Personal notes on gravitational waves

Jose Perdiguero Garate

April 2, 2025

Contents

Notation and conventions

I am working with the following conventions, for the metric tensor

$$(-1, +1, +1, +1) \tag{1}$$

1 Linear general relativity

In this section I am presenting a general overview on how to build-up the Einstein's field equation, given a background metric tensor and a perturbation tensor.

1.1 General overview

Decomposed the metric tensor as the sum of a background metric, in this case the flat spacetime Minkowski metric $\eta_{\mu\nu}$,plus a perturbation $h_{\mu\nu}$ as follows

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{2}$$

Using the definition of the Kronecker delta object it is straightforward to obtain the inverse tensor of the perturbation

$$\delta^{\mu}_{\nu} = g^{\mu\sigma} g_{\sigma\nu}. \tag{3}$$

Replacing Eq.(2) in to Eq.(3), and neglecting second order terms in the perturbation field, leads to

$$h^{\mu\nu} = -\eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}, \tag{4}$$

notice that, in order to upper/lower the indices of the perturbation I am only using the background metric.[1] Schematically, second order terms are neglected

$$h_{\mu\nu}h_{\alpha\beta} \sim 0, \qquad h_{\mu\nu}\partial_{\gamma}h_{\alpha\beta} \sim 0, \qquad \partial_{\delta}h_{\mu\nu}\partial_{\gamma}h_{\alpha\beta} \sim 0. \tag{5}$$

This is the general overview of the fundamental field of general relativity, which at its core is the metric tensor. In the following subsection, I will be computing the Einstein's field equations for the metric tensor written in Eq.(2).

1.2 Geometrical objects

The first object that is required to compute the Einstein's field equations is the connection. Working on a torsion-free manifold, the Levi-Civita connection is written as

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2}g^{\mu\rho}\left(\partial_{\alpha}g_{\rho\beta} + \partial_{\beta}g_{\rho\alpha} - \partial_{\rho}g_{\alpha\beta}.\right) \tag{6}$$

Replacing Eq.(2) in to Eq.(6) leads to

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2}\left(\eta^{\mu\rho} - h^{\mu\rho}\right)\left(\partial_{\alpha}\eta_{\rho\beta} + \partial_{\beta}\eta_{\rho\alpha} - \partial_{\rho}\eta_{\alpha\beta} + \partial_{\alpha}h_{\rho\beta} + \partial_{\beta}h_{\rho\alpha} - \partial_{\rho}h_{\alpha\beta}\right). \tag{7}$$

Notice that, the only non-trivial contributions are the ones that are linear in the perturbation, additionally, the partial derivatives of the Minkowski's metric tensor vanishes, therefore, the connection coefficients are reduced to

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2}\eta^{\mu\rho}\left(\partial_{\alpha}h_{\rho\beta} + \partial_{\beta}h_{\rho\alpha} - \partial_{\rho}h_{\alpha\beta}\right). \tag{8}$$

[1] Including the perturbation tensor leads to second order terms, which I am ignoring.

Next, compute the Riemann curvature tensor

$$\mathcal{R}^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}{}_{\sigma\nu} - \partial_{\nu}\Gamma^{\rho}{}_{\sigma\mu} + \Gamma^{\gamma}{}_{\nu\sigma}\Gamma^{\rho}{}_{\mu\gamma} + \Gamma^{\gamma}{}_{\mu\sigma}\Gamma^{\rho}{}_{\nu\gamma}, \tag{9}$$

however, instead of computing directly from the above equation, it is convenient to notice the structure of the curvature tensor. The last two terms are quadratic in the Levi-Civita connection, and, since the connection is written with perturbation, then, square terms in the connection vanishes, reducing the Riemann curvature tensor to

$$\mathcal{R}^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\Gamma^{\rho}{}_{\sigma\nu} - \partial_{\nu}\Gamma^{\rho}{}_{\sigma\mu}. \tag{10}$$

Replacing Eq.(8) in to Eq.(10) leads to

$$\mathcal{R}^{\rho}{}_{\sigma\mu\nu} = \frac{1}{2}\eta^{\rho\alpha}\partial_{\mu}\left(\partial_{\sigma}h_{\alpha\nu} + \partial_{\nu}h_{\sigma\alpha} - \partial_{\alpha}h_{\sigma\nu}\right) + \frac{1}{2}\eta^{\rho\alpha}\partial_{\nu}\left(\partial_{\sigma}h_{\alpha\mu} + \partial_{\mu}h_{\sigma\alpha} - \partial_{\alpha}h_{\sigma\mu}\right), \tag{11}$$

the above expression, can be simplified to

$$\mathcal{R}^{\rho}{}_{\sigma\mu\nu} = \frac{1}{2}\eta^{\rho\alpha}\left(\partial_{\mu}\partial_{\sigma}h_{\alpha\nu} - \partial_{\mu}\partial_{\alpha}h_{\sigma\nu} - \partial_{\nu}\partial_{\sigma}h_{\mu\alpha} + \partial_{\nu}\partial_{\alpha}h_{\mu\sigma}\right). \tag{12}$$

