Spherically and static solutions of Polynomial Affine Gravity in the torsion-free sector

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In the following subsections, I present a brief introduction to the polynomial affine model of gravity, how to build the ansatz compatible with the spherical symmetry and the field equations.

2.1 Model review

The Polynomial Affine Gravity uses the affine connection to mediated gravitational interactions, instead of the metric tensor. To build the action, we decompose the affine connection into its irreducible fields

$$
\begin{aligned}
\hat{\Gamma}_\alpha{}^\beta{}_\gamma &= \hat{\Gamma}_{(\alpha}{}^\beta{}_{\gamma)} + \hat{\Gamma}_{[\alpha}{}^\beta{}_{\gamma]}, \\
&= \Gamma_\alpha{}^\beta{}_\gamma + \mathcal{B}_\alpha{}^\beta{}_\gamma + \delta^\beta_{[\gamma}\mathcal{A}_{\alpha]},
\end{aligned} \tag{1}
$$

where the first term Γ stands for the symmetric part and $\mathcal{A}$, $\mathcal{B}$ fields are related to the skew symmetric part of the affine connection.

Additionally, is necessary to introduce a volume form without the use of the metric tensor. In order to achieve this, we used the volume element defined as $\mathrm{d}V^{\alpha\beta\gamma\delta} = \mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma \wedge \mathrm{d}x^\delta$, which is completely antisymmetric.

Moreover, we want to preserve the invariance under diffeomorphism, which is why the symmetric part of the affine connection, goes indirectly through the covariant derivative ∇^Γ.

The action is built up using a sort of *dimensional analysis* technique. This strategy allows us to generate every scalar density composed by powers of the irreducible fields of the affine connection. This method has been using to build the action in four dimensions, see Refs. and in three dimensions Ref [].

The most general action (up to topological invariants and boundary terms) in four dimensions is given by

