Torsion effects in Polynomial Affine Gravity as a dynamical system

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1 Introduction

2 The model

2.1 The formulation

The most general action (up to topological invariants and boundary terms) in four dimensions is given by

$$\begin{aligned} S = \int \mathrm{d}V^{\alpha\beta\gamma\delta} \Big[& B_1 \mathcal{R}_{\mu\nu}{}^{\mu}{}_{\rho} \mathcal{B}_{\alpha}{}^{\nu}{}_{\beta} \mathcal{B}_{\gamma}{}^{\rho}{}_{\delta} + B_2 \mathcal{R}_{\alpha\beta}{}^{\mu}{}_{\rho} \mathcal{B}_{\gamma}{}^{\nu}{}_{\delta} \mathcal{B}_{\mu}{}^{\rho}{}_{\nu} + B_3 \mathcal{R}_{\mu\nu}{}^{\mu}{}_{\alpha} \mathcal{B}_{\beta}{}^{\nu}{}_{\gamma} \mathcal{A}_{\delta} \\ & + B_4 \mathcal{R}_{\alpha\beta}{}^{\sigma}{}_{\rho} \mathcal{B}_{\gamma}{}^{\rho}{}_{\delta} \mathcal{A}_{\sigma} + B_5 \mathcal{R}_{\alpha\beta}{}^{\rho}{}_{\rho} \mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta} \mathcal{A}_{\sigma} + C_1 \mathcal{R}_{\mu\alpha}{}^{\mu}{}_{\nu} \nabla_{\beta} \mathcal{B}_{\gamma}{}^{\nu}{}_{\delta} \\ & + C_2 \mathcal{R}_{\alpha\beta}{}^{\rho}{}_{\rho} \nabla_{\sigma} \mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta} + D_1 \mathcal{B}_{\nu}{}^{\mu}{}_{\lambda} \mathcal{B}_{\mu}{}^{\nu}{}_{\alpha} \nabla_{\beta} \mathcal{R}_{\gamma}{}^{\lambda}{}_{\delta} + D_2 \mathcal{B}_{\alpha}{}^{\mu}{}_{\beta} \mathcal{B}_{\mu}{}^{\lambda}{}_{\nu} \nabla_{\lambda} \mathcal{B}_{\gamma}{}^{\nu}{}_{\delta} \\ & + D_3 \mathcal{B}_{\alpha}{}^{\mu}{}_{\nu} \mathcal{B}_{\beta}{}^{\lambda}{}_{\gamma} \nabla_{\lambda} \mathcal{B}_{\mu}{}^{\nu}{}_{\delta} + D_4 \mathcal{B}_{\alpha}{}^{\lambda}{}_{\beta} \mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta} \nabla_{\lambda} \mathcal{A}_{\sigma} + D_5 \mathcal{B}_{\alpha}{}^{\lambda}{}_{\beta} \mathcal{A}_{\sigma} \nabla_{\lambda} \mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta} \\ & + D_6 \mathcal{B}_{\alpha}{}^{\lambda}{}_{\beta} \mathcal{A}_{\gamma} \nabla_{\lambda} A_{\delta} + D_7 \mathcal{B}_{\alpha}{}^{\lambda}{}_{\beta} \mathcal{A}_{\lambda} \nabla_{\gamma} A_{\delta} + E_1 \nabla_{\rho} \mathcal{B}_{\alpha}{}^{\rho}{}_{\beta} \nabla_{\sigma} \mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta} \\ & + E_2 \nabla_{\rho} \mathcal{B}_{\alpha}{}^{\rho}{}_{\beta} \nabla_{\gamma} \mathcal{A}_{\delta} + F_1 \mathcal{B}_{\alpha}{}^{\mu}{}_{\beta} \mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta} \mathcal{B}_{\mu}{}^{\lambda}{}_{\rho} \mathcal{B}_{\sigma}{}^{\rho}{}_{\lambda} + F_2 \mathcal{B}_{\alpha}{}^{\mu}{}_{\beta} \mathcal{B}_{\gamma}{}^{\nu}{}_{\lambda} \mathcal{B}_{\delta}{}^{\lambda}{}_{\rho} \mathcal{B}_{\mu}{}^{\rho}{}_{\nu} \\ & + F_3 \mathcal{B}_{\nu}{}^{\mu}{}_{\lambda} \mathcal{B}_{\mu}{}^{\nu}{}_{\alpha} \mathcal{B}_{\beta}{}^{\lambda}{}_{\gamma} \mathcal{A}_{\delta} + F_4 \mathcal{B}_{\alpha}{}^{\mu}{}_{\beta} \mathcal{B}_{\gamma}{}^{\nu}{}_{\delta} \mathcal{A}_{\mu} \mathcal{A}_{\nu} \Big]. \end{aligned} \tag{1}$$

