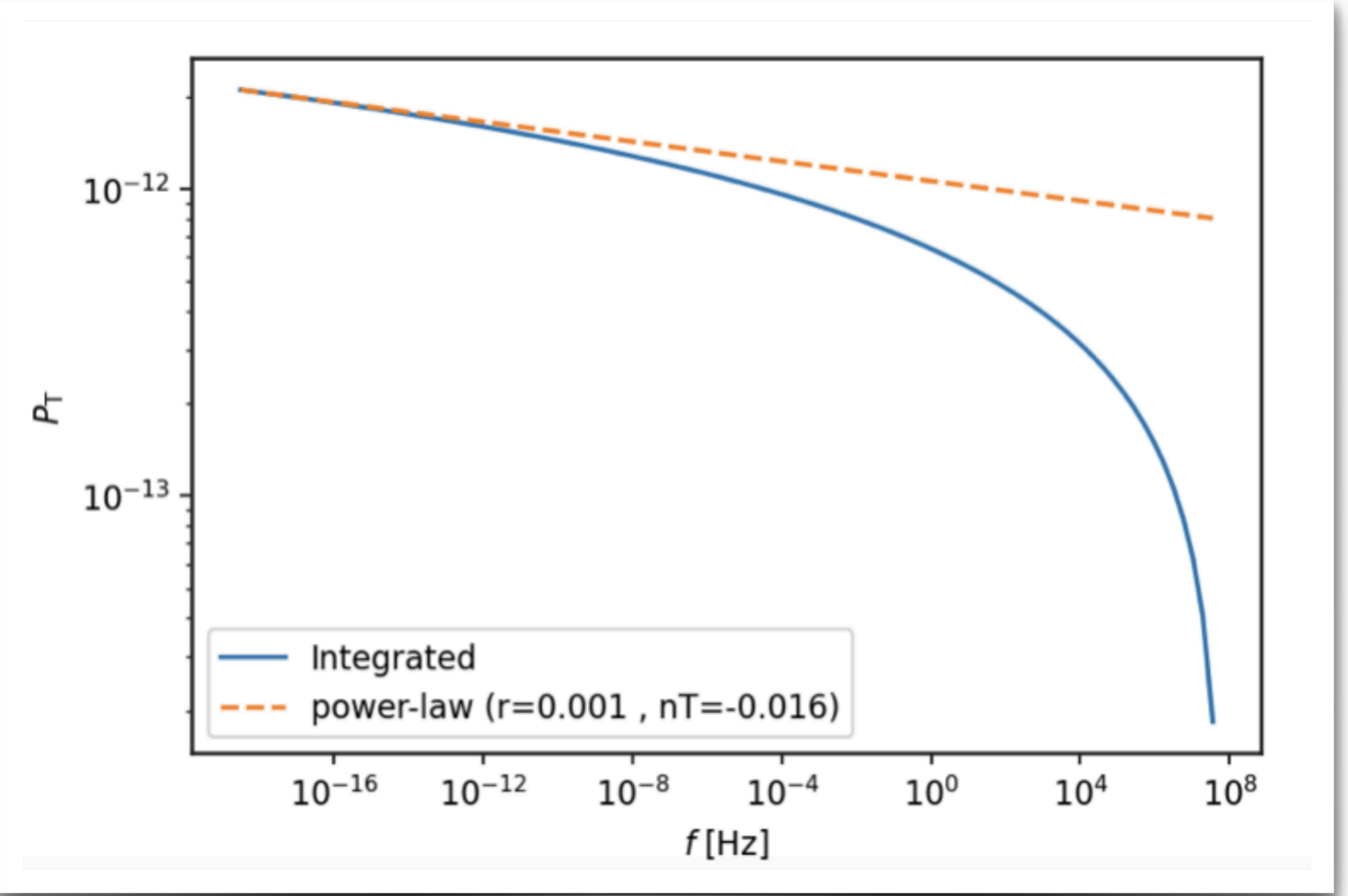
- The total spectrum of h can be derived as

$$\mathcal{P}_h = 2 \times \frac{4}{M_p^2} \times \left(\frac{H^2}{(2\pi)^2}\right)$$

where the 2 counts the polarization states and $\frac{4}{M_p^2}$ is coming from the normalization factor.