From the Riemann tensor, it is straightforward to compute the Ricci tensor, by contracting their respective indices

$$\mathcal{R}_{\sigma\nu} = \mathcal{R}^{\mu}{}_{\sigma\mu\nu}. \tag{13}$$

A direct computation shows the structure of the Ricci tensor

$$\mathcal{R}_{\sigma\nu} = \frac{1}{2}\left(\partial^{\alpha}\partial_{\sigma}h_{\alpha\nu} - \Box h_{\sigma\nu} - \partial_{\nu}\partial_{\sigma}h + \partial_{\nu}\partial^{\alpha}h_{\alpha\sigma}\right), \tag{14}$$

where $\Box$ is the d'Alembert operator and h is the trace of the perturbation.

In the same spirit, the curvature scalar can be obtained directly through the contraction of the Ricci tensor

$$\mathcal{R} = g^{\mu\sigma}\mathcal{R}_{\mu\sigma}. \tag{15}$$

This computation is straightforward

$$\mathcal{R} = \left(\eta^{\mu\sigma} - h^{\mu\sigma}\right)\frac{1}{2}\left(\partial^{\alpha}\partial_{\sigma}h_{\alpha\nu} - \Box h_{\sigma\nu} - \partial_{\nu}\partial_{\sigma}h + \partial_{\nu}\partial^{\alpha}h_{\alpha\sigma}\right). \tag{16}$$

Neglecting second order terms in the perturbation field, the scalar curvature is given by

$$\mathcal{R} = \partial_{\mu}\partial_{\sigma}h^{\mu\sigma} - \Box h \tag{17}$$

Now, we can compute the Einstein's field equations without a cosmological constant

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = \frac{8\pi G}{c^4}\mathcal{T}_{\mu\nu} \tag{18}$$

where $\mathcal{T}_{\mu\nu}$ is the energy momentum tensor. Replacing Eq.(14) and Eq.(17) in to Eq.(18) leads to

$$\partial^\alpha \partial_\mu h_{\alpha\nu} - \Box h_{\mu\nu} - \partial_\nu \partial_\mu h + \partial_\nu \partial^\alpha h_{\alpha\mu} - \left(\partial_\alpha \partial_\beta h^{\alpha\beta} - \Box h\right)(\eta_{\mu\nu} + h_{\mu\nu}) = \frac{16\pi G}{c^4}\mathcal{T}_{\mu\nu}. \tag{19}$$

Just like before, neglecting second order terms in the perturbation, the above equation can be reduced to

$$\partial^\alpha \partial_\mu h_{\alpha\nu} - \Box h_{\mu\nu} - \partial_\nu \partial_\mu h + \partial_\nu \partial^\alpha h_{\alpha\mu} - \eta_{\mu\nu}\partial_\alpha \partial_\beta h^{\alpha\beta} - \eta_{\mu\nu}\Box h = \frac{16\pi G}{c^4}\mathcal{T}_{\mu\nu}. \tag{20}$$

The above equation, can be written in a much more compact manner by using the following variable change[2]

$$X_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h, \tag{21}$$

which can be inverted through standard methods

$$h_{\mu\nu} = X_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} X \tag{22}$$

where X is the trace of the tensor $X_{\mu\nu}$, and the tensor $X_{\mu\nu}$, also satisfies the relation $X = -h$. Replacing the variable change written in Eq.(22) in Eq.(20) and simplifying terms, leads to

$$\partial^\alpha \partial_\mu X_{\alpha\nu} + \partial_\nu \partial^\alpha X_{\alpha\mu} - \Box X_{\mu\nu} - \eta_{\mu\nu}\partial_\alpha \partial_\beta X^{\alpha\beta} = \frac{16\Pi G}{c^4}\mathcal{T}_{\mu\nu}, \tag{23}$$

which, in some sense has a more simple structure that Eq.(20), and also contains the wave operator. Nonetheless, this does not look like a gravitational wave equations. In the next subsection, I will show you, how can you derived the gravitational wave equation from the above expression using a gauge transformation.

