

# Personal notes on gravitational waves

Jose Perdiguero Garate

April 7, 2025

# Contents

<b>1</b>	<b>Linear general relativity</b>	<b>4</b>
1.1	General overview . . . . .	4
1.2	Geometrical objects . . . . .	4
1.3	Gauge transformation . . . . .	6
1.4	Degrees of freedom . . . . .	8

## Notation and conventions

I am working with the following conventions, for the metric tensor

$$(-1, +1, +1, +1) \tag{1}$$

# 1 Linear general relativity

In this section I am presenting a general overview on how to build-up the Einstein's field equation, given a background metric tensor and a perturbation tensor.

## 1.1 General overview

Decomposed the metric tensor as the sum of a background metric, in this case the flat spacetime Minkowski metric  $\eta_{\mu\nu}$ , plus a perturbation  $h_{\mu\nu}$  as follows

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \quad (2)$$

Using the definition of the Kronecker delta object it is straightforward to obtain the inverse tensor of the perturbation

$$\delta^\mu_\nu = g^{\mu\sigma} g_{\sigma\nu}. \quad (3)$$

Replacing Eq.(2) in to Eq.(3), and neglecting second order terms in the perturbation field, leads to

$$h^{\mu\nu} = -\eta^{\mu\alpha} \eta^{\nu\beta} h_{\alpha\beta}, \quad (4)$$

notice that, in order to upper/lower the indices of the perturbation I am only using the background metric.<sup>1</sup> Schematically, second order terms are neglected

$$h_{\mu\nu} h_{\alpha\beta} \sim 0, \quad h_{\mu\nu} \partial_\gamma h_{\alpha\beta} \sim 0, \quad \partial_\delta h_{\mu\nu} \partial_\gamma h_{\alpha\beta} \sim 0. \quad (5)$$

This is the general overview of the fundamental field of general relativity, which at its core is the metric tensor. In the following subsection, I will be computing the Einstein's field equations for the metric tensor written in Eq.(2).

## 1.2 Geometrical objects

The first object that is required to compute the Einstein's field equations is the connection. Working on a torsion-free manifold, the Levi-Civita connection is written as

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\rho} (\partial_\alpha g_{\rho\beta} + \partial_\beta g_{\rho\alpha} - \partial_\rho g_{\alpha\beta}). \quad (6)$$

Replacing Eq.(2) in to Eq.(6) leads to

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} (\eta^{\mu\rho} - h^{\mu\rho}) (\partial_\alpha \eta_{\rho\beta} + \partial_\beta \eta_{\rho\alpha} - \partial_\rho \eta_{\alpha\beta} + \partial_\alpha h_{\rho\beta} + \partial_\beta h_{\rho\alpha} - \partial_\rho h_{\alpha\beta}). \quad (7)$$

Notice that, the only non-trivial contributions are the ones that are linear in the perturbation, additionally, the partial derivatives of the Minkowski's metric tensor vanishes, therefore, the connection coefficients are reduced to

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} \eta^{\mu\rho} (\partial_\alpha h_{\rho\beta} + \partial_\beta h_{\rho\alpha} - \partial_\rho h_{\alpha\beta}). \quad (8)$$

---

<sup>1</sup>Including the perturbation tensor leads to second order terms, which I am ignoring.

Next, compute the Riemann curvature tensor

$$\mathcal{R}^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\sigma\nu} - \partial_\nu \Gamma^\rho_{\sigma\mu} + \Gamma^\gamma_{\nu\sigma} \Gamma^\rho_{\mu\gamma} + \Gamma^\gamma_{\mu\sigma} \Gamma^\rho_{\nu\gamma}, \quad (9)$$

however, instead of computing directly from the above equation, it is convenient to notice the structure of the curvature tensor. The last two terms are quadratic in the Levi-Civita connection, and, since the connection is written with perturbation, then, square terms in the connection vanishes, reducing the Riemann curvature tensor to

$$\mathcal{R}^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\sigma\nu} - \partial_\nu \Gamma^\rho_{\sigma\mu}. \quad (10)$$

Replacing Eq.(8) in to Eq.(10) leads to

$$\mathcal{R}^\rho_{\sigma\mu\nu} = \frac{1}{2} \eta^{\rho\alpha} \partial_\mu (\partial_\sigma h_{\alpha\nu} + \partial_\nu h_{\sigma\alpha} - \partial_\alpha h_{\sigma\nu}) + \frac{1}{2} \eta^{\rho\alpha} \partial_\nu (\partial_\sigma h_{\alpha\mu} + \partial_\mu h_{\sigma\alpha} - \partial_\alpha h_{\sigma\mu}), \quad (11)$$

the above expression, can be simplified to

$$\mathcal{R}^\rho_{\sigma\mu\nu} = \frac{1}{2} \eta^{\rho\alpha} (\partial_\mu \partial_\sigma h_{\alpha\nu} - \partial_\mu \partial_\alpha h_{\sigma\nu} - \partial_\nu \partial_\sigma h_{\mu\alpha} + \partial_\nu \partial_\alpha h_{\mu\sigma}). \quad (12)$$

