Personal notes on gravitational waves

Jose Perdiguero Garate

May 5, 2025

Contents

Notation and conventions

Throughout this notes I will be using the metric signature

$$(-1, +1, +1, +1)\,. \tag{1}$$

Greek indices run over the four dimensional manifold spacetime, from 0 to 3, whereas latin indices run over only the spatial dimensions, from 1 to 3. Additionally, I am using the Einstein summation convention, where, repeated indices, indicate a summation.

The Minkowski metric tensor is written as $\eta_{\mu\nu}$ and an arbitrary metric tensor is $g_{\mu\nu}$. Partial derivatives are denoted by ∂ which can be acting on the four dimensions or three dimensions, and can be written with a commas $g_{\mu\nu,\alpha}$. Covariant derivatives are defined with the symmetric standard connection ∇, which is compatible with the metric tensor such that $\nabla_\alpha g_{\beta\gamma} = g_{\beta\gamma;\alpha} = 0$ where I used semicolons for its denotation.

1 Linear general relativity

In this section I am presenting a general overview on how to build-up the Einstein's field equation, given a background metric tensor and a perturbation tensor.

1.1 General overview

Decomposed the metric tensor as the sum of a background metric, in this case the flat spacetime Minkowski metric $\eta_{\mu\nu}$,plus a perturbation $h_{\mu\nu}$ as follows

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{2}$$

Using the definition of the Kronecker delta object it is straightforward to obtain the inverse tensor of the perturbation

$$\delta^\mu_\nu = g^{\mu\sigma} g_{\sigma\nu}. \tag{3}$$

Replacing Eq.(2) in to Eq.(3), and neglecting second order terms in the perturbation field, leads to

$$h^{\mu\nu} = -\eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}, \tag{4}$$

notice that, in order to upper/lower the indices of the perturbation I am only using the background metric.[1] Schematically, second order terms are neglected

$$h_{\mu\nu}h_{\alpha\beta} \sim 0, \qquad h_{\mu\nu}\partial_\gamma h_{\alpha\beta} \sim 0, \qquad \partial_\delta h_{\mu\nu}\partial_\gamma h_{\alpha\beta} \sim 0. \tag{5}$$

This is the general overview of the fundamental field of general relativity, which at its core is the metric tensor. In the following subsection, I will be computing the Einstein's field equations for the metric tensor written in Eq.(2).

1.2 Geometrical objects

The first object that is required to compute the Einstein's field equations is the connection. Working on a torsion-free manifold, the Levi-Civita connection is written as

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2} g^{\mu\rho} \left(\partial_\alpha g_{\rho\beta} + \partial_\beta g_{\rho\alpha} - \partial_\rho g_{\alpha\beta}.\right) \tag{6}$$

Replacing Eq.(2) in to Eq.(6) leads to

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2} \left(\eta^{\mu\rho} - h^{\mu\rho}\right) \left(\partial_\alpha \eta_{\rho\beta} + \partial_\beta \eta_{\rho\alpha} - \partial_\rho \eta_{\alpha\beta} + \partial_\alpha h_{\rho\beta} + \partial_\beta h_{\rho\alpha} - \partial_\rho h_{\alpha\beta}\right). \tag{7}$$

Notice that, the only non-trivial contributions are the ones that are linear in the perturbation, additionally, the partial derivatives of the Minkowski's metric tensor vanishes, therefore, the connection coefficients are reduced to

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2} \eta^{\mu\rho} \left(\partial_\alpha h_{\rho\beta} + \partial_\beta h_{\rho\alpha} - \partial_\rho h_{\alpha\beta}\right). \tag{8}$$

[1]Including the perturbation tensor leads to second order terms, which I am ignoring.

Next, compute the Riemann curvature tensor

$$\mathcal{R}^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}{}_{\sigma\nu} - \partial_\nu \Gamma^{\rho}{}_{\sigma\mu} + \Gamma^{\gamma}{}_{\nu\sigma}\Gamma^{\rho}{}_{\mu\gamma} + \Gamma^{\gamma}{}_{\mu\sigma}\Gamma^{\rho}{}_{\nu\gamma}, \tag{9}$$

however, instead of computing directly from the above equation, it is convenient to notice the structure of the curvature tensor. The last two terms are quadratic in the Levi-Civita connection, and, since the connection is written with perturbation, then, square terms in the connection vanishes, reducing the Riemann curvature tensor to

$$\mathcal{R}^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}{}_{\sigma\nu} - \partial_\nu \Gamma^{\rho}{}_{\sigma\mu}. \tag{10}$$

Replacing Eq.(8) in to Eq.(10) leads to

$$\mathcal{R}^{\rho}{}_{\sigma\mu\nu} = \frac{1}{2}\eta^{\rho\alpha}\partial_\mu \left(\partial_\sigma h_{\alpha\nu} + \partial_\nu h_{\sigma\alpha} - \partial_\alpha h_{\sigma\nu}\right) + \frac{1}{2}\eta^{\rho\alpha}\partial_\nu \left(\partial_\sigma h_{\alpha\mu} + \partial_\mu h_{\sigma\alpha} - \partial_\alpha h_{\sigma\mu}\right), \tag{11}$$

the above expression, can be simplified to

$$\mathcal{R}^{\rho}{}_{\sigma\mu\nu} = \frac{1}{2}\eta^{\rho\alpha}\left(\partial_\mu\partial_\sigma h_{\alpha\nu} - \partial_\mu\partial_\alpha h_{\sigma\nu} - \partial_\nu\partial_\sigma h_{\mu\alpha} + \partial_\nu\partial_\alpha h_{\mu\sigma}\right). \tag{12}$$

