Spherically solutions of Polynomial Affine Gravity in the torsion-free sector

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1 Introduction

2 Polynomial Affine Gravity

2.1 Model review

The Polynomial Affine Gravity uses the affine connection to mediated gravitational interactions, instead of the metric tensor. To build the action, we decompose the affine connection into its irreducible fields

$$\begin{aligned} \hat{\Gamma}_{\alpha}{}^{\beta}{}_{\gamma} &= \hat{\Gamma}_{(\alpha}{}^{\beta}{}_{\gamma)} + \hat{\Gamma}_{[\alpha}{}^{\beta}{}_{\gamma]}, \\ &= \Gamma_{\alpha}{}^{\beta}{}_{\gamma} + \mathcal{B}_{\alpha}{}^{\beta}{}_{\gamma} + \delta^{\beta}_{[\gamma}\mathcal{A}_{\alpha]}, \end{aligned} \tag{1}$$

where the first term Γ stands for the symmetric part and $\mathcal{A}$, $\mathcal{B}$ fields are related to the skew symmetric part of the affine connection.

Additionally, is necessary to introduce a volume form without the use of the metric tensor. In order to achieve this, we used the volume element defined as $\mathrm{d}V^{\alpha\beta\gamma\delta} = \mathrm{d}x^{\alpha} \wedge \mathrm{d}x^{\beta} \wedge \mathrm{d}x^{\gamma} \wedge \mathrm{d}x^{\delta}$, which is completely antisymmetric.

Moreover, we want to preserve the invariance under diffeomorphism, which is why the symmetric part of the affine connection, goes indirectly through the covariant derivative ∇^{Γ}.

The action is built up using a sort of *dimensional analysis* technique. This strategy allows us to generate every scalar density composed by powers of the irreducible fields of the affine connection. This method has been using to build the action in four dimensions, see Refs. and in three dimensions Ref [].

The most general action (up to topological invariants and boundary terms) in four dimensions is given by

