Personal notes on gravitational waves

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April 21, 2025

Contents

Notation and conventions

Throughout this notes I will be using the metric signature

$$(-1, +1, +1, +1)\,. \tag{1}$$

Greek indices run over the four dimensional manifold spacetime, from 0 to 3, whereas latin indices run over only the spatial dimensions, from 1 to 3. Additionally, I am using the Einstein summation convention, where, repeated indices, indicate a summation.

The Minkowski metric tensor is written as $\eta_{\mu\nu}$ and an arbitrary metric tensor is $g_{\mu\nu}$. Partial derivatives are denoted by ∂ which can be acting on the four dimensions or three dimensions, and can be written with a commas $g_{\mu\nu,\alpha}$. Covariant derivatives are defined with the symmetric standard connection ∇, which is compatible with the metric tensor such that $\nabla_\alpha g_{\beta\gamma} = g_{\beta\gamma;\alpha} = 0$ where I used semicolons for its denotation.

1 Linear general relativity

In this section I am presenting a general overview on how to build-up the Einstein's field equation, given a background metric tensor and a perturbation tensor.

1.1 General overview

Decomposed the metric tensor as the sum of a background metric, in this case the flat spacetime Minkowski metric $\eta_{\mu\nu}$,plus a perturbation $h_{\mu\nu}$ as follows

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{2}$$

Using the definition of the Kronecker delta object it is straightforward to obtain the inverse tensor of the perturbation

$$\delta^{\mu}_{\nu} = g^{\mu\sigma} g_{\sigma\nu}. \tag{3}$$

Replacing Eq.(2) in to Eq.(3), and neglecting second order terms in the perturbation field, leads to

$$h^{\mu\nu} = -\eta^{\mu\alpha}\eta^{\nu\beta} h_{\alpha\beta}, \tag{4}$$

notice that, in order to upper/lower the indices of the perturbation I am only using the background metric.[1] Schematically, second order terms are neglected

$$h_{\mu\nu}h_{\alpha\beta} \sim 0, \qquad h_{\mu\nu}\partial_\gamma h_{\alpha\beta} \sim 0, \qquad \partial_\delta h_{\mu\nu}\partial_\gamma h_{\alpha\beta} \sim 0. \tag{5}$$

This is the general overview of the fundamental field of general relativity, which at its core is the metric tensor. In the following subsection, I will be computing the Einstein's field equations for the metric tensor written in Eq.(2).

1.2 Geometrical objects

The first object that is required to compute the Einstein's field equations is the connection. Working on a torsion-free manifold, the Levi-Civita connection is written as

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2} g^{\mu\rho} \left(\partial_\alpha g_{\rho\beta} + \partial_\beta g_{\rho\alpha} - \partial_\rho g_{\alpha\beta}.\right) \tag{6}$$

Replacing Eq.(2) in to Eq.(6) leads to

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2} \left(\eta^{\mu\rho} - h^{\mu\rho}\right) \left(\partial_\alpha \eta_{\rho\beta} + \partial_\beta \eta_{\rho\alpha} - \partial_\rho \eta_{\alpha\beta} + \partial_\alpha h_{\rho\beta} + \partial_\beta h_{\rho\alpha} - \partial_\rho h_{\alpha\beta}\right). \tag{7}$$

Notice that, the only non-trivial contributions are the ones that are linear in the perturbation, additionally, the partial derivatives of the Minkowski's metric tensor vanishes, therefore, the connection coefficients are reduced to

$$\Gamma^{\mu}{}_{\alpha\beta} = \frac{1}{2} \eta^{\mu\rho} \left(\partial_\alpha h_{\rho\beta} + \partial_\beta h_{\rho\alpha} - \partial_\rho h_{\alpha\beta}\right). \tag{8}$$

[1] Including the perturbation tensor leads to second order terms, which I am ignoring.

Next, compute the Riemann curvature tensor

$$\mathcal{R}^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}{}_{\sigma\nu} - \partial_\nu \Gamma^{\rho}{}_{\sigma\mu} + \Gamma^{\gamma}{}_{\nu\sigma}\Gamma^{\rho}{}_{\mu\gamma} + \Gamma^{\gamma}{}_{\mu\sigma}\Gamma^{\rho}{}_{\nu\gamma}, \tag{9}$$

however, instead of computing directly from the above equation, it is convenient to notice the structure of the curvature tensor. The last two terms are quadratic in the Levi-Civita connection, and, since the connection is written with perturbation, then, square terms in the connection vanishes, reducing the Riemann curvature tensor to

$$\mathcal{R}^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}{}_{\sigma\nu} - \partial_\nu \Gamma^{\rho}{}_{\sigma\mu}. \tag{10}$$