From the Riemann tensor, it is straightforward to compute the Ricci tensor, by contracting their respective indices

$$\mathcal{R}_{\sigma\nu} = \mathcal{R}^{\mu}{}_{\sigma\mu\nu}. \tag{13}$$

A direct computation shows the structure of the Ricci tensor

$$\mathcal{R}_{\sigma\nu} = \frac{1}{2}\left(\partial^\alpha\partial_\sigma h_{\alpha\nu} - \Box h_{\sigma\nu} - \partial_\nu\partial_\sigma h + \partial_\nu\partial^\alpha h_{\alpha\sigma}\right), \tag{14}$$

where $\Box$ is the d'Alembert operator and h is the trace of the perturbation.

In the same spirit, the curvature scalar can be obtained directly through the contraction of the Ricci tensor

$$\mathcal{R} = g^{\mu\sigma}\mathcal{R}_{\mu\sigma}. \tag{15}$$

This computation is straightforward

$$\mathcal{R} = \left(\eta^{\mu\sigma} - h^{\mu\sigma}\right)\frac{1}{2}\left(\partial^\alpha\partial_\sigma h_{\alpha\nu} - \Box h_{\sigma\nu} - \partial_\nu\partial_\sigma h + \partial_\nu\partial^\alpha h_{\alpha\sigma}\right). \tag{16}$$

Neglecting second order terms in the perturbation field, the scalar curvature is given by

$$\mathcal{R} = \partial_\mu\partial_\sigma h^{\mu\sigma} - \Box h \tag{17}$$

Now, we can compute the Einstein's field equations without a cosmological constant

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = \frac{8\pi G}{c^4}\mathcal{T}_{\mu\nu} \tag{18}$$

where $\mathcal{T}_{\mu\nu}$ is the energy momentum tensor. Replacing Eq.(14) and Eq.(17) in to Eq.(18) leads to

$$\partial^\alpha\partial_\mu h_{\alpha\nu} - \Box h_{\mu\nu} - \partial_\nu\partial_\mu h + \partial_\nu\partial^\alpha h_{\alpha\mu} - \left(\partial_\alpha\partial_\beta h^{\alpha\beta} - \Box h\right)(\eta_{\mu\nu} + h_{\mu\nu}) = \frac{16\pi G}{c^4}\mathcal{T}_{\mu\nu}. \tag{19}$$

Just like before, neglecting second order terms in the perturbation, the above equation can be reduced to

$$\partial^\alpha\partial_\mu h_{\alpha\nu} - \Box h_{\mu\nu} - \partial_\nu\partial_\mu h + \partial_\nu\partial^\alpha h_{\alpha\mu} - \eta_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta} - \eta_{\mu\nu}\Box h = \frac{16\pi G}{c^4}\mathcal{T}_{\mu\nu}. \tag{20}$$

The above equation, can be written in a much more compact manner by using the following variable change[2]

$$X_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \tag{21}$$

which can be inverted through standard methods

$$h_{\mu\nu} = X_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}X \tag{22}$$

where X is the trace of the tensor $X_{\mu\nu}$, and the tensor $X_{\mu\nu}$, also satisfies the relation $X = -h$. Replacing the variable change written in Eq.(22) in Eq.(20) and simplifying terms, leads to

$$\partial^\alpha\partial_\mu X_{\alpha\nu} + \partial_\nu\partial^\alpha X_{\alpha\mu} - \Box X_{\mu\nu} - \eta_{\mu\nu}\partial_\alpha\partial_\beta X^{\alpha\beta} = \frac{16\Pi G}{c^4}\mathcal{T}_{\mu\nu}, \tag{23}$$

which, in some sense has a more simple structure that Eq.(20), and also contains the wave operator. Nonetheless, this does not look like a gravitational wave equations. In the next subsection, I will show you, how can you derived the gravitational wave equation from the above expression using a gauge transformation.

1.3 Gauge transformation

Consider the infinitesimal gauge coordinate transformation

$$x^\mu \longrightarrow x'^\mu = x^\mu + \xi^\mu, \tag{24}$$

where ξ^μ is a small vector. Then, it is possible to obtain the relation between and the inverse relation of the coordinate transformation of their respective derivatives

$$\frac{\partial x'^\alpha}{\partial x^\beta} = \delta^\alpha_\beta + \partial_\beta\xi^\alpha \qquad\qquad \frac{\partial x^\alpha}{\partial x'^\beta} = \delta^\alpha_\beta - \partial_\beta\xi^\alpha. \tag{25}$$

[2] In the standard literature $X_{\mu\nu}$ is written as $\bar{h}_{\mu\nu}$, but I strongly believed this leads to confusions.