$$\begin{aligned}
S = \int \mathrm{d}V^{\alpha\beta\gamma\delta}\Big[& B_1 \mathcal{R}_{\mu\nu}{}^{\mu}{}_{\rho} \mathcal{B}_{\alpha}{}^{\nu}{}_{\beta} \mathcal{B}_{\gamma}{}^{\rho}{}_{\delta} + B_2 \mathcal{R}_{\alpha\beta}{}^{\mu}{}_{\rho} \mathcal{B}_{\gamma}{}^{\nu}{}_{\delta} \mathcal{B}_{\mu}{}^{\rho}{}_{\nu} + B_3 \mathcal{R}_{\mu\nu}{}^{\mu}{}_{\alpha} \mathcal{B}_{\beta}{}^{\nu}{}_{\gamma} \mathcal{A}_{\delta} \\
& + B_4 \mathcal{R}_{\alpha\beta}{}^{\sigma}{}_{\rho} \mathcal{B}_{\gamma}{}^{\rho}{}_{\delta} \mathcal{A}_{\sigma} + B_5 \mathcal{R}_{\alpha\beta}{}^{\rho}{}_{\rho} \mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta} \mathcal{A}_{\sigma} + C_1 \mathcal{R}_{\mu\alpha}{}^{\mu}{}_{\nu} \nabla_{\beta} \mathcal{B}_{\gamma}{}^{\nu}{}_{\delta} \\
& + C_2 \mathcal{R}_{\alpha\beta}{}^{\rho}{}_{\rho} \nabla_{\sigma} \mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta} + D_1 \mathcal{B}_{\nu}{}^{\mu}{}_{\lambda} \mathcal{B}_{\mu}{}^{\nu}{}_{\alpha} \nabla_{\beta} \mathcal{R}_{\gamma}{}^{\lambda}{}_{\delta} + D_2 \mathcal{B}_{\alpha}{}^{\mu}{}_{\beta} \mathcal{B}_{\mu}{}^{\lambda}{}_{\nu} \nabla_{\lambda} \mathcal{B}_{\gamma}{}^{\nu}{}_{\delta} \\
& + D_3 \mathcal{B}_{\alpha}{}^{\mu}{}_{\nu} \mathcal{B}_{\beta}{}^{\lambda}{}_{\gamma} \nabla_{\lambda} \mathcal{B}_{\mu}{}^{\nu}{}_{\delta} + D_4 \mathcal{B}_{\alpha}{}^{\lambda}{}_{\beta} \mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta} \nabla_{\lambda} \mathcal{A}_{\sigma} + D_5 \mathcal{B}_{\alpha}{}^{\lambda}{}_{\beta} \mathcal{A}_{\sigma} \nabla_{\lambda} \mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta} \\
& + D_6 \mathcal{B}_{\alpha}{}^{\lambda}{}_{\beta} \mathcal{A}_{\gamma} \nabla_{\lambda} A_{\delta} + D_7 \mathcal{B}_{\alpha}{}^{\lambda}{}_{\beta} \mathcal{A}_{\lambda} \nabla_{\gamma} A_{\delta} + E_1 \nabla_{\rho} \mathcal{B}_{\alpha}{}^{\rho}{}_{\beta} \nabla_{\sigma} \mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta} \\
& + E_2 \nabla_{\rho} \mathcal{B}_{\alpha}{}^{\rho}{}_{\beta} \nabla_{\gamma} \mathcal{A}_{\delta} + F_1 \mathcal{B}_{\alpha}{}^{\mu}{}_{\beta} \mathcal{B}_{\gamma}{}^{\sigma}{}_{\delta} \mathcal{B}_{\mu}{}^{\lambda}{}_{\rho} \mathcal{B}_{\sigma}{}^{\rho}{}_{\lambda} + F_2 \mathcal{B}_{\alpha}{}^{\mu}{}_{\beta} \mathcal{B}_{\gamma}{}^{\nu}{}_{\lambda} \mathcal{B}_{\delta}{}^{\lambda}{}_{\rho} \mathcal{B}_{\mu}{}^{\rho}{}_{\nu} \\
& + F_3 \mathcal{B}_{\nu}{}^{\mu}{}_{\lambda} \mathcal{B}_{\mu}{}^{\nu}{}_{\alpha} \mathcal{B}_{\beta}{}^{\lambda}{}_{\gamma} \mathcal{A}_{\delta} + F_4 \mathcal{B}_{\alpha}{}^{\mu}{}_{\beta} \mathcal{B}_{\gamma}{}^{\nu}{}_{\delta} \mathcal{A}_{\mu} \mathcal{A}_{\nu} \Big].
\end{aligned} \tag{2}$$

Notice the Riemann curvature and the Ricci tensor are defined with respect to the symmetric part of the connection.

The action written in Eq. (14) is purely affine and does not required the existence of a metric tensor to be defined. Additionally is polynomial in the connection and its covariant derivative, unlike the Einstein-Hilbert action, where there is the factor $\sqrt{-g}$.

As a consequence of the lack of metric tensor in our model, the numbers of terms that can go to the action, is limited due to the geometrical constraint coming from its formulation. We call this property, the *rigidity* of the model.

Moreover, is possible to coupled a scalar field using the *dimensional analysis* technique. This provide a non-standard procedure to coupled the *kinetic term* of the scalar field to the irreducible fields coming from the antisymmetric part of the affine connection and the volume form, without the use of a metric tensor. The effects of a scalar field in polynomial affine gravity has been studied in the torsion-free sector in Ref.[].

Interestingly, all coupling constant are dimensionless, which suggest some sort of conformal symmetry, at least at a classical level, and also indicates that the model is power-counting renormalizable. This, is a necessary condition but not sufficient condition to guarantee that the model is renormalizable.

Finally, in the torsion-free limit it is possible to recover all Einstein vacuum solutions, meaning that it is a subspace of solutions of of polynomial affine gravity.