POWER LAW PARAMETERIZATION

Image Taken from arXiv:2210.14159



The primordial spectra can be well parametrized by power laws

$$\mathscr{P}_s = \left(\frac{1}{8\pi^2 M_{\rm pl}^2}\right)\left(\frac{H^2}{\epsilon}\right) \simeq A_s \left(\frac{k}{k_*}\right)^{n_s - 1}$$

$$\mathscr{P}_{\rm t}(k) = \frac{8}{M_p^2}\left(\frac{H}{2\pi}\right)^2 \simeq rA_s \left(\frac{k}{k_*}\right)^{n_{\rm t}}$$

These relations are nothing but a Taylor expansion of $\ln \mathscr{P}_{\rm s,t}(k)$ around k_*, so:

$$n_s - 1 = \left.\frac{d\ln \mathscr{P}_{\rm s}}{d\ln k}\right|_{k=k_*} = 2\eta_V - 6\epsilon_V$$

$$r = \frac{A_t}{A_s} = -8n_{\rm t}$$

$$n_{\rm t} = \left.\frac{d\ln \mathscr{P}_{\rm t}}{d\ln k}\right|_{k=k_*} = -2\epsilon_v$$

(More details in Lecture 2)

Linking Inflation and the CMB

INFLATION

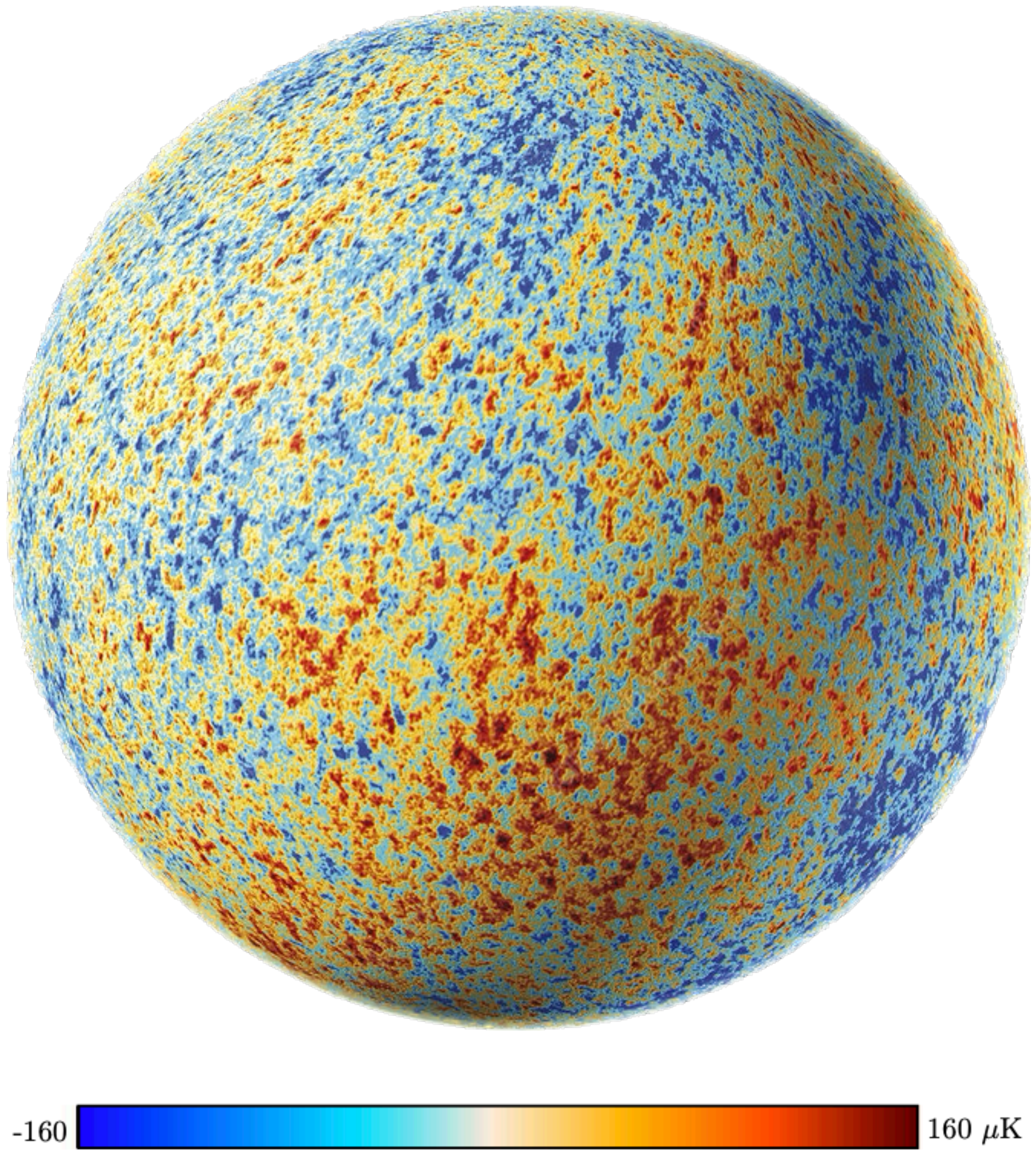
Primordial Scalar Spectrum

$$\mathscr{P}_s(k) = A_s \left(\frac{k}{k_*}\right)^{n_s-1}$$

Primordial Tensor Spectrum

$$\mathscr{P}_{\rm t}(k) = rA_s \left(\frac{k}{k_*}\right)^{n_{\rm t}}$$

Planck 2018



Linking Inflation and the CMB

CMB Temperature Anisotropies

We split the temperature in the average value and perturbations

$$T(\eta,\bar{x},\hat{p}) = T(\eta)\left[1+\frac{\delta T(\eta,\bar{x},\hat{p})}{T(\eta)}\right] \equiv T(\eta)\big[1+\Theta(\eta,\bar{x},\hat{p})\big]$$

We can expand in spherical harmonics

$$\Theta(\eta,\bar{x},\hat{p}) = \sum_{\ell=1}^{\infty}\sum_{m=-\ell}^{\ell} a_{\ell m}(\eta,\bar{x})Y_{\ell m}(\hat{p})$$

Planck 2018

Linking Inflation and the CMB

CMB Temperature Anisotropies

$$\Theta(\eta, \bar{x}, \hat{p}) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}(\eta, \bar{x}) Y_{\ell m}(\hat{p})$$