$$S = \int \mathrm{d}V^{\alpha\beta\gamma\delta} \Big[B_1 \mathcal{R}_{\mu\nu}{}^{\mu}{}_{\rho} \mathcal{B}_{\alpha}{}^{\nu}{}_{\beta} \mathcal{B}_{\gamma}{}^{\rho}{}_{\delta} + B_2 \mathcal{R}_{\alpha\beta}{}^{\mu}{}_{\rho} \mathcal{B}_{\gamma}{}^{\nu}{}_{\delta} \mathcal{B}_{\mu}{}^{\rho}{}_{\nu} + B_3 \mathcal{R}_{\mu\nu}{}^{\mu}{}_{\alpha} \mathcal{B}_{\beta}{}^{\nu}{}_{\gamma} \mathcal{A}_{\delta}$$
$$+ B_4 \mathcal{R}_{\alpha\beta}{}^{\sigma}{}_{\rho} \mathcal{B}_{\gamma}{}^{\rho}{}_{\delta} \mathcal{A}_{\sigma} + B_5 \mathcal{R}_{\alpha\beta}{}^{\rho}{}_{\rho} \mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta} \mathcal{A}_{\sigma} + C_1 \mathcal{R}_{\mu\alpha}{}^{\mu}{}_{\nu} \nabla_{\beta} \mathcal{B}_{\gamma}{}^{\nu}{}_{\delta}$$
$$+ C_2 \mathcal{R}_{\alpha\beta}{}^{\rho}{}_{\rho} \nabla_{\sigma} \mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta} + D_1 \mathcal{B}_{\nu}{}^{\mu}{}_{\lambda} \mathcal{B}_{\mu}{}^{\nu}{}_{\alpha} \nabla_{\beta} \mathcal{R}_{\gamma}{}^{\lambda}{}_{\delta} + D_2 \mathcal{B}_{\alpha}{}^{\mu}{}_{\beta} \mathcal{B}_{\mu}{}^{\lambda}{}_{\nu} \nabla_{\lambda} \mathcal{B}_{\gamma}{}^{\nu}{}_{\delta}$$
$$+ D_3 \mathcal{B}_{\alpha}{}^{\mu}{}_{\nu} \mathcal{B}_{\beta}{}^{\lambda}{}_{\gamma} \nabla_{\lambda} \mathcal{B}_{\mu}{}^{\nu}{}_{\delta} + D_4 \mathcal{B}_{\alpha}{}^{\lambda}{}_{\beta} \mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta} \nabla_{\lambda} \mathcal{A}_{\sigma} + D_5 \mathcal{B}_{\alpha}{}^{\lambda}{}_{\beta} \mathcal{A}_{\sigma} \nabla_{\lambda} \mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta}$$
$$+ D_6 \mathcal{B}_{\alpha}{}^{\lambda}{}_{\beta} \mathcal{A}_{\gamma} \nabla_{\lambda} A_{\delta} + D_7 \mathcal{B}_{\alpha}{}^{\lambda}{}_{\beta} \mathcal{A}_{\lambda} \nabla_{\gamma} A_{\delta} + E_1 \nabla_{\rho} \mathcal{B}_{\alpha}{}^{\rho}{}_{\beta} \nabla_{\sigma} \mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta}$$
$$+ E_2 \nabla_{\rho} \mathcal{B}_{\alpha}{}^{\rho}{}_{\beta} \nabla_{\gamma} \mathcal{A}_{\delta} + F_1 \mathcal{B}_{\alpha}{}^{\mu}{}_{\beta} \mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta} \mathcal{B}_{\mu}{}^{\lambda}{}_{\rho} \mathcal{B}_{\sigma}{}^{\rho}{}_{\lambda} + F_2 \mathcal{B}_{\alpha}{}^{\mu}{}_{\beta} \mathcal{B}_{\gamma}{}^{\nu}{}_{\lambda} \mathcal{B}_{\delta}{}^{\lambda}{}_{\rho} \mathcal{B}_{\mu}{}^{\rho}{}_{\nu}$$
$$+ F_3 \mathcal{B}_{\nu}{}^{\mu}{}_{\lambda} \mathcal{B}_{\mu}{}^{\nu}{}_{\alpha} \mathcal{B}_{\beta}{}^{\lambda}{}_{\gamma} \mathcal{A}_{\delta} + F_4 \mathcal{B}_{\alpha}{}^{\mu}{}_{\beta} \mathcal{B}_{\gamma}{}^{\nu}{}_{\delta} \mathcal{A}_{\mu} \mathcal{A}_{\nu} \Big]. \tag{2}$$

Notice the Riemann curvature and the Ricci tensor are defined with respect to the symmetric part of the connection.

The action written in Eq. (2) is purely affine and does not required the existence of a metric tensor to be defined. Additionally is polynomial in the connection and its covariant derivative, unlike the Einstein-Hilbert action, where there is the factor $\sqrt{-g}$.

As a consequence of the lack of metric tensor in our model, the numbers of terms that can go to the action, is limited due to the geometrical constraint coming from its formulation. We call this property, the *rigidity* of the model.

Moreover, is possible to coupled a scalar field using the *dimensional analysis* technique. This provide a non-standard procedure to coupled the *kinetic term* of the scalar field to the irreducible fields coming from the antisymmetric part of the affine connection and the volume form, without the use of a metric tensor. The effects of a scalar field in polynomial affine gravity has been studied in the torsion-free sector in Ref.[].

Even though the manifold is only endowed with an affine connection as its fundamental field, is possible to still obtain descendent metric structures, the first comes from its symmetric part, which is the Ricci tensor $\mathcal{R}_{\mu\nu}$, coming from the natural contraction of the Riemann curvature

$$\mathcal{R}_{\mu\nu}(\Gamma) = \mathcal{R}_{\mu\rho}{}^{\rho}{}_{\nu}(\Gamma), \tag{3}$$

whereas, the second comes from the antisymmetric part of the connection, specifically, from the contraction of two torsion tensors, defined as

$$\mathcal{P}_{\mu\nu} = \left(\mathcal{B}_{\alpha}{}^{\beta}{}_{\mu} + \delta^{\beta}_{[\alpha]\mathcal{A}_{\mu}} \right) \left(\mathcal{B}_{\beta}{}^{\alpha}{}_{\nu} + \delta^{\alpha}_{[\beta]\mathcal{A}_{\nu}} \right). \tag{4}$$

Therefore, we can use Eq. (3) or Eq. (4) to define the notion of distance, and allowing a classification of vectors into time-lie, space-lie or null-like.[1]

[1]This is only valid when the tensors are well behaved, meaning that, they can not be degenerate, they must be invertible.