2.2 Building the ansatz

To build up the ansatz of the affine connection Γ, we compute its Lie derivative $\mathcal{L}_{\xi_j}\Gamma_\alpha{}^\beta{}_\gamma$ along the Killing vectors ξ_j that generate the desired symmetry, in this case a spherical symmetry. The Lie derivative of a a connection is written as

$$\mathcal{L}_{\xi_i}\Gamma_\alpha{}^\beta{}_\gamma = \xi_i^\delta\partial_\delta\Gamma_\alpha{}^\beta{}_\gamma - \Gamma_\alpha{}^\delta{}_\beta\partial_\delta\xi^\beta + \Gamma_\alpha{}^\beta{}_\delta\partial_\gamma\xi^\delta + \Gamma_\delta{}^\beta{}_\gamma\partial_\alpha\xi^\delta + \frac{\partial^2\xi^\beta}{\partial x^\alpha\partial x^\gamma}, \tag{3}$$

notice this is the standard definition of a Lie derivative of a tensor $(1,2)$, however, there is an extra term (non-homogeneous), because the affine connection does not transform as a tensor. The Killing vectors are

$$\xi_1 = \sigma\left(0, \cos\phi\sin\theta, \frac{\cos\phi\cos\theta}{r}, -\frac{\sin\phi}{r\sin\theta}\right), \tag{4}$$

$$\xi_2 = \sigma\left(0, \sin\phi\sin\theta, \frac{\sin\phi\cos\theta}{r}, \frac{\cos\phi}{r\sin\theta}\right), \tag{5}$$

$$\xi_3 = \sigma\left(0, \cos\theta, -\frac{\sin\theta}{r}, 0\right), \tag{6}$$

where σ is defined as

$$\sigma = \sqrt{1-\kappa r^2}. \tag{7}$$

This procedure has been cover in Refs. [], and, an explicit computation of every term can be found in Ref.[]

The affine connection is completely defined by twelve function[1], where each function has a time and radial dependence as follow

$$\begin{array}{llll}
\Gamma_t{}^t{}_t = V(t,r) & \Gamma_t{}^r{}_t = B(t,r) & \Gamma_t{}^\theta{}_\theta = Z(t,r) & \Gamma_t{}^\phi{}_\theta = \dfrac{D(t,r)}{\sin\theta} \\
\Gamma_t{}^t{}_r = A(t,r) & \Gamma_t{}^r{}_r = Y(t,r) & \Gamma_t{}^\theta{}_\phi = -D(t,r)\sin\theta & \Gamma_t{}^\phi{}_\phi = Z(t,r) \\
\Gamma_r{}^t{}_r = W(t,r) & \Gamma_r{}^r{}_r = C(t,r) & \Gamma_r{}^\theta{}_\theta = G(t,r) & \Gamma_r{}^\phi{}_\theta = \dfrac{H(t,r)}{\sin\theta} \\
\Gamma_\theta{}^t{}_\theta = X(t,r) & \Gamma_\theta{}^r{}_\theta = F(t,r) & \Gamma_r{}^\theta{}_\phi = -H(t,r)\sin\theta & \Gamma_r{}^\phi{}_\phi = G(t,r) \\
\Gamma_\phi{}^t{}_\phi = X(t,r)\sin^2\theta & \Gamma_\phi{}^r{}_\phi = F(t,r)\sin^2\theta & \Gamma_\phi{}^\theta{}_\phi = -\cos\theta\sin\theta & \Gamma_\theta{}^\phi{}_\phi = \dfrac{\cos\theta}{\sin\theta}
\end{array} \tag{8}$$

Notice that, under a parametrization of the time coordinate, the first coefficient can be set equal to zero. This transformation has been extensively use in the

[1] This is valid in the affine geometry without torsion. If we introduce a non trivial torsion field, the affine connection is defined by twenty time and radial dependent functions.

frame of cosmology and can also be applied to the spherical case. For more information on this type of transformation, refer to Ref. [1].