Using the above information, the metric tensor under a gauge coordinate transformation changes as

$$g'_{\alpha\beta} = \frac{\partial x^{\mu}}{\partial x'^{\alpha}} \frac{\partial x^{\nu}}{\partial x'^{\beta}} g_{\mu\nu}. \tag{26}$$

Using Eq.(25) in the above equation, and neglecting second order terms in the perturbation, leads to

$$g'_{\alpha\beta} = g_{\alpha\beta} - g_{\alpha\nu}\partial_{\beta}\xi^{\nu} - g_{\mu\beta}\partial_{\alpha}\xi^{\mu} \tag{27}$$

replacing the expressions for the metric tensor, see Eq.(2), leads to

$$h'_{\alpha\beta} = h_{\alpha\beta} - \partial_{\beta}\xi_{\alpha} - \partial_{\alpha}\xi_{\beta}. \tag{28}$$

Is worth mention that the Riemann tensor is invariant under a gauge transformation. Therefore, we have the freedom to choose or fix the vector ξ^{μ} as we liked. Additionally, Eq.(28) is only valid using a Minkowski background, if we were working on a curved spacetime background, there will be additional terms, the most general expression can be found in at the end of this section. As the rule of gauge transformation is written in Eq.(28), then is trivial to compute the gauge transformation of the auxiliary variable $X_{\mu\nu}$

$$X_{\mu\nu} \rightarrow X'_{\mu\nu} = h'_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}h'_{\alpha\beta}, \tag{29}$$

replacing the transformation rule leads to

$$X'_{\mu\nu} = h_{\mu\nu} - \partial_{\nu}\xi_{\mu} - \partial_{\mu}\xi_{\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\left(h_{\alpha\beta} - \partial_{\beta}\xi_{\alpha} - \partial_{\alpha}\xi_{\beta}\right). \tag{30}$$

The above expression can be simplified

$$X'_{\mu\nu} = X_{\mu\nu} - \partial_{\mu}\xi_{\nu} - \partial_{\nu}\xi_{\mu} + \eta_{\mu\nu}\partial^{\sigma}\xi_{\sigma}, \tag{31}$$

where I used the definition of the $X_{\mu\nu}$ tensor written in Eq.(22). Now, it is convenient to work with upper index, thus, using the inverse Minkowski metric we can upper every index

$$X'^{\alpha\beta} = X^{\alpha\beta} - \eta^{\mu\alpha}\partial_{\mu}\xi^{\beta} - \eta^{\nu\beta}\partial_{\nu}\xi^{\alpha} + \eta^{\alpha\beta}\partial^{\sigma}\xi_{\sigma}. \tag{32}$$

At this point we take the divergence of $X_{\mu\nu}$

$$\partial_{\beta}X'^{\alpha\beta} = \partial_{\beta}X^{\alpha\beta} - \eta^{\mu\alpha}\partial_{\beta}\partial_{\mu}\xi^{\beta} - \eta^{\nu\beta}\partial_{\beta}\partial_{\nu}\xi^{\alpha} + \eta^{\alpha\beta}\partial_{\beta}\partial^{\sigma}\xi_{\sigma}, \tag{33}$$

after simplification of terms, leads to

$$\partial_{\beta}X'^{\alpha\beta} = \partial_{\beta}X^{\alpha\beta} - \Box\xi^{\alpha}. \tag{34}$$

At this point, recall that we still have the freedom to choose the ξ^{μ}. Therefore, fixing

$$\Box\xi^{\alpha} = \partial_{\beta}X^{\alpha\beta} \rightarrow \partial_{\beta}X'^{\alpha\beta} = 0, \tag{35}$$

this is known as the Lorentz gauge, which vanishes the vast majorities of terms of Eq.(23). The only, non-trivial term comes from the wave operator, leading to

$$\Box X_{\mu\nu} = -\frac{16\Pi G}{c^4}\mathcal{T}_{\mu\nu}, \tag{36}$$

which for the special vacuum case $\mathcal{T}_{\mu\nu} = 0$, reduces the above equation to

$$\Box X_{\mu\nu} = 0. \tag{37}$$

The above result, is known as the gravitational wave equation. In the next subsection we will be dealing with the problem of counting the degrees of freedom of a gravitational wave.

1.4 Degrees of freedom

The perturbation has 16 independent components which must be determined. However, because it is defined as a symmetric tensor it only has 10 independent components. Moreover, due to the gauge condition Eq.(35) there are four additional equations/constraint that must be satisfied, leaving only 6 independent components. Nonetheless, this choice, does not completely fixes the gauge, we still have the freedom to choose to fix the four components of the displacement vector ξ^μ, and hence, reduced even further the number of independent components of the perturbation to 2 independent components. This is known as the residual gauge.

An intuitive way of seen this reduction of the independent components of the perturbation tensor comes from knowing that initially there are 10 independent components of a symmetric tensor of rank 2 tensor. Then by using the Lorentz gauge restriction

$$\partial_\beta X'^{\alpha\beta} = 0, \tag{38}$$

which lead to four different equations, meaning that, there are four more constraint to impose, reducing the number of 10 independent components to 6. Then, we can perform another gauge transformation by an infinitesimal vector displacement ξ^μ

$$X'_{\mu\nu} \to X'_{\mu\nu} + \xi_{\mu\nu}, \tag{39}$$

where $\xi_{\mu\nu}$ is defined as

$$\xi_{\mu\nu} \equiv \eta_{\mu\nu}\partial^\alpha\xi_\alpha - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu. \tag{40}$$

Therefore, choosing properly the vector ξ_μ such that $\Box\xi_{\mu\nu} = 0$, it is possible to reduced 4 more degrees of freedom of the perturbation tensor. Leaving only 2 independent components, which are known as the 2 polarization, $+$ and x.