$$\Gamma_{t}{}^{t}{}_{t} = f(t) \qquad \Gamma_{i}{}^{t}{}_{j} = g(t) S_{ij} \tag{2}$$

$$\Gamma_{t}{}^{i}{}_{j} = h(t) \delta^{i}_{j} \qquad \Gamma_{i}{}^{j}{}_{k} = \gamma_{i}{}^{j}{}_{k}, \tag{3}$$

where S_{ij} is the three-dimensional rank two symmetric tensor defined as follow

$$S_{ij} = \begin{pmatrix} \frac{1}{1-\kappa r^2} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}, \tag{4}$$

and γ is the symmetric connection compatible with desired symmetries written as

$$\gamma_r{}^r{}_r = \frac{\kappa r}{1-\kappa r^2} \qquad \gamma_\theta{}^r{}_\theta = \kappa r^3 - r \quad \gamma_\varphi{}^r{}_\varphi = \left(\kappa r^3 - r\right)\sin^2\theta \quad \gamma_r{}^\theta{}_\theta = \frac{1}{r} \tag{5}$$

$$\gamma_\varphi{}^\theta{}_\varphi = -\cos\theta\sin\theta \quad \gamma_r{}^\varphi{}_\varphi = \frac{1}{r} \qquad \gamma_\theta{}^\varphi{}_\varphi = \frac{\cos\theta}{\sin\theta}. \tag{6}$$

Interestingly, the affine function $f(t)$ can be set equal to zero, under a reparametrisation of the time coordinate, for more information on this type of transformation, please refer to Ref. [**?**]. Therefore, there are only two non trivial functions to define completely the symmetric part of the connection.

$$\mathcal{B}_\theta{}^r{}_\varphi = \psi(t) r^2 \sin\theta\sqrt{1-\kappa r^2} \quad \mathcal{B}_r{}^\theta{}_\varphi = \frac{\psi(t)\sin\theta}{\sqrt{1-\kappa r^2}} \quad \mathcal{B}_r{}^\varphi{}_\theta = \frac{\psi(t)}{\sqrt{1-\kappa r^2}\sin\theta}. \tag{7}$$

Notice the trace-less part of the torsion tensor has only one time-dependent function to defined the tensor completely.

$$\mathcal{A}_t = \eta(t). \tag{8}$$

There is only one time-dependent function to define completely the vectorial part of the torsion tensor.

Finally, the complete set of field equations are obtained through Kijowski's formalism, see Refs.[XX] for each irreducible field of the affine connection. The explicit form of the field equations can be found in Ref. [1].

3 Dynamical system technique

$$\left(B_3\left(\dot{g} + gh + 2\kappa\right) - 2B_4\left(\dot{g} - gh\right) + 2D_6\eta g - 2F_3\psi^2\right)\psi = 0, \tag{9}$$

$$\left(B_3\eta\psi - 2B_4\eta\psi + C_1\left(\dot{\psi} - 2h\psi\right)\right)g = 0, \tag{10}$$

$$\left(B_3 + 2B_4\right)\eta g\psi + 2C_1\left(\kappa\psi + 4gh\psi - g\dot{\psi} - \psi\dot{g}\right) + 2\psi^3\left(2D_2 - D_1 - D_3\right) = 0, \tag{11}$$

$$\begin{aligned} &B_3\left(\eta\left(h\psi - \dot{\psi}\right) - \psi\dot{\eta}\right) - 2B_4\left(\eta\left(-h\psi - \dot{\psi}\right) - \psi\dot{\eta}\right) \\ &\quad + C_1\left(4h^2\psi + 2\psi\dot{h} - \ddot{\psi}\right) + D_6\eta^2\psi = 0, \end{aligned} \tag{12}$$

$$\begin{aligned} &B_3\left(\dot{g} + gh + 2\kappa\right)\eta - 2B_4\left(\dot{g} - gh\right)\eta + C_1\left(2\kappa h + 4gh^2 + 2g\dot{h} - \ddot{g}\right) \\ &\quad + 6h\psi^2\left(2D_2 - D_1 - D_3\right) + D_6\eta^2 g - 6F_3\eta\psi^2 = 0 \end{aligned} \tag{13}$$