From the Riemann tensor, it is straightforward to compute the Ricci tensor, by contracting their respective indices

$$\mathcal{R}_{\sigma\nu} = \mathcal{R}^\mu_{\sigma\mu\nu}. \quad (13)$$

A direct computation shows the structure of the Ricci tensor

$$\mathcal{R}_{\sigma\nu} = \frac{1}{2} (\partial^\alpha \partial_\sigma h_{\alpha\nu} - \square h_{\sigma\nu} - \partial_\nu \partial_\sigma h + \partial_\nu \partial^\alpha h_{\alpha\sigma}), \quad (14)$$

where  $\square$  is the d'Alembert operator and  $h$  is the trace of the perturbation.

In the same spirit, the curvature scalar can be obtained directly through the contraction of the Ricci tensor

$$\mathcal{R} = g^{\mu\sigma} \mathcal{R}_{\mu\sigma}. \quad (15)$$

This computation is straightforward

$$\mathcal{R} = (\eta^{\mu\sigma} - h^{\mu\sigma}) \frac{1}{2} (\partial^\alpha \partial_\sigma h_{\alpha\nu} - \square h_{\sigma\nu} - \partial_\nu \partial_\sigma h + \partial_\nu \partial^\alpha h_{\alpha\sigma}). \quad (16)$$

Neglecting second order terms in the perturbation field, the scalar curvature is given by

$$\mathcal{R} = \partial_\mu \partial_\sigma h^{\mu\sigma} - \square h \quad (17)$$

Now, we can compute the Einstein's field equations without a cosmological constant

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} \mathcal{R} g_{\mu\nu} = \frac{8\pi G}{c^4} \mathcal{T}_{\mu\nu} \quad (18)$$

where  $\mathcal{T}_{\mu\nu}$  is the energy momentum tensor. Replacing Eq.(14) and Eq.(17) in to Eq.(18) leads to

$$\partial^\alpha \partial_\mu h_{\alpha\nu} - \square h_{\mu\nu} - \partial_\nu \partial_\mu h + \partial_\nu \partial^\alpha h_{\alpha\mu} - (\partial_\alpha \partial_\beta h^{\alpha\beta} - \square h) (\eta_{\mu\nu} + h_{\mu\nu}) = \frac{16\pi G}{c^4} \mathcal{T}_{\mu\nu}. \quad (19)$$

Just like before, neglecting second order terms in the perturbation, the above equation can be reduced to

$$\partial^\alpha \partial_\mu h_{\alpha\nu} - \square h_{\mu\nu} - \partial_\nu \partial_\mu h + \partial_\nu \partial^\alpha h_{\alpha\mu} - \eta_{\mu\nu} \partial_\alpha \partial_\beta h^{\alpha\beta} - \eta_{\mu\nu} \square h = \frac{16\pi G}{c^4} \mathcal{T}_{\mu\nu}. \quad (20)$$

The above equation, can be written in a much more compact manner by using the following variable change<sup>2</sup>

$$X_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h, \quad (21)$$

which can be inverted through standard methods

$$h_{\mu\nu} = X_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} X \quad (22)$$

where  $X$  is the trace of the tensor  $X_{\mu\nu}$ , and the tensor  $X_{\mu\nu}$ , also satisfies the relation  $X = -h$ . Replacing the variable change written in Eq.(22) in Eq.(20) and simplifying terms, leads to

$$\partial^\alpha \partial_\mu X_{\alpha\nu} + \partial_\nu \partial^\alpha X_{\alpha\mu} - \square X_{\mu\nu} - \eta_{\mu\nu} \partial_\alpha \partial_\beta X^{\alpha\beta} = \frac{16\pi G}{c^4} \mathcal{T}_{\mu\nu}, \quad (23)$$

which, in some sense has a more simple structure than Eq.(20), and also contains the wave operator. Nonetheless, this does not look like a gravitational wave equation. In the next subsection, I will show you, how can you derived the gravitational wave equation from the above expression using a gauge transformation.