Replacing Eq.(8) in to Eq.(10) leads to

$$\mathcal{R}^{\rho}{}_{\sigma\mu\nu} = \frac{1}{2}\eta^{\rho\alpha}\partial_\mu \left(\partial_\sigma h_{\alpha\nu} + \partial_\nu h_{\sigma\alpha} - \partial_\alpha h_{\sigma\nu}\right) + \frac{1}{2}\eta^{\rho\alpha}\partial_\nu \left(\partial_\sigma h_{\alpha\mu} + \partial_\mu h_{\sigma\alpha} - \partial_\alpha h_{\sigma\mu}\right), \tag{11}$$

the above expression, can be simplified to

$$\mathcal{R}^{\rho}{}_{\sigma\mu\nu} = \frac{1}{2}\eta^{\rho\alpha}\left(\partial_\mu\partial_\sigma h_{\alpha\nu} - \partial_\mu\partial_\alpha h_{\sigma\nu} - \partial_\nu\partial_\sigma h_{\mu\alpha} + \partial_\nu\partial_\alpha h_{\mu\sigma}\right). \tag{12}$$

From the Riemann tensor, it is straightforward to compute the Ricci tensor, by contracting their respective indices

$$\mathcal{R}_{\sigma\nu} = \mathcal{R}^{\mu}{}_{\sigma\mu\nu}. \tag{13}$$

A direct computation shows the structure of the Ricci tensor

$$\mathcal{R}_{\sigma\nu} = \frac{1}{2}\left(\partial^\alpha\partial_\sigma h_{\alpha\nu} - \Box h_{\sigma\nu} - \partial_\nu\partial_\sigma h + \partial_\nu\partial^\alpha h_{\alpha\sigma}\right), \tag{14}$$

where $\Box$ is the d'Alembert operator and h is the trace of the perturbation.

In the same spirit, the curvature scalar can be obtained directly through the contraction of the Ricci tensor

$$\mathcal{R} = g^{\mu\sigma}\mathcal{R}_{\mu\sigma}. \tag{15}$$

This computation is straightforward

$$\mathcal{R} = \left(\eta^{\mu\sigma} - h^{\mu\sigma}\right)\frac{1}{2}\left(\partial^\alpha\partial_\sigma h_{\alpha\nu} - \Box h_{\sigma\nu} - \partial_\nu\partial_\sigma h + \partial_\nu\partial^\alpha h_{\alpha\sigma}\right). \tag{16}$$

Neglecting second order terms in the perturbation field, the scalar curvature is given by

$$\mathcal{R} = \partial_\mu\partial_\sigma h^{\mu\sigma} - \Box h \tag{17}$$

Now, we can compute the Einstein's field equations without a cosmological constant

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = \frac{8\pi G}{c^4}\mathcal{T}_{\mu\nu} \tag{18}$$

where $\mathcal{T}_{\mu\nu}$ is the energy momentum tensor. Replacing Eq.(14) and Eq.(17) in to Eq.(18) leads to

$$\partial^{\alpha}\partial_{\mu}h_{\alpha\nu}-\Box h_{\mu\nu}-\partial_{\nu}\partial_{\mu}h+\partial_{\nu}\partial^{\alpha}h_{\alpha\mu}-\left(\partial_{\alpha}\partial_{\beta}h^{\alpha\beta}-\Box h\right)\left(\eta_{\mu\nu}+h_{\mu\nu}\right)=\frac{16\pi G}{c^{4}}\mathcal{T}_{\mu\nu}. \tag{19}$$

Just like before, neglecting second order terms in the perturbation, the above equation can be reduced to

$$\partial^{\alpha}\partial_{\mu}h_{\alpha\nu}-\Box h_{\mu\nu}-\partial_{\nu}\partial_{\mu}h+\partial_{\nu}\partial^{\alpha}h_{\alpha\mu}-\eta_{\mu\nu}\partial_{\alpha}\partial_{\beta}h^{\alpha\beta}-\eta_{\mu\nu}\Box h=\frac{16\pi G}{c^{4}}\mathcal{T}_{\mu\nu}. \tag{20}$$