Exploiting the residual gauge symmetry, it is possible to eliminate directly components of the perturbation tensor. Using Eq.(31) it is possible to vanishes the trace and the spatial-temporal components of the perturbation $X = 0$ and

$X_{0i} = 0$. As a consequence of this, the perturbation and the variable change are the same $X_{\mu\nu} = h_{\mu\nu}$. From the Lorentz gauge condition Eq.(35)

$$\dot{h}_{00} + \partial_i h_{i0} = 0, \tag{41}$$

from which, we infer that the temporal-temporal components is a function of only the spatial coordinates. As this component does not depend on the time coordinate, (which is our concern, because gravitational waves are time-dependent), we can set $h_{00} = 0$. In a nutshell, the constraint are

$$h_{\mu 0} = 0, \qquad h = 0, \qquad \partial_i h_{ij} = 0. \tag{42}$$

This is known as the *transverse-traceless* gauge (TT).

1.5 Scalar-Vector-Tensor decomposition

Although we have found so far the gravitational wave equation coupled with a energy-momentum tensor, in principle we are done. Now, the next task should be to compute the Einstein's field equation for a given symmetry of the metric tensor. However, we can take advantage of the *scalar-vector-tensor* decomposition to write down the independent components of Einstein's equations in a much more simple form.

First, lets decomposed the perturbation tensor as follows

$$h_{00} = -2\phi, \tag{43}$$

$$h_{0i} = \partial_i B + S_i, \tag{44}$$

$$h_{ij} = h_{ij} + \partial_i F_j + \partial_j F_i - 2\psi\delta_{ij}, + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\nabla^2\right) E, \tag{45}$$

where ϕ, B, ψ and E are scalars functions, S_i and F_i are vectors whose divergence vanishes $\partial_i S_i = \partial_i F_i = 0$, and h_{ij} is a tensor whose divergence is trivial $\partial_i h_{ij} = 0$ and also by construction must be traceless $h = 0$. Notice that, in principle, the perturbation tensor has 10 independent components, and from the decomposition, we have 4 scalars, 6 components for the two vectors -2 constraints, and from the tensorial part there are only 6 independent components -4 constraints. Therefore, the decomposition leads to $4+(6-2)+(6-4) = 10$, which, of course, matches the number of independent components of the perturbation. Details on how to derived this decomposition can be found in Ref.[1] in section 5.5.

In the same way, the vector displacement ξ_μ can also be decomposed

$$\xi_\mu = (d_0, \partial_i d + d_i)\,, \tag{46}$$

and using the gauge transformation rule written in Eq.(28), it is straightforward to derived how the components of the perturbation tensor transform under a coordinate gauge transformation. Take for example the h_{tt} component

$$h'_{tt} = h_{tt} - 2\partial_t \xi_t, \tag{47}$$

which, replacing the values leads to

$$2\phi' = 2\phi + 2\dot{d}_0, \tag{48}$$

then, the rule of transformation can be written as

$$\phi' = \phi + \dot{d}_0. \tag{49}$$

The same idea should be applied to the different components of the SVT decomposition. The scalar components transform as follow

$$\phi' = \phi + \dot{d}_0, \qquad B' = B - d_0 - \dot{d} \tag{50}$$

$$\psi' = \psi + \frac{1}{3}\nabla^2 d \qquad E = E - 2d, \tag{51}$$

the vector parts transfrom as

$$S_i' = S_i - \dot{d}_i, \qquad F_i = F_i - d_i. \tag{52}$$

and finally the tensor part

$$h_{ij}' = h_{ij} \tag{53}$$

Once the rules of gauge transformation are known for the STV decomposition, it is straightforward to build-up gauge invariants using the components of the perturbation, these gauge invariants are

$$\Phi \equiv \phi + \dot{B} - \frac{1}{2}\ddot{E}, \quad \Theta \equiv -2\psi - \frac{1}{3}\nabla^2 E, \quad \Sigma_i \equiv S_i - \dot{F}_i, \quad h_{ij} \equiv h_{ij}, \tag{54}$$

there are two scalar gauge invariants, one vector invariant and one tensor invariant quantity. It is straightforward the proof that these are gauge invariants, lets consider the first gauge invariant object and perform the coordinate gauge transformation

$$\Phi \to \Phi' = \phi' + \dot{B}' - \frac{1}{2}\ddot{E}', \tag{55}$$

replacing the rules of transformation for each object leads to

$$\Phi' = \Phi + \dot{d}_0 + \dot{B} - \dot{d}_0 - \ddot{d} - \frac{1}{2}\left(\ddot{E} - 2\ddot{d}\right), \tag{56}$$

which, after simple algebra simplification turns to

$$\Phi' = \Phi + \dot{B} - \frac{1}{2}\ddot{E}. \tag{57}$$

The proof for the other three gauge invariants follows the same idea.

Lets compute the geometrical part of Einstein's field Eq. (20), to make this more easy to see, I will write down the equation

$$\partial^\alpha\partial_\mu h_{\alpha\nu} - \Box h_{\mu\nu} - \partial_\nu\partial_\mu h + \partial_\nu\partial^\alpha h_{\alpha\mu} - \eta_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta} - \eta_{\mu\nu}\Box h = \frac{16\pi G}{c^4}\mathcal{T}_{\mu\nu}. \tag{58}$$