### 1.3 Gauge transformation

Consider the infinitesimal gauge coordinate transformation

$$x^\mu \longrightarrow x'^\mu = x^\mu + \xi^\mu, \quad (24)$$

where  $\xi^\mu$  is a small vector. Then, it is possible to obtain the relation between and the inverse relation of the coordinate transformation of their respective derivatives

$$\frac{\partial x'^\alpha}{\partial x^\beta} = \delta_\beta^\alpha + \partial_\beta \xi^\alpha \qquad \frac{\partial x^\alpha}{\partial x'^\beta} = \delta_\beta^\alpha - \partial_\beta \xi^\alpha. \quad (25)$$

---

<sup>2</sup>In the standard literature  $X_{\mu\nu}$  is written as  $\bar{h}_{\mu\nu}$ , but I strongly believed this leads to confusions.

Using the above information, the metric tensor under a gauge coordinate transformation changes as

$$g'_{\alpha\beta} = \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} g_{\mu\nu}. \quad (26)$$

Using Eq.(25) in the above equation, and neglecting second order terms in the perturbation, leads to

$$g'_{\alpha\beta} = g_{\alpha\beta} - g_{\alpha\nu} \partial_\beta \xi^\nu - g_{\mu\beta} \partial_\alpha \xi^\mu \quad (27)$$

replacing the expressions for the metric tensor, see Eq.(2), leads to

$$h'_{\alpha\beta} = h_{\alpha\beta} - \partial_\beta \xi_\alpha - \partial_\alpha \xi_\beta. \quad (28)$$

Is worth mention that the Riemann tensor is invariant under a gauge transformation. Therefore, we have the freedom to choose or fix the vector  $\xi^\mu$  as we liked. Additionally, Eq.(28) is only valid using a Minkowski background, if we were working on a curved spacetime background, there will be additional terms. As the rule of gauge transformation is written in Eq.(28), then is trivial to compute the gauge transformation of the auxiliary variable  $X_{\mu\nu}$

$$X_{\mu\nu} \rightarrow X'_{\mu\nu} = h'_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \eta^{\alpha\beta} h'_{\alpha\beta}, \quad (29)$$

replacing the transformation rule leads to

$$X'_{\mu\nu} = h_{\mu\nu} - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu - \frac{1}{2} \eta_{\mu\nu} \eta^{\alpha\beta} (h_{\alpha\beta} - \partial_\beta \xi_\alpha - \partial_\alpha \xi_\beta). \quad (30)$$

The above expression can be simplified

$$X'_{\mu\nu} = X_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu} \partial^\sigma \xi_\sigma, \quad (31)$$

where I used the definition of the  $X_{\mu\nu}$  tensor written in Eq.(22). Now, it is convenient to work with upper index, thus, using the inverse Minkowski metric we can upper every index

$$X'^{\alpha\beta} = X^{\alpha\beta} - \eta^{\mu\alpha} \partial_\mu \xi^\beta - \eta^{\nu\beta} \partial_\nu \xi^\alpha + \eta^{\alpha\beta} \partial^\sigma \xi_\sigma. \quad (32)$$

At this point we take the divergence of  $X_{\mu\nu}$

$$\partial_\beta X'^{\alpha\beta} = \partial_\beta X^{\alpha\beta} - \eta^{\mu\alpha} \partial_\beta \partial_\mu \xi^\beta - \eta^{\nu\beta} \partial_\beta \partial_\nu \xi^\alpha + \eta^{\alpha\beta} \partial_\beta \partial^\sigma \xi_\sigma, \quad (33)$$

after simplification of terms, leads to

$$\partial_\beta X'^{\alpha\beta} = \partial_\beta X^{\alpha\beta} - \square \xi^\alpha. \quad (34)$$

At this point, recall that we still have the freedom to choose the  $\xi^\mu$ . Therefore, fixing

$$\square \xi^\alpha = \partial_\beta X^{\alpha\beta} \rightarrow \partial_\beta X'^{\alpha\beta} = 0, \quad (35)$$

this is known as the Lorentz gauge, which vanishes the vast majorities of terms of Eq.(23). The only, non-trivial term comes from the wave operator, leading to

$$\square X_{\mu\nu} = -\frac{16\Pi G}{c^4}\mathcal{T}_{\mu\nu}, \quad (36)$$

which for the special vacuum case  $\mathcal{T}_{\mu\nu} = 0$ , reduces the above equation to

$$\square X_{\mu\nu} = 0. \quad (37)$$

The above result, is known as the gravitational wave equation. In the next subsection we will be dealing with the problem of counting the degrees of freedom of a gravitational wave.

## 1.4 Degrees of freedom

The perturbation has 16 independent components which must be determined. However, because it is defined as a symmetric tensor it only has 10 independent components. Moreover, due to the gauge condition Eq.(35) there are four additional equations/constraint that must be satisfied, leaving only 6 independent components. Nonetheless, this choice, does not completely fixes the gauge, we still have the freedom to choose to fix the four components of the displacement vector  $\xi^\mu$ , and hence, reduced even further the number of independent components of the perturbation to 2 independent components. This is known as the residual gauge.

A