1.3 Gauge transformation

Consider the infinitesimal gauge coordinate transformation

$$x^\mu \longrightarrow x'^\mu = x^\mu + \xi^\mu, \tag{24}$$

where ξ^μ is a small vector. Then, it is possible to obtain the relation between and the inverse relation of the coordinate transformation of their respective derivatives

$$\frac{\partial x'^\alpha}{\partial x^\beta} = \delta^\alpha_\beta + \partial_\beta \xi^\alpha \qquad \frac{\partial x^\alpha}{\partial x'^\beta} = \delta^\alpha_\beta - \partial_\beta \xi^\alpha. \tag{25}$$

[2]In the standard literature $X_{\mu\nu}$ is written as $\bar{h}_{\mu\nu}$, but I strongly believed this leads to confusions.

Using the above information, the metric tensor under a gauge coordinate transformation changes as

$$g'_{\alpha\beta} = \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} g_{\mu\nu}. \tag{26}$$

Using Eq.(25) in the above equation, and neglecting second order terms in the perturbation, leads to

$$g'_{\alpha\beta} = g_{\alpha\beta} - g_{\alpha\nu}\partial_{\beta}\xi^{\nu} - g_{\mu\beta}\partial_{\alpha}\xi^{\mu} \tag{27}$$

replacing the expressions for the metric tensor, see Eq.(2), leads to

$$h'_{\alpha\beta} = h_{\alpha\beta} - \partial_{\beta}\xi_{\alpha} - \partial_{\alpha}\xi_{\beta}. \tag{28}$$

Is worth mention that the Riemann tensor is invariant under a gauge transformation. Therefore, we have the freedom to choose or fix the vector ξ^{μ} as we liked. Additionally, Eq.(28) is only valid using a Minkowski background, if we were working on a curved spacetime background, there will be additional terms. As the rule of gauge transformation is written in Eq.(28), then is trivial to compute the gauge transformation of the auxiliary variable $X_{\mu\nu}$

$$X_{\mu\nu} \to X'_{\mu\nu} = h'_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}h'_{\alpha\beta}, \tag{29}$$

replacing the transformation rule leads to

$$X'_{\mu\nu} = h_{\mu\nu} - \partial_{\nu}\xi_{\mu} - \partial_{\mu}\xi_{\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\left(h_{\alpha\beta} - \partial_{\beta}\xi_{\alpha} - \partial_{\alpha}\xi_{\beta}\right). \tag{30}$$

THe above expression can be simplified

$$X'_{\mu\nu} = X_{\mu\nu} - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu} + \eta_{\mu\nu}\partial^{\sigma}\xi_{\sigma}, \tag{31}$$

where I used the definition of the $X_{\mu\nu}$ tensor written in Eq.(22). Now, it is convenient to work with upper index, thus, using the inverse Minkowski metric we can upper every index

$$X'^{\alpha\beta} = X^{\alpha\beta} - \eta^{\mu\alpha}\partial_{\mu}\xi^{\beta} - \eta^{\nu\beta}\partial_{\nu}\xi^{\alpha} + \eta^{\alpha\beta}\partial^{\sigma}\xi_{\sigma}. \tag{32}$$

At this point we take the divergence of $X_{\mu\nu}$

$$\partial_{\beta}X'^{\alpha\beta} = \partial_{\beta}X^{\alpha\beta} - \eta^{\mu\alpha}\partial_{\beta}\partial_{\mu}\xi^{\beta} - \eta^{\nu\beta}\partial_{\beta}\partial_{\nu}\xi^{\alpha} + \eta^{\alpha\beta}\partial_{\beta}\partial^{\sigma}\xi_{\sigma}, \tag{33}$$

after simplification of terms, leads to

$$\partial_{\beta}X'^{\alpha\beta} = \partial_{\beta}X^{\alpha\beta} - \Box\xi^{\alpha}. \tag{34}$$

At this point, recall that we still have the freedom to choose the ξ^{μ}. Therefore, fixing

$$\Box\xi^{\alpha} = \partial_{\beta}X^{\alpha\beta}, \tag{35}$$

vanishes the vast majorities of terms of Eq.(23). The only, non-trivial term comes from the wave operator, leading to

$$\Box X_{\mu\nu} = -\frac{16\Pi G}{c^4}\mathcal{T}_{\mu\nu}, \tag{36}$$

which for the special vacuum case $\mathcal{T}_{\mu\nu} = 0$, reduces the above equation to

$$\Box X_{\mu\nu} = 0. \tag{37}$$

The above result, is known as the gravitational wave equation. In the next subsection we will be dealing with the problem of counting the degrees of freedom of a gravitational wave.