The above equation, can be written in a much more compact manner by using the following variable change[2]

$$X_{\mu\nu}=h_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}h, \tag{21}$$

which can be inverted through standard methods

$$h_{\mu\nu}=X_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}X \tag{22}$$

where X is the trace of the tensor $X_{\mu\nu}$, and the tensor $X_{\mu\nu}$, also satisfies the relation $X=-h$. Replacing the variable change written in Eq.(22) in Eq.(20) and simplifying terms, leads to

$$\partial^{\alpha}\partial_{\mu}X_{\alpha\nu}+\partial_{\nu}\partial^{\alpha}X_{\alpha\mu}-\Box X_{\mu\nu}-\eta_{\mu\nu}\partial_{\alpha}\partial_{\beta}X^{\alpha\beta}=\frac{16\Pi G}{c^{4}}\mathcal{T}_{\mu\nu}, \tag{23}$$

which, in some sense has a more simple structure that Eq.(20), and also contains the wave operator. Nonetheless, this does not look like a gravitational wave equations. In the next subsection, I will show you, how can you derived the gravitational wave equation from the above expression using a gauge transformation.

1.3 Gauge transformation

Consider the infinitesimal gauge coordinate transformation

$$x^{\mu}\longrightarrow x'^{\mu}=x^{\mu}+\xi^{\mu}, \tag{24}$$

where ξ^{μ} is a small vector. Then, it is possible to obtain the relation between and the inverse relation of the coordinate transformation of their respective derivatives

$$\frac{\partial x'^{\alpha}}{\partial x^{\beta}}=\delta^{\alpha}_{\beta}+\partial_{\beta}\xi^{\alpha} \qquad\qquad \frac{\partial x^{\alpha}}{\partial x'^{\beta}}=\delta^{\alpha}_{\beta}-\partial_{\beta}\xi^{\alpha}. \tag{25}$$

[2] In the standard literature $X_{\mu\nu}$ is written as $\bar{h}_{\mu\nu}$, but I strongly believed this leads to confusions.

Using the above information, the metric tensor under a gauge coordinate transformation changes as

$$g'_{\alpha\beta} = \frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta} g_{\mu\nu}. \tag{26}$$

Using Eq.(25) in the above equation, and neglecting second order terms in the perturbation, leads to

$$g'_{\alpha\beta} = g_{\alpha\beta} - g_{\alpha\nu}\partial_\beta \xi^\nu - g_{\mu\beta}\partial_\alpha \xi^\mu \tag{27}$$

replacing the expressions for the metric tensor, see Eq.(2), leads to

$$h'_{\alpha\beta} = h_{\alpha\beta} - \partial_\beta \xi_\alpha - \partial_\alpha \xi_\beta. \tag{28}$$

Is worth mention that the Riemann tensor is invariant under a gauge transformation. Therefore, we have the freedom to choose or fix the vector ξ^μ as we liked. Additionally, Eq.(28) is only valid using a Minkowski background, if we were working on a curved spacetime background, there will be additional terms. As the rule of gauge transformation is written in Eq.(28), then is trivial to compute the gauge transformation of the auxiliary variable $X_{\mu\nu}$

$$X_{\mu\nu} \to X'_{\mu\nu} = h'_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}h'_{\alpha\beta}, \tag{29}$$

replacing the transformation rule leads to

$$X'_{\mu\nu} = h_{\mu\nu} - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\left(h_{\alpha\beta} - \partial_\beta \xi_\alpha - \partial_\alpha \xi_\beta\right). \tag{30}$$

The above expression can be simplified

$$X'_{\mu\nu} = X_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu}\partial^\sigma \xi_\sigma, \tag{31}$$

where I used the definition of the $X_{\mu\nu}$ tensor written in Eq.(22). Now, it is convenient to work with upper index, thus, using the inverse Minkowski metric we can upper every index

$$X'^{\alpha\beta} = X^{\alpha\beta} - \eta^{\mu\alpha}\partial_\mu \xi^\beta - \eta^{\nu\beta}\partial_\nu \xi^\alpha + \eta^{\alpha\beta}\partial^\sigma \xi_\sigma. \tag{32}$$

At this point we take the divergence of $X_{\mu\nu}$

$$\partial_\beta X'^{\alpha\beta} = \partial_\beta X^{\alpha\beta} - \eta^{\mu\alpha}\partial_\beta\partial_\mu \xi^\beta - \eta^{\nu\beta}\partial_\beta\partial_\nu \xi^\alpha + \eta^{\alpha\beta}\partial_\beta\partial^\sigma \xi_\sigma, \tag{33}$$

after simplification of terms, leads to

$$\partial_\beta X'^{\alpha\beta} = \partial_\beta X^{\alpha\beta} - \Box \xi^\alpha. \tag{34}$$

