

Personal notes on gravitational waves

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Notation and conventions

I am working with the following conventions, for the metric tensor

$$(-1, +1, +1, +1) \tag{1}$$

1 Linear general relativity

In this section I am presenting a general overview on how to build-up the Einstein's field equation, given a background metric tensor and a perturbation tensor.

1.1 General overview

Decomposed the metric tensor as the sum of a background metric, in this case the flat spacetime Minkowski metric $\eta_{\mu\nu}$, plus a perturbation $h_{\mu\nu}$ as follows

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \quad (2)$$

Using the definition of the Kronecker delta object it is straightforward to obtain the inverse tensor of the perturbation

$$\delta^\mu_\nu = g^{\mu\sigma} g_{\sigma\nu}. \quad (3)$$

Replacing Eq.(2) in to Eq.(3), and neglecting second order terms in the perturbation field, leads to

$$h^{\mu\nu} = -\eta^{\mu\alpha} \eta^{\nu\beta} h_{\alpha\beta}, \quad (4)$$

notice that, in order to upper/lower the indices of the perturbation I am only using the background metric.¹ Schematically, second order terms are neglected

$$h_{\mu\nu} h_{\alpha\beta} \sim 0, \quad h_{\mu\nu} \partial_\gamma h_{\alpha\beta} \sim 0, \quad \partial_\delta h_{\mu\nu} \partial_\gamma h_{\alpha\beta} \sim 0. \quad (5)$$

This is the general overview of the fundamental field of general relativity, which at its core is the metric tensor. In the following subsection, I will be computing the Einstein's field equations for the metric tensor written in Eq.(2).

1.2 Geometrical objects

The first object that is required to compute the Einstein's field equations is the connection. Working on a torsion-free manifold, the Levi-Civita connection is written as

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\rho} (\partial_\alpha g_{\rho\beta} + \partial_\beta g_{\rho\alpha} - \partial_\rho g_{\alpha\beta}). \quad (6)$$

Replacing Eq.(2) in to Eq.(6) leads to

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} (\eta^{\mu\rho} - h^{\mu\rho}) (\partial_\alpha \eta_{\rho\beta} + \partial_\beta \eta_{\rho\alpha} - \partial_\rho \eta_{\alpha\beta} + \partial_\alpha h_{\rho\beta} + \partial_\beta h_{\rho\alpha} - \partial_\rho h_{\alpha\beta}). \quad (7)$$

Notice that, the only non-trivial contributions are the ones that are linear in the perturbation, additionally, the partial derivatives of the Minkowski's metric tensor vanishes, therefore, the connection coefficients are reduced to

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} \eta^{\mu\rho} (\partial_\alpha h_{\rho\beta} + \partial_\beta h_{\rho\alpha} - \partial_\rho h_{\alpha\beta}). \quad (8)$$

¹Including the perturbation tensor leads to second order terms, which I am ignoring.

Next, compute the Riemann curvature tensor

$$\mathcal{R}^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho{}_{\sigma\nu} - \partial_\nu \Gamma^\rho{}_{\sigma\mu} + \Gamma^\gamma{}_{\nu\sigma} \Gamma^\rho{}_{\mu\gamma} + \Gamma^\gamma{}_{\mu\sigma} \Gamma^\rho{}_{\nu\gamma}, \quad (9)$$

however, instead of computing directly from the above equation, it is convenient to notice the structure of the curvature tensor. The last two terms are quadratic in the Levi-Civita connection, and, since the connection is written with perturbation, then, square terms in the connection vanishes, reducing the Riemann curvature tensor to

$$\mathcal{R}^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho{}_{\sigma\nu} - \partial_\nu \Gamma^\rho{}_{\sigma\mu}. \quad (10)$$

Replacing Eq.(8) in to Eq.(10) leads to

$$\mathcal{R}^\rho{}_{\sigma\mu\nu} = \frac{1}{2} \eta^{\rho\alpha} \partial_\mu (\partial_\sigma h_{\alpha\nu} + \partial_\nu h_{\sigma\alpha} - \partial_\alpha h_{\sigma\nu}) + \frac{1}{2} \eta^{\rho\alpha} \partial_\nu (\partial_\sigma h_{\alpha\mu} + \partial_\mu h_{\sigma\alpha} - \partial_\alpha h_{\sigma\mu}), \quad (11)$$

the above expression, can be simplified to

$$\mathcal{R}^\rho{}_{\sigma\mu\nu} = \frac{1}{2} \eta^{\rho\alpha} (\partial_\mu \partial_\sigma h_{\alpha\nu} - \partial_\mu \partial_\alpha h_{\sigma\nu} - \partial_\nu \partial_\sigma h_{\mu\alpha} + \partial_\nu \partial_\alpha h_{\mu\sigma}). \quad (12)$$