Let's compute the tt components of the LHS of the above equation. A straightforward computation leads to

$$\left(2\ddot{\phi}+\nabla^2\dot{B}\right)+\left(-2\ddot{\phi}+2\nabla^2\phi\right)+\left(6\ddot{\psi}+2\ddot{\phi}\right)+\left(2\ddot{\phi}+\nabla^2\dot{B}\right) \tag{59}$$

$$+\left(-2\ddot{\phi}-2\nabla^2\dot{B}-2\nabla^2\psi-\frac{2}{3}\nabla^2\nabla^2E\right)+\left(-6\ddot{\psi}+6\nabla^2\psi-2\ddot{\phi}-2\nabla^2\phi\right), \tag{60}$$

which after simplification is reduced to

$$G_{00}=\frac{1}{3}\nabla^2\left(-2\psi-\nabla^2E\right)\rightarrow G_{00}=\nabla^2\Theta, \tag{61}$$

alternatively, it can be written in terms of gauge invariant as seen in the above right side. in a similar manner, it can be computed out all the other components of the Einstein tensor

$$G_{00}=-\nabla^2\theta, \tag{62}$$

$$G_{0i}=-\frac{1}{2}\nabla^2\Sigma_i-\partial_i\dot{\Theta}, \tag{63}$$

$$G_{ij}=-\frac{1}{2}\Box h_{ij}-\partial_{(i}\dot{\Sigma}_{j)}-\frac{1}{2}\partial_i\partial_j\left(2\Phi+\Theta\right)+\delta_{ij}\left(\frac{1}{2}\nabla^2\left(2\Phi+\Theta\right)-\ddot{\Theta}\right) \tag{64}$$

A similar procedure can be applied to the stress energy momentum tensor

$$\mathcal{T}_{00}=\rho, \tag{65}$$

$$\mathcal{T}_{0i}=\partial_iu+u_i, \tag{66}$$

$$\mathcal{T}_{ij}=\Pi_{ij}+\partial_iv_j+\partial_jv_i+p\delta_{ij}+\left(\partial_i\partial_j-\frac{1}{3}\delta_{ij}\nabla^2\right)\sigma, \tag{67}$$

where ρ, u, p and σ are scalar functions, u_i and v_i are vectors whose divergence vanishes $\partial_iu_i=0$ and $\partial_iv_i=0$, and Π_{ij} is a traceless tensor (by construction), therefore $\Pi=0$, whose divergence vanishes $\partial_i\Pi_{ij}=0$. The natural next step, would be to build-up the rules of gauge transformation of the components of the energy-momentum tensor, however because the energy-momentum tensor must be zero in the background, all the components (perturbed) are gauge invariants, this is known as the Steward Walker lemma [2].

From the conservation law,

$$\nabla_\mu\mathcal{T}^{\mu\nu}=\partial_\mu\mathcal{T}^{\mu\nu}+\Gamma^\mu{}_{\mu\lambda}\mathcal{T}^{\lambda\nu}+\Gamma^\nu{}_{\mu\lambda}\mathcal{T}^{\mu\lambda}\sim\partial_\mu\mathcal{T}^{\mu\nu}. \tag{68}$$

Notice that the energy-momentum tensor is defined as a perturbation and the background energy momentum tensor vanishes, therefore the second and third term can be neglected, leaving only

$$\partial^t\mathcal{T}_{t\nu}+\partial^i\mathcal{T}_{i\nu}=0 \tag{69}$$

. Working with the t component leads to

$$\partial^t \mathcal{T}_{tt} + \partial^i \mathcal{T}_{it} = 0, \tag{70}$$

Replacing Eq.(65) in the above equation leads to

$$-\dot{\rho} + \partial^i \left(\partial_i u + u_i\right) = 0, \tag{71}$$

however, the last term vanishes out because its divergence is zero. Therefore, the above equation is simplified to

$$\nabla^2 u = \dot{\rho}. \tag{72}$$

To obtain the other continuity equations from the conservation of the energy-momentum tensor follows a similar idea. Replacing Eq.(65) into the above equation leads to the following constraints

$$\nabla^2 u = \dot{\rho}, \qquad \nabla^2 \sigma = \frac{3}{2}\left(\dot{u} - p\right), \qquad \nabla^2 v_i = \dot{u}_i, \tag{73}$$

The next step, is to put all this ingredients together into Einstein's field equation. A straightforward computation leads to the following field equations

$$\nabla^2 \Theta = -\rho, \tag{74}$$
$$\nabla^2 \Phi = \left(\rho + 3p - 3\dot{u}\right), \tag{75}$$
$$\nabla^2 \Sigma_i = -2S_i, \tag{76}$$
$$\Box h_{ij} = -2\Pi_{ij}. \tag{77}$$

These are the set of differential equations, where there is only one wave equations coming from the h_{ij}. The other fields obey a Poisson-like equation. Notice that, both sides of the Eqs.(74) are gauge invariants, as they should be.

1.6 Gravitational waves in a curved spacetime

Now, lets move on one step a head and use the FRW metric as the background metric

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{78}$$

where the bar quantities make references to background objects. The distance between two points can be compute using

$$\mathrm{d}s^2 = a^2(\eta)\left(-\mathrm{d}\eta^2 + \delta_{ij}\mathrm{d}x^i\mathrm{d}x^j + h_{ij}\mathrm{d}x^i\mathrm{d}x^j\right), \tag{79}$$

where h_{ij} is in the TT-gauge, and I am also working in conformal time, defined by the relation

$$a\mathrm{d}\eta = \mathrm{d}t. \tag{80}$$

Next, you can compute the Christoffel symbols, the Riemann curvature tensor and the Ricci scalar. These are the geometrical objects that you need to compute the Einstein's field equations. The perturbed Einstein's field equation leads to just one equation

$$h''_{ij} + 2\mathcal{H}h'_{ij} - \nabla^2 h_{ij} = 8\pi G a^2 \bar{p}\Pi_{ij}, \tag{81}$$

where the time derivatives are take with respect to the conformal time, and the Hubble function $\mathcal{H}$ is defined also in conformal time. Using Eq.(80) you can work the above equation into physics time

$$\ddot{h}_{ij} + 3H\dot{h}_{ij} - \frac{\nabla^2}{a^2}h_{ij} = 8\pi G\bar{p}\Pi_{ij}. \tag{82}$$