Spherical harmonics are orthonormalized, so this relation can be reversed

$$a_{\ell m}(\eta, \bar{x}) = \int d\Omega \, \Theta(\eta, \bar{x}, \hat{p}) \, Y^*_{\ell m}(\hat{p})$$

Therefore the coefficients $a_{\ell m}$ can be directly connected with the (measured) temperature fluctuations, although only here ($\bar{x} = \bar{x}_0$) and today ($\eta = \eta_0$)

Step-by-step derivation

- We start from:

$$\Theta(\eta, \bar{x}, \hat{p}) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}(\eta, \bar{x}) Y_{\ell m}(\hat{p})$$

- We note that spherical harmonics are orthonormalized:

$$\int d\Omega Y_{\ell m}(\hat{p}) Y^*_{\ell' m'}(\hat{p}) = \delta_{\ell \ell'} \delta_{m m'}$$

where Ω is the solid angle spanned by $\hat{p}$ on a unit sphere

- We multiply both sides by $Y^*_{\ell' m'}(\hat{p})$

$$\Theta(\eta, \bar{x}, \hat{p}) Y^*_{\ell' m'}(\hat{p}) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}(\eta, \bar{x}) Y_{\ell m}(\hat{p}) Y^*_{\ell' m'}(\hat{p})$$

- We integrate over the solid angle and

$$\int d\Omega \, \Theta(\eta, \bar{x}, \hat{p}) Y^*_{\ell' m'}(\hat{p}) = \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}(\eta, \bar{x}) \int d\Omega \, Y_{\ell m}(\hat{p}) Y^*_{\ell' m'}(\hat{p})$$
$$= \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}(\eta, \bar{x}) \delta_{\ell \ell'} \delta_{m m'}$$

- From this we obtain:

$$\int d\Omega \, \Theta(\eta, \bar{x}, \hat{p}) Y^*_{\ell m}(\hat{p}) = a_{\ell m}(\eta, \bar{x})$$

Linking Inflation and the CMB

Planck 2018

CMB Temperature Anisotropies

We cannot make predictions about any particular value of $a_{\ell m}$, but we can say more about their *statistical* properties.