From the Riemann tensor, it is straightforward to compute the Ricci tensor, by contracting their respective indices

$$\mathcal{R}_{\sigma\nu} = \mathcal{R}^\mu{}_{\sigma\mu\nu}. \quad (13)$$

A direct computation shows the structure of the Ricci tensor

$$\mathcal{R}_{\sigma\nu} = \frac{1}{2} (\partial^\alpha \partial_\sigma h_{\alpha\nu} - \square h_{\sigma\nu} - \partial_\nu \partial_\sigma h + \partial_\nu \partial^\alpha h_{\alpha\sigma}), \quad (14)$$

where \square is the d'Alembert operator and h is the trace of the perturbation.

In the same spirit, the curvature scalar can be obtained directly through the contraction of the Ricci tensor

$$\mathcal{R} = g^{\mu\sigma} \mathcal{R}_{\mu\sigma}. \quad (15)$$

This computation is straightforward

$$\mathcal{R} = (\eta^{\mu\sigma} - h^{\mu\sigma}) \frac{1}{2} (\partial^\alpha \partial_\sigma h_{\alpha\nu} - \square h_{\sigma\nu} - \partial_\nu \partial_\sigma h + \partial_\nu \partial^\alpha h_{\alpha\sigma}). \quad (16)$$

Neglecting second order terms in the perturbation field, the scalar curvature is given by

$$\mathcal{R} = \partial_\mu \partial_\sigma h^{\mu\sigma} - \square h \quad (17)$$

Now, we can compute the Einstein's field equations without a cosmological constant

$$\mathcal{R}_{\mu\nu} - \frac{1}{2} \mathcal{R} g_{\mu\nu} = \frac{8\pi G}{c^4} \mathcal{T}_{\mu\nu} \quad (18)$$

where $\mathcal{T}_{\mu\nu}$ is the energy momentum tensor. Replacing Eq.(14) and Eq.(17) in to Eq.(18) leads to

$$\partial^\alpha \partial_\mu h_{\alpha\nu} - \square h_{\mu\nu} - \partial_\nu \partial_\mu h + \partial_\nu \partial^\alpha h_{\alpha\mu} - (\partial_\alpha \partial_\beta h^{\alpha\beta} - \square h) (\eta_{\mu\nu} + h_{\mu\nu}) = \frac{16\pi G}{c^4} \mathcal{T}_{\mu\nu}. \quad (19)$$

Just like before, neglecting second order terms in the perturbation, the above equation can be reduced to

$$\partial^\alpha \partial_\mu h_{\alpha\nu} - \square h_{\mu\nu} - \partial_\nu \partial_\mu h + \partial_\nu \partial^\alpha h_{\alpha\mu} - \eta_{\mu\nu} \partial_\alpha \partial_\beta h^{\alpha\beta} - \eta_{\mu\nu} \square h = \frac{16\pi G}{c^4} \mathcal{T}_{\mu\nu}. \quad (20)$$

The above equation, can be written in a much more compact manner by using the following variable change²

$$X_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h, \quad (21)$$

which can be inverted through standard methods

$$h_{\mu\nu} = X_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} X \quad (22)$$

where X is the trace of the tensor $X_{\mu\nu}$, and the tensor $X_{\mu\nu}$, also satisfies the relation $X = -h$. Replacing the variable change written in Eq.(22) in Eq.(20) and simplifying terms, leads to

$$\partial^\alpha \partial_\mu X_{\alpha\nu} + \partial_\nu \partial^\alpha X_{\alpha\mu} - \square X_{\mu\nu} - \eta_{\mu\nu} \partial_\alpha \partial_\beta X^{\alpha\beta} = \frac{16\pi G}{c^4} \mathcal{T}_{\mu\nu}, \quad (23)$$

which, in some sense has a more simple structure than Eq.(20), and also contains the wave operator. Nonetheless, this does not look like a gravitational wave equation. In the next subsection, I will show you, how can you derive the gravitational wave equation from the above expression using a gauge transformation.

²In the standard literature $X_{\mu\nu}$ is written as \bar{h} , but I strongly believe this leads to confusions.