From the above expression it is not obvious the classification type of solutions. However, by using the following variable change

$$X_{ij}\left(\mathbf{k},\eta\right) = a h_{ij}\left(\mathbf{k},\eta\right), \tag{83}$$

the differential equation turns to

$$X_{ij}\left(\mathbf{k},\eta\right) + \left(k^2 - \frac{a''}{a}\right)X_{ij}\left(\mathbf{k},\eta\right) = 16\pi G a^3 \Pi_{ij}. \tag{84}$$

Looking for analytical solutions in vacuum $\Pi_{ij} = 0$, there is a clear distinction between two branches of solutions. The first case is given by the condition $k^2 \gg \mathcal{H}^2$ known as the sub-Hubble sphere

$$X''\left(\mathbf{k},\eta\right) + k^2 X\left(\mathbf{k},\eta\right) = 0, \tag{85}$$

which is the equation of the well known harmonic oscillator, whose solution is

$$X\left(\mathbf{k},\eta\right) = A_r(\mathbf{k})e^{ik\eta} + B_r(\mathbf{k})e^{-ik\eta}, \tag{86}$$

or in terms of the original variable

$$h_r\left(\mathbf{k},\eta\right) = \frac{A_r}{a(\eta)}(\mathbf{k})e^{ik\eta} + \frac{B_r}{a(\eta)}(\mathbf{k})e^{-ik\eta}. \tag{87}$$

The second case is $k^2 \ll \mathcal{H}^2$, this are the super-Hubble spheres

$$X''\left(\mathbf{k},\eta\right) - \frac{a''}{a}\left(\mathbf{k},\eta\right) = 0, \tag{88}$$

whose solution in terms of the original variable is

$$h_r\left(\mathbf{k},\eta\right) = A_r(\mathbf{k}) + B_r(\mathbf{k})\int \frac{\mathrm{d}\eta}{a^2(\eta')}. \tag{89}$$

To finalized this section, I just want to deduced the most general gauge transformation of a rank 2 tensor, having a curved background. This is a more general expression that the one found in Eq.(28). To obtain this expression consider the coordinate transformation

$$B'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x^\mu}\frac{\partial x^\beta}{\partial x^\nu}B_{\alpha\beta}(x), \tag{90}$$

under a gauge transformation it follows that

$$B'_{\mu\nu}(x') = \left(\delta^{\alpha}_{\mu} - \partial_{\mu}\xi^{\alpha}\right)\left(\delta^{\beta}_{\nu} - \partial_{\nu}\xi^{\beta}\right) B_{\alpha\beta}(x), \tag{91}$$

expanding the above expression and neglecting ξ^2 terms leads to

$$B'_{\mu\nu}(x') = B_{\mu\nu}(x) - B_{\mu\alpha}(x)\partial_{\nu}\xi^{\alpha} - B_{\beta\nu}(x)\partial_{\mu}\xi^{\beta}. \tag{92}$$

Notice that, the LHS is in the x' coordinate system, whereas the RHS is written in x coordinate system. In order solve the discrepancy, it is required to perform a Taylor series expansion

$$B'_{\mu\nu}(x') = B_{\mu\nu}(x'-\xi) - B_{\mu\alpha}(x'-\xi)\partial_{\nu}\xi^{\alpha} - B_{\beta\nu}(x'-\xi)\partial_{\mu}\xi^{\beta}, \tag{93}$$

$$B'_{\mu\nu}(x') = B_{\mu\nu}(x') - \xi^{\alpha}\partial_{\alpha}B_{\mu\nu}(x') - B_{\mu\alpha}(x')\partial_{\nu}\xi^{\alpha} - B_{\beta\nu}(x')\partial_{\mu}\xi^{\beta}. \tag{94}$$

At this point, the B tensor can be decomposed with as a background plus a perturbation, and considering that the background in both coordinate system are similar, leads to

$$\delta B'_{\mu\nu} = \delta B_{\mu\nu} - \bar{B}_{\mu\alpha}\partial_{\nu}\xi^{\alpha} - \bar{B}_{\beta\nu}\partial_{\mu}\xi^{\beta} - \xi^{\alpha}\partial_{\alpha}\bar{B}_{\mu\nu}. \tag{95}$$

Notice that, if the background quantity is trivial (vanishes), then, all the perturbation are by construction gauge invariants. This is a direct proof of the Steward-Walker lemma. Additionally, Eq. (95) was derived for an arbitrary background, if we use this result to compute how the perturbation $h_{\mu\nu}$ transform leads to

$$h'_{\mu\nu} = h_{\mu\nu} - h_{\mu\alpha}\partial_{\nu}\xi^{\alpha} - h_{\beta\nu}\partial_{\mu}\xi^{\beta} - \xi^{\alpha}\partial_{\alpha}h_{\mu\nu}, \tag{96}$$

however, the last term vanishes due to its higher order in the perturbation series. Therefore, the above expression is reduced to one already found in Eq. (28).