From this, we solve eq.(**??**) to find an expression for $\eta(t)$

$$\eta(t) = \left(\frac{2h\psi - \dot{\psi}}{\psi}\right)\left(\frac{C_1}{B_3 - 2B_4}\right), \tag{14}$$

replacing the above expression for $\eta(t)$ into eq. (**??**), leads to two sub-branches for the $h(t)$ function

$$h(t) = \frac{\dot{\psi}}{2\psi} \qquad h(t) = \frac{\dot{\psi}}{\psi}\left(\frac{C_1 D_6}{3B_3^2 - 8B_3B_4 + B_4^2 + 2C_1D_6}\right) \tag{15}$$

using the simplest form of $h(t)$ function, then eq. (**??**) leads to

$$-2\left(D_1 - 2D_2 + D_3\right)\psi^3 + 2C_1\left(\psi\left(\kappa - \dot{g}\right) + g\dot{\psi}\right) = 0, \tag{16}$$

which is a first order differential equation which can be solve for $g(t)$ in terms of the ψ function

$$g(t) = \psi(t)\left(g_0 + \int_1^t \left(\frac{\kappa}{\psi(\tau)} - \psi(\tau)\left(\frac{D_1 - 2D_2 + D_3}{C_1}\right)\right)\mathrm{d}\tau\right) \tag{17}$$

the above solution also solves eq. (**??**), where g_0 is an integration constant. Then, eq. (**??**) becomes an integro-differential equation of first order

$$\begin{aligned}\dot{\psi}&\left(g_0 + \int_1^t \left(\frac{\kappa}{\psi(\tau)} - \psi(\tau)\left(\frac{D_1 - 2D_2 + D_3}{C_1}\right)\right)\mathrm{d}\tau\right)\left(\frac{3B_3 - 2B_4}{2}\right)\\ &- \psi^2\frac{\left(B_3 - 2B_4\right)\left(D_1 - 2D_2 + D_3\right) + 2C_1F_3}{C_1} + \kappa\left(3B_3 - 2B_4\right) = 0.\end{aligned} \tag{18}$$

The above equation can be solved for the special case $\kappa = 0$, with the variable change $\psi(t) = \dot{\phi}(t)$, then

$$\ddot{\phi}\left(g_0 - \phi\alpha\right)\beta - \dot{\phi}^2\gamma = 0, \tag{19}$$

where

$$\alpha = \left(\frac{D_1 - 2D_2 + D_3}{C_1}\right) \qquad \beta = \left(\frac{3B_3 - 2B_4}{2}\right) \tag{20}$$

$$\gamma = \frac{\left(B_3 - 2B_4\right)\left(D_1 - 2D_2 + D_3\right) + 2C_1F_3}{C_1}, \tag{21}$$

whose solution

$$\phi(t) = \frac{g_0}{\alpha} + \frac{\left(\phi_0\left(\alpha\beta + \gamma\right)\left(t + \phi_1\right)\right)^{\frac{\alpha\beta}{\alpha\beta+\gamma}}}{\alpha\beta}, \tag{22}$$

where ϕ_0 and ϕ_1 are integration constant. From this

$$\psi(t) = \phi_0\left(\phi_0\left(\alpha\beta + \gamma\right)\left(t + \phi_1\right)\right)^{-\frac{\gamma}{\alpha\beta+\gamma}}. \tag{23}$$

From the above expression and using the relations in eqs. (14), (15) and (17) it is straightforward to find the rest of the affine functions

$$\eta(t) = 0, \tag{24}$$

$$h(t) = -\frac{\gamma}{2\left(\alpha\beta+\gamma\right)\left(t+\phi_1\right)}, \tag{25}$$

$$g(t) = \phi_0\left(\left(\alpha\beta+\gamma\right)\phi_0\left(t+\phi_1\right)\right)^{-\frac{\gamma}{\alpha\beta+\gamma}}\left(g_0 - \frac{\left(\left(\alpha\beta+\gamma\right)\phi_0\left(t+\phi_1\right)\right)^{\frac{\alpha\beta}{\alpha\beta+\gamma}}}{\beta}\right) \tag{26}$$

4 Final remarks

References

[1] Oscar Castillo-Felisola, Bastian Grez, Oscar Orellana, José Perdiguero, Aureliano Skirzewski, and Alfonso R Zerwekh. Corrigendum: Emergent metric and geodesic analysis in cosmological solutions of (torsion-free) polynomial affine gravity (2020 class. quantum grav.37 075013). *Classical and Quantum Gravity*, 40(24):249501, nov 2023.