At this point, recall that we still have the freedom to choose the ξ^μ. Therefore, fixing

$$\Box \xi^\alpha = \partial_\beta X^{\alpha\beta} \to \partial_\beta X'^{\alpha\beta} = 0, \tag{35}$$

this is known as the Lorentz gauge, which vanishes the vast majorities of terms of Eq.(23). The only, non-trivial term comes from the wave operator, leading to

$$\Box X_{\mu\nu} = -\frac{16\Pi G}{c^4}\mathcal{T}_{\mu\nu}, \tag{36}$$

which for the special vacuum case $\mathcal{T}_{\mu\nu} = 0$, reduces the above equation to

$$\Box X_{\mu\nu} = 0. \tag{37}$$

The above result, is known as the gravitational wave equation. In the next subsection we will be dealing with the problem of counting the degrees of freedom of a gravitational wave.

1.4 Degrees of freedom

The perturbation has 16 independent components which must be determined. However, because it is defined as a symmetric tensor it only has 10 independent components. Moreover, due to the gauge condition Eq.(35) there are four additional equations/constraint that must be satisfied, leaving only 6 independent components. Nonetheless, this choice, does not completely fixes the gauge, we still have the freedom to choose to fix the four components of the displacement vector ξ^μ, and hence, reduced even further the number of independent components of the perturbation to 2 independent components. This is known as the residual gauge.

An intuitive way of seen this reduction of the independent components of the perturbation tensor comes from knowing that initially there are 10 independent components of a symmetric tensor of rank 2 tensor. Then by using the Lorentz gauge restriction

$$\partial_\beta X'^{\alpha\beta} = 0, \tag{38}$$

which lead to four different equations, meaning that, there are four more constraint to impose, reducing the number of 10 independent components to 6. Then, we can perform another gauge transformation by an infinitesimal vector displacement ξ^μ

$$X'_{\mu\nu} \to X'_{\mu\nu} + \xi_{\mu\nu}, \tag{39}$$

where $\xi_{\mu\nu}$ is defined as

$$\xi_{\mu\nu} \equiv \eta_{\mu\nu}\partial^\alpha\xi_\alpha - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu. \tag{40}$$

Therefore, choosing properly the vector ξ_μ such that $\Box\xi_{\mu\nu} = 0$, it is possible to reduced 4 more degrees of freedom of the perturbation tensor. Leaving only 2 independent components, which are known as the 2 polarization, $+$ and x.

Exploiting the residual gauge symmetry, it is possible to eliminate directly components of the perturbation tensor. Using Eq.(31) it is possible to vanishes the trace and the spatial-temporal components of the perturbation $X = 0$ and

$X_{0i} = 0$. As a consequence of this, the perturbation and the variable change are the same $X_{\mu\nu} = h_{\mu\nu}$. From the Lorentz gauge condition Eq.(35)

$$\dot{h}_{00} + \partial_i h_{i0} = 0, \tag{41}$$

from which, we infer that the temporal-temporal components is a function of only the spatial coordinates. As this component does not depend on the time coordinate, (which is our concern, because gravitational waves are time-dependent), we can set $h_{00} = 0$. In a nutshell, the constraint are

$$h_{\mu 0} = 0, \qquad h = 0, \qquad \partial_i h_{ij} = 0. \tag{42}$$

This is known as the *transverse-traceless* gauge (TT).

1.5 Scalar-Vector-Tensor decomposition

Although we have found so far the gravitational wave equation coupled with a energy-momentum tensor, in principle we are done. Now, the next task should be to compute the Einstein's field equation for a given symmetry of the metric tensor. However, we can take advantage of the *scalar-vector-tensor* decomposition to write down the independent components of Einstein's equations in a much more simple form.

First, lets decomposed the perturbation tensor as follows

$$h_{00} = -2\phi, \tag{43}$$

$$h_{0i} = \partial_i B + S_i, \tag{44}$$

$$h_{ji} = h_{ij} + \partial_i F_j + \partial_j F_i - 2\psi\delta_{ij}, + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right) E, \tag{45}$$

where ϕ, B, ψ and E are scalars functions, S_i and F_i are vectors whose divergence vanishes $\partial_i S_i = \partial_i F_i = 0$, and h_{ij} is a tensor whose divergence is trivial $\partial_i h_{ij} = 0$ and also by construction must be traceless $h = 0$. Notice that, in principle, the perturbation tensor has 10 independent components, and from the decomposition, we have 4 scalars, 6 components for the two vectors -2 constraints, and from the tensorial part there are only 6 independent components -4 constraints. Therefore, the decomposition leads to $4+(6-2)+(6-4) = 10$, which, of course, matches the number of independent components of the perturbation. Details on how to derived this decomposition can be found in Ref.[1] in section 5.5.