2 Chern-Simons gravity

Consider the Chern-Simons action coupled with the Einstein-Hilbert action is

$$S = \int \mathrm{d}^4x \sqrt{-g}\mathcal{R} \tag{97}$$

3 Polynomial parity breaking

The most general gravitational wave propagation equation is of the form

$$Ah'' + Bh' + Ch' = 0, \tag{98}$$

where time derivatives are taking in conformal time. Without any loss of generalities and assuming $A \neq 0$ the above equation can be written as

$$h'' + \bar{B}h' + \bar{C}h = 0. \tag{99}$$

The idea is to write down the most generality gravitational wave propagation equation that breaks the symmetry of parity, and the fundamental building blocks we can use are

$$k, \mathcal{H}, \Lambda a, \tag{100}$$

which are the only elements that have units of inverse of conformal time, this is a necessary constraint that must be satisfied. Writing down the most general polynomial expression using those fields are

$$\bar{B} = k\bar{B}_k + \mathcal{H}\bar{B}_{\mathcal{H}} + \Lambda a\bar{B}_\Lambda + f(\varphi')\bar{B}_\varphi, \tag{101}$$

$$\bar{C} = k\bar{C}_{kk} + \mathcal{H}\bar{C}_{\mathcal{H}\mathcal{H}} + \Lambda a\bar{C}_{\Lambda\Lambda} + f(\varphi')\bar{C}_{\varphi\varphi}. \tag{102}$$

It is possible to write down the coefficients as a dimensionless quantities

$$\bar{B}_i = \sum b^i_{nm}(\eta)\left(\frac{k}{\Lambda a}\right)^n\left(\frac{\mathcal{H}}{\Lambda a}\right)^m, \tag{103}$$

$$\bar{C}_{ij} = \sum c^{ij}_{nm}(\eta)\left(\frac{k}{\Lambda a}\right)^n\left(\frac{\mathcal{H}}{\Lambda a}\right)^m, \tag{104}$$

where $n > 0$ and $m > 0$, additionally, for consistency, the polynomial functions must be well defined in the limits, such that it is possible to recover the Minkowski spacetime

$$\mathcal{H} \to 0, \qquad k \to 0, \qquad f(\varphi') \to 0 \tag{105}$$

Notice that, in Eq. (101) there are three fundamental building blocks, however, in its respective equation Eq. (103) there are only two fundamental fields, the reason for that, is the degeneracy of the fields. It is possible to compute all configurations of those three fields using only those two written down in Eq. (103), to see this, consider the following example: lets compute one coefficient

$$n = 1, m = 0 \to k\bar{B}_k = kb^k_{1,0}\left(\frac{k}{\Lambda a}\right) = b^k_{1,0}\frac{k^2}{\Lambda a} \tag{106}$$

and compute the degenerate term that produces the same contribution

$$n = 2, m = 0 \to \Lambda a\bar{B}_\Lambda = \Lambda a b^\Lambda_{2,0}\left(\frac{k}{\Lambda a}\right)^2 = b^\Lambda_{2,0}\frac{k^2}{\Lambda a}, \tag{107}$$

as you can see, the term b^{Λ}_{20} is already contained in b^{κ}_{10}. Lets work another example

$$n=0, m=1 \to k\bar{B}_k = kb^k_{0,1}\left(\frac{\mathcal{H}}{\Lambda a}\right) = b^k_{0,1}\frac{k\mathcal{H}}{\Lambda a}, \tag{108}$$

and the degenerate term that leads to the same result

$$n=1, m=0 \to \mathcal{H}\bar{B}_{\mathcal{H}} = \mathcal{H}b^{\mathcal{H}}_{1,0}\left(\frac{k}{\Lambda a}\right) = b^k_{1,0}\frac{k\mathcal{H}}{\Lambda a}, \tag{109}$$

as expected, $b^k_{10} = b^k_{01}$. Lets work one final example,

$$n=3, m=0 \to k\bar{B}_k = kb^k_{3,0}\left(\frac{k}{\Lambda a}\right)^3 = b^k_{3,0}\frac{k^4}{(\Lambda a)^3}, \tag{110}$$

and the other term

$$n=4, m=0 \to \Lambda a\bar{B}_\Lambda = \Lambda a b^k_{4,0}\left(\frac{k}{\Lambda a}\right)^4 = b^k_{4,0}\frac{k^4}{(\Lambda a)^3}. \tag{111}$$

This should make clear, that, the only two fundamental building blocks necessary to define the most general polynomial term is

$$\bar{B} = \mathcal{H}\bar{B}_{\mathcal{H}} + \Lambda a\bar{B}_\Lambda, \tag{112}$$

and also neglecting second order terms in $\mathcal{H}$. A similar analysis can be done for the $\bar{C}$ contribution, leading two only two types of term

$$\bar{C} = k^2\bar{C}_{kk} + (\Lambda a)^2\,\bar{C}_{\Lambda\Lambda}. \tag{113}$$

Assuming small deviations from general relativity due to the parity breaking symmetry, we can expand the coefficients as follows

$$b^i_{n,m} = \bar{b}^i_{n,m} + \lambda^n_{r,l}\lambda^m_{r,l}\delta b^i_{n,m}, \tag{114}$$
$$c^{ij}_{n,m} = \bar{c}^{ij}_{n,m} + \lambda^n_{r,l}\lambda^m_{r,l}\delta c^{ij}_{n,m}. \tag{115}$$