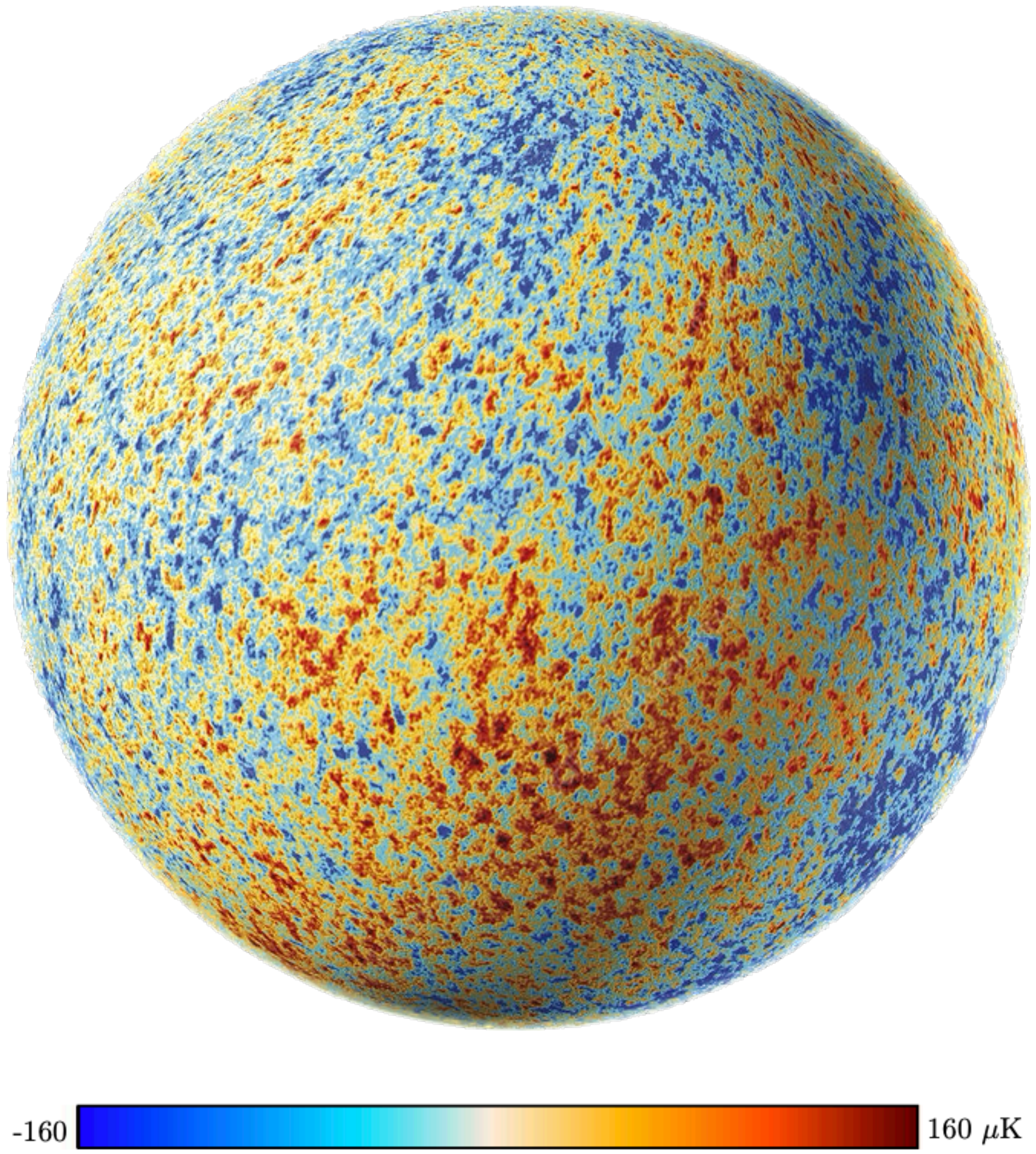
1) For an isotropic Universe, invariance under rotations implies