Interestingly, all coupling constant are dimensionless, which suggest some sort of conformal symmetry, at least at a classical level, and also indicates that the model is power-counting renormalizable. This, is a necessary condition but not sufficient condition to guarantee that the model is renormalizable.

2.2 Building the ansatz

To build the ansatz of the affine connection Γ we compute its Lie derivative $\mathcal{L}_{\xi_j}\Gamma_\alpha{}^\beta{}_\gamma$ along the Killing vectors ξ_j that generate the desired symmetry, in this case a spherical symmetry. This procedure has been extensively cover in Refs. []

The final affine connection's coefficient in the torsion-free sector are given by twelve independent functions defined as follow:

$$\begin{array}{lllll}
\Gamma_t{}^t{}_t = V(t,r) & \Gamma_t{}^t{}_r = A(t,r) & \Gamma_r{}^t{}_r = W(t,r) & \Gamma_\theta{}^t{}_\theta = X(t,r) & \Gamma_\phi{}^t{}_\phi = X(t,r)\sin^2\theta \\
\Gamma_t{}^r{}_t = B(t,r) & \Gamma_t{}^r{}_r = Y(t,r) & \Gamma_r{}^r{}_r = C(t,r) & \Gamma_\theta{}^r{}_\theta = F(t,r) & \Gamma_\phi{}^r{}_\phi = F(t,r)\sin^2\theta \\
\Gamma_t{}^\theta{}_\theta = Z(t,r) & \Gamma_t{}^\theta{}_\phi = -D(t,r)\sin\theta & \Gamma_r{}^\theta{}_\theta = G(t,r) & \Gamma_r{}^\theta{}_\phi = -H(t,r)\sin\theta & \Gamma_\phi{}^\theta{}_\phi = -\cos\theta\sin\theta \\
\Gamma_t{}^\phi{}_\theta = \dfrac{D(t,r)}{\sin\theta} & \Gamma_t{}^\phi{}_\phi = Z(t,r) & \Gamma_r{}^\phi{}_\theta = \dfrac{H(t,r)}{\sin\theta} & \Gamma_r{}^\phi{}_\phi = G(t,r) & \Gamma_\theta{}^\phi{}_\phi = \dfrac{\cos\theta}{\sin\theta}
\end{array} \tag{5}$$

Notice that, under a parametrization of the time coordinate, the first coefficient can be set equal to zero, see Ref []. The above ansatz can be simplified even further by imposing additional symmetries on the affine connection. First, lets consider stationary solutions

$$\nabla_{e_t} e_t = e_t \Gamma_t{}^t{}_t + e_r \Gamma_t{}^r{}_t \tag{6}$$

$$\nabla_{-e_t}(-e_t) = -e_t \Gamma_t{}^t{}_t + e_r \Gamma_t{}^r{}_t \tag{7}$$

the consistent condition requires that $\Gamma_t{}^t{}_t = 0$. Applying the same principle to the other basis vectors, then

$$\Gamma_t{}^j{}_i = 0 \qquad\qquad \Gamma_i{}^t{}_j = 0 \tag{8}$$

where i, j are restricted to space index.

Next, we demand an azimuthal angle symmetry, in which case the

$$\Gamma_\phi{}^\phi{}_\phi = 0 \qquad\qquad \Gamma_t{}^j{}_\phi = 0 \qquad\qquad \Gamma_i{}^\phi{}_j = 0 \tag{9}$$

2.3 Field equations

3 Solutions

4 Final remarks