In the same way, the vector displacement ξ_μ can also be decomposed

$$\xi_\mu = \left(d_0, \partial_i d + d_i\right), \tag{46}$$

and using the gauge transformation rule written in Eq.(31), it is straightforward to derived how the components of the perturbation tensor transform under a coordinate gauge transformation. Take for example the h_{tt} component

$$h'_{tt} = h_{tt} - 2\partial_t \xi_t, \tag{47}$$

which, replacing the values leads to

$$2\phi' = 2\phi + 2\dot{d}_0, \tag{48}$$

then, the rule of transformation can be written as

$$\phi' = \phi + \dot{d}_0. \tag{49}$$

Applying the same idea to the other scalars, vectors and tensor components of the tensor perturbation, we find that

$$\phi' = \phi + \dot{d}_0, \qquad B' = B - d_0 - \dot{d}, \qquad E = E - 2d, \tag{50}$$

$$S'_i = S_i - \dot{d}_i, \qquad F_i = F_i - d_i. \tag{51}$$

Once the rules of gauge transformation are known for the STV decomposition, it is straightforward to build-up gauge invariants using the components of the perturbation, these are gauge invariants are

$$\Phi \equiv \phi + \dot{B} - \frac{1}{2}\ddot{E}, \quad \Theta \equiv -2\psi - \frac{1}{3}\nabla^2 E, \quad \Sigma_i \equiv S_i - \dot{F}_i, \quad h_{ij} \equiv h_{ij}, \tag{52}$$

there are two scalar gauge invariants, one vector invariant and one tensor invariant quantity.

A similar procedure can be applied to the stress energy momentum tensor

$$\mathcal{T}_{00} = \rho, \tag{53}$$

$$\mathcal{T}_{0i} = \partial_i u + u_i, \tag{54}$$

$$\mathcal{T}_{ij} = \Pi_{ij} + \partial_i v_j + \partial_j v_i + p\delta_{ij} + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right)\sigma, \tag{55}$$

where ρ, u, p and σ are scalar functions, u_i and v_i are vectors whose divergence vanishes $\partial_i u_i = 0$ and $\partial_i v_i = 0$, and Π_{ij} is a traceless tensor (by construction), therefore $\Pi = 0$, whose divergence vanishes $\partial_i \Pi_{ij} = 0$. The natural next step, would be to build-up the rules of gauge transformation of the components of the energy-momentum tensor, however because the energy-momentum tensor must be zero in the background, all the components (perturbed) are gauge invariants, this is known as the Steward Walker lemma [2].

From the conservation law,

$$\nabla_\mu \mathcal{T}^{\mu\nu} = \partial_\mu \mathcal{T}^{\mu\nu} + \Gamma^\mu{}_{\mu\lambda}\mathcal{T}^{\lambda\nu} + \Gamma^\nu{}_{\mu\lambda}\mathcal{T}^{\mu\lambda} \sim \partial_\mu \mathcal{T}^{\mu\nu}. \tag{56}$$

Replacing Eq.(53) into the above equation leads to the following constraints

$$\nabla^2 u = \dot{\rho}, \qquad \nabla^2\sigma = \frac{3}{2}\left(\dot{u} - p\right), \qquad \nabla^2 v_i = \dot{u}_i, \tag{57}$$

The next step, is to put all this ingredients together into Einstein's field equation. A straightforward computation leads to the following field equations

$$\nabla^2\Theta = -\rho, \tag{58}$$

$$\nabla^2\Phi = \left(\rho + 3p - 3\dot{u}\right), \tag{59}$$

$$\nabla^2\Sigma_i = -2S_i, \tag{60}$$

$$\Box h_{ij} = -2\Pi_{ij}. \tag{61}$$

These are the set of differential equations, where there is only one wave equations coming from the h_{ij}. The other fields obey a Poisson-like equation. Notice that, both sides of the Eqs.(58) are gauge invariants, as they should be.

2 Gravitational waves in a curved spacetime

References

[1] E. Poisson and C. Will, *Gravity: Newtonian, Post-Newtonian, Relativistic*. Cambridge University Press, 2014.

[2] J. Stewart and J. Stewart, *Advanced General Relativity*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1993.