$$\langle a_{\ell m} \rangle = 0$$

2) However they will have a non-zero variance

$$\langle a^*_{\ell m} a_{\ell' m'} \rangle = C_\ell^{TT} \, \delta_{\ell \ell'} \, \delta_{m m'}$$

3) The distribution from which they are drawn traces its origin to the primordial perturbations produced during inflation. If these are Gaussian, so are $a_{\ell m}$.



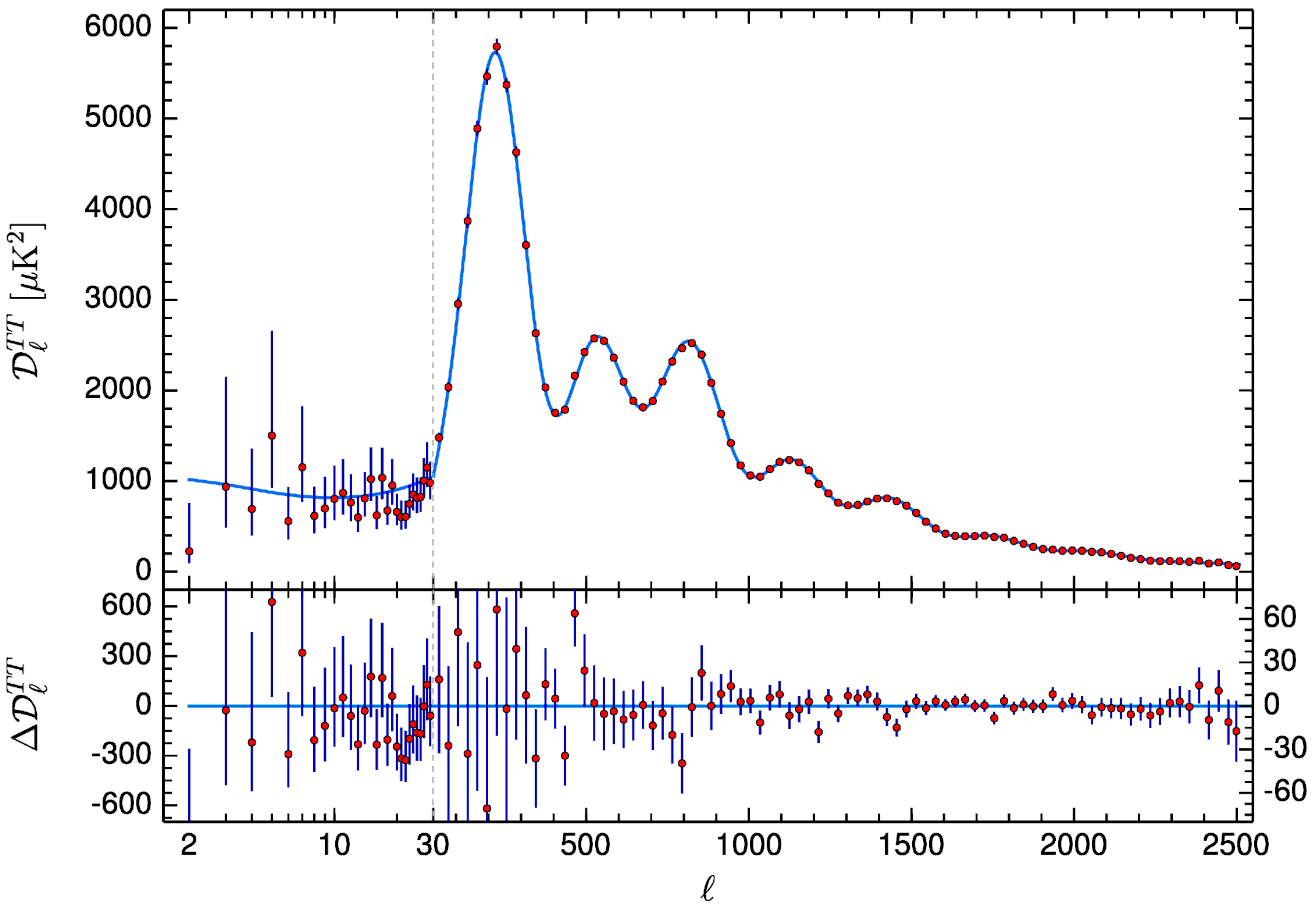
Linking Inflation and the CMB

CMB Temperature Anisotropies