Replacing the above expression for $b^i_{n,m}$ in Eq. (112) leads to

$$\begin{aligned}\bar{B} = &\sum_{n,m}\mathcal{H}\left(\bar{b}^{\mathcal{H}}_{n,m} + \lambda^n_{r,l}\lambda^m_{r,l}\delta b^{\mathcal{H}}_{n,m}\right)\left(\frac{k}{\Lambda a}\right)^n\left(\frac{\mathcal{H}}{\Lambda a}\right)^m \\ &+\sum_{n,m}\Lambda a\left(\bar{b}^{\Lambda}_{n,m} + \lambda^n_{r,l}\lambda^m_{r,l}\delta b^{\Lambda}_{n,m}\right)\left(\frac{k}{\Lambda a}\right)^n\left(\frac{\mathcal{H}}{\Lambda a}\right)^m.\end{aligned} \tag{116}$$

It is possible to extract more information from the above expression by considering the general relativity background. Consider the case $n = m = 0$ then

$$\bar{B} = \mathcal{H}\bar{b}^{\mathcal{H}}_{0,0} + \mathcal{H}\lambda^n_{r,l}\lambda^m_{r,l}\delta b^{\mathcal{H}}_{0,0} + \Lambda a\bar{b}^{\Lambda}_{0,0} + \lambda^n_{r,l}\lambda^m_{r,l}\delta\bar{b}^{\Lambda}_{0,0}, \tag{117}$$

from which is clear that, in order to recover general relativity it is required that (this is only by working with the background)

$$\bar{b}^{\mathcal{H}}_{0,0} = 2, \qquad \bar{b}^{\mathcal{H}}_{n,m} = 0, \qquad \bar{b}^{\Lambda}_{n,m} = 0, \tag{118}$$

where the background object is completely defined. Moving to the perturbation part, it is required that $k \gg \mathcal{H}$, meaning that gravitational wavelengths are smaller compared with the Hubble scale, therefore, we will keep $\mathcal{H}$ at linear order, coupled to higher order of k.

$$\delta b^{\mathcal{H}}_{0,0} = 0 \qquad \bar{b}^{\Lambda}_{0,0} = 0 \qquad \delta \bar{b}^{\Lambda}_{0,0} = 0 \tag{119}$$

Using the above identification, we can derived the most general gravitational wave equation whose parity symmetry es broken

$$\begin{aligned} h''_{r,l} + h'_{r,l} \left(2\mathcal{H} + \lambda_{r,l} k^n \left(\frac{\alpha_n}{\Lambda^n a^n} \mathcal{H} + \frac{\beta\left(\eta\right)}{\Lambda^{(n-1)} a^{(n-1)}} \right) \right) \\ + k^2 h_{r,l} \left(1 + \lambda_{r,l} k^{m-1} \left(\frac{\gamma(\eta)}{\Lambda^m a^m} \mathcal{H} \right) + \frac{\delta_m(\eta)}{\Lambda^{m-1} a^{m-1}} \right) = 0. \end{aligned} \tag{120}$$

The general solution to the above equation is

$$h_{r,l}(\eta) = A_{r,l}(\eta) e^{-i\left(\phi(\eta) - k_i x^i\right)}, \tag{121}$$

ands its first and second derivatives can be compute directly from the above expression

$$h'_{r,l}(\eta) = -i\phi'(\eta) A_{r,l}(\eta) e^{-i\left(\phi(\eta) - k_i x^i\right)}, \tag{122}$$

$$h''_{r,l}(\eta) = -A_{r,l}(\eta) e^{-i\left(\phi(\eta) - k_i x^i\right)} \left(i\phi''(\eta) + \phi'^2(\eta) \right), \tag{123}$$

where it was assume that the amplitude of the gravitational wave varies on much longer timescales than the phase, therefore it is possible to neglected its time-conformal derivatives. Replacing the solution in Eq. (116) leads to

$$\begin{aligned} -\left(i\phi''(\eta) + \phi'^2(\eta) \right) - i\phi'(\eta) \left(2\mathcal{H} + \lambda_{r,l} k^n \left(\frac{\alpha_n}{\Lambda^n a^n} \mathcal{H} + \frac{\beta\left(\eta\right)}{\Lambda^{(n-1)} a^{(n-1)}} \right) \right) \\ k^2 \left(1 + \lambda_{r,l} k^{m-1} \left(\frac{\gamma(\eta)}{\Lambda^m a^m} \mathcal{H} \right) + \frac{\delta_m(\eta)}{\Lambda^{m-1} a^{m-1}} \right) = 0. \end{aligned} \tag{124}$$

which can be rewritten as

$$\begin{aligned} \phi''(\eta) - i\phi'^2(\eta) + \phi'(\eta) \left(2\mathcal{H} + \lambda_{r,l} k^n \left(\frac{\alpha_n}{\Lambda^n a^n} \mathcal{H} + \frac{\beta\left(\eta\right)}{\Lambda^{(n-1)} a^{(n-1)}} \right) \right) \\ ik^2 \left(1 + \lambda_{r,l} k^{m-1} \left(\frac{\gamma(\eta)}{\Lambda^m a^m} \mathcal{H} \right) + \frac{\delta_m(\eta)}{\Lambda^{m-1} a^{m-1}} \right) = 0. \end{aligned} \tag{125}$$

Now, assuming that parity breaking terms are small deviations from general relativity, it is possible to linearized the wave equation by taking

$$\phi = \bar{\phi} + \delta\phi, \tag{126}$$

where $\bar{\phi}$ is the usual background solution of general relativity which has the form of $\bar{\phi}' = \pm k - i\mathcal{H}$.

References

[1] E. Poisson and C. Will, *Gravity: Newtonian, Post-Newtonian, Relativistic*. Cambridge University Press, 2014.

[2] J. Stewart and J. Stewart, *Advanced General Relativity.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1993.