$$C_\ell = \frac{1}{2\ell + 1} \sum_{m=-\ell}^{m=\ell} \left| a_{\ell m} \right|^2$$

$$a_{\ell m}(\eta, \bar{x}) = \int d\Omega \, \Theta(\eta, \bar{x}, \hat{p}) \, Y^*_{\ell m}(\hat{p})$$

From temperature anisotropies, we can measure C_ℓ^{TT}



Linking Inflation and the CMB

CMB Temperature Anisotropies

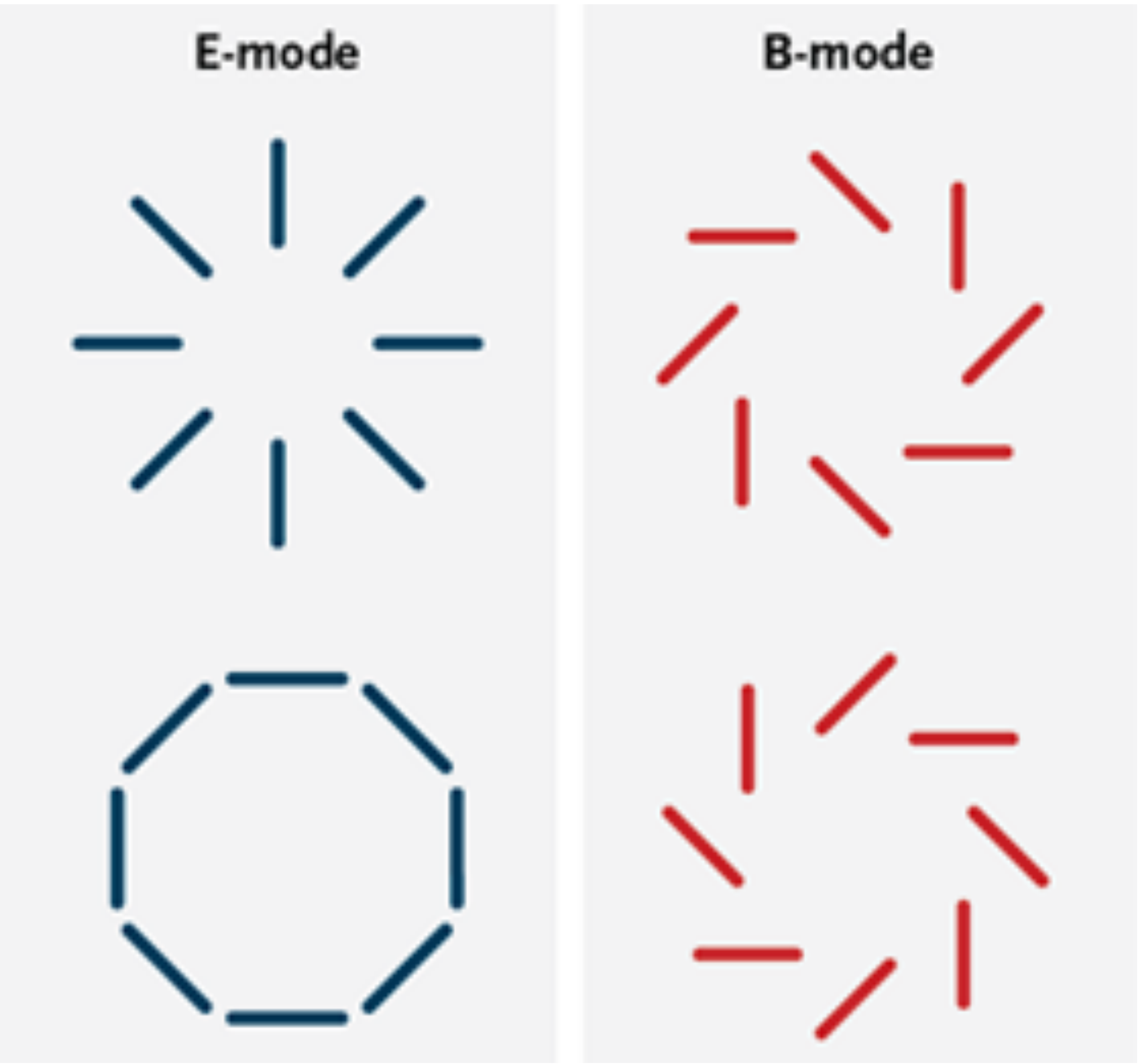
$$\langle a^*_{\ell m} a_{\ell' m'} \rangle = C_\ell^{TT}\, \delta_{\ell \ell'}\, \delta_{mm'}$$

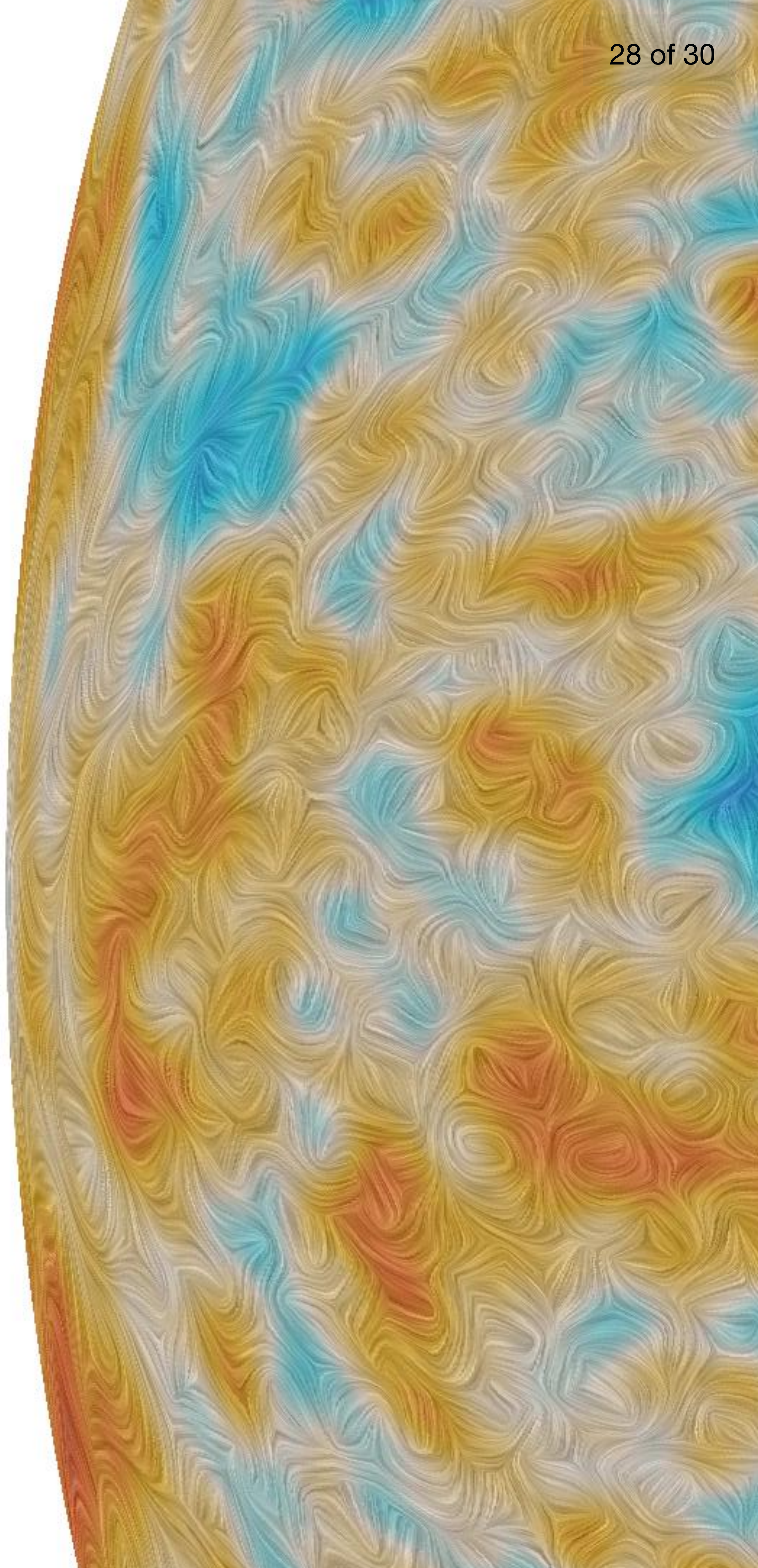
CMB Polarization

$$\langle E^*_{\ell m} E_{\ell' m'} \rangle = C_\ell^{EE}\, \delta_{\ell \ell'}\, \delta_{mm'}$$

$$\langle a^*_{\ell m} E_{\ell' m'} \rangle = C_\ell^{TE}\, \delta_{\ell \ell'}\, \delta_{mm'}$$

$$\langle B^*_{\ell m} B_{\ell' m'} \rangle = C_\ell^{BB}\, \delta_{\ell \ell'}\, \delta_{mm'}$$





Linking Inflation and the CMB

CMB Spectra

$$\langle a^*_{\ell m} a_{\ell' m'} \rangle = C_\ell^{TT}\, \delta_{\ell\ell'}\, \delta_{mm'}$$

$$\langle E^*_{\ell m} E_{\ell' m'} \rangle = C_\ell^{EE}\, \delta_{\ell\ell'}\, \delta_{mm'}$$

$$\langle a^*_{\ell m} E_{\ell' m'} \rangle = C_\ell^{TE}\, \delta_{\ell\ell'}\, \delta_{mm'}$$

$$\langle B^*_{\ell m} B_{\ell' m'} \rangle = C_\ell^{BB}\, \delta_{\ell\ell'}\, \delta_{mm'}$$

Primordial Spectra from Inflation

Primordial Scalar Spectrum

$$\mathscr{P}_s(k) = A_s \left(\frac{k}{k_*}\right)^{n_s - 1}$$

Primordial Tensor Spectrum

$$\mathscr{P}_\mathrm{t}(k) = rA_s \left(\frac{k}{k_*}\right)^{n_\mathrm{t}}$$

Linking Inflation and the CMB

Transfer Functions

$$\left[C_\ell^{XY}\right]_{\text{scalar}} = \frac{2\pi}{\ell(\ell+1)} \int_0^\infty d\ln k \; \underbrace{T_\ell^X(k)\, T_\ell^Y(k)}_{\text{Scalar Transfer functions}} \; \underbrace{\mathscr{P}_{\rm s}(k)}_{\text{Scalar spectrum}}$$

$$\left[C_\ell^{XY}\right]_{\text{tensor}} = \frac{2\pi}{\ell(\ell+1)} \int_0^\infty d\ln k \; \underbrace{T_\ell^X(k)\, T_\ell^Y(k)}_{\text{Tensor Transfer functions}} \; \underbrace{\mathscr{P}_{\rm t}(k)}_{\text{Tensor spectrum}}$$

Note:

- Scalar and Tensor transfer functions are different
- $C_\ell^{\rm tot} = \left[C_\ell\right]_{\text{scalar}} + \left[C_\ell\right]_{\text{tensor}}$
- In $\left[C_\ell^{XY}\right]_{\text{scalar}}$ we have: $X, Y = \{T, E\}$
- In $\left[C_\ell^{XY}\right]_{\text{tensor}}$ we have: $X, Y = \{T, E, B\}$
- Transfer functions are different for T, E, B

END OF LECTURE 1

Thank You!