The above ansatz can be simplified even further by imposing additional symmetries on the affine connection. First, lets consider time reversal

$$\nabla_t e_t = e_t \Gamma_t{}^t{}_t + e_r \Gamma_t{}^r{}_t \tag{9}$$

$$\nabla_{-t}\left(-e_t\right) = -e_t \Gamma_t{}^t{}_t + e_r \Gamma_t{}^r{}_t \tag{10}$$

the consistent condition requires that $\Gamma_t{}^t{}_t = 0$. Applying the same principle to the other basis vectors, then

$$\Gamma_t{}^j{}_i = 0 \qquad\qquad \Gamma_i{}^t{}_j = 0 \tag{11}$$

where i, j are restricted to space index.

Next, we demand an azimuthal angle symmetry, in which case the

$$\Gamma_\phi{}^\phi{}_\phi = 0 \qquad\qquad \Gamma_t{}^j{}_\phi = 0 \qquad\qquad \Gamma_i{}^\phi{}_j = 0 \tag{12}$$

Finally, we restrict the affine coefficient to be time independent (static). The final form of the ansatz is written as

$$\begin{array}{lll} \Gamma_t{}^t{}_r = A(r) & \Gamma_t{}^r{}_t = B(r) & \Gamma_r{}^r{}_r = C(r) \\ \Gamma_\theta{}^r{}_\theta = F(r) & \Gamma_\phi{}^r{}_\phi = F(r)\sin^2\theta & \Gamma_r{}^\theta{}_\theta = G(r) \\ \Gamma_\phi{}^\theta{}_\phi = -\cos\theta\sin\theta & \Gamma_r{}^\phi{}_\phi = G(r) & \Gamma_\theta{}^\phi{}_\phi = \dfrac{\cos\theta}{\sin\theta} \end{array} \tag{13}$$

Notice that, originally we have twelve time and radial dependence, which was reduced to only five radial dependent functions.

2.3 Field equations

In the torsion-free limit, the only non trivial contribution are coming from the terms that are liner in the irreducible fields of the torsion tensor $\mathcal{A}$ and $\mathcal{B}$. In this case, the effective action is written as

$$S = \int \mathrm{d}V^{\alpha\beta\gamma\delta}\left[C_1 \mathcal{R}_{\mu\alpha}{}^\mu{}_\nu \nabla_\beta \mathcal{B}_\gamma{}^\nu{}_\delta\right], \tag{14}$$

whose variation with respect to the $\mathcal{B}$ leads to the field equation

$$\nabla_{[\sigma}\mathcal{R}_{\mu]\nu} = 0, \tag{15}$$

where is said that the Ricci tensor is a Codazzi tensor. From the field equation written in Eq. (15) we distinguish three branches of solutions: the first type of solutions requires the Ricci tensor to vanish, meaning $\mathcal{R}_{\mu\nu} = 0$, which written using Eq. (13) leads to

$$\frac{\partial B}{\partial r} + B\left(2G + C - A\right) = 0, \tag{16}$$

$$\frac{\partial A}{\partial r}+2\frac{\partial G}{\partial r}+A^{2}+2G^{2}-AC-2CG=0, \tag{17}$$

$$\frac{\partial F}{\partial r}+F\left(A+C\right)+1=0, \tag{18}$$

where we have three first order differential equations for five unknown functions. Therefore, the system is underdetermined.

The second type of solution is the subspace of parallel Ricci, which implies that $\nabla_{\sigma}\mathcal{R}_{\mu\nu}=0$, whose field equation under the ansatz in Eq. (13) can be written as follow

The third type is to solve directly $\nabla_{[\sigma}\mathcal{R}_{\mu]\nu}=0$, which is known as *harmonic curvature*. Using Eq. (13), the *harmonic curvature* is defined as

$$\frac{\partial^{2}B}{\partial r^{2}}+B\left(\frac{\partial C}{\partial r}-2\frac{\partial A}{\partial r}\right)-\frac{\partial B}{\partial r}\left(2A-C-2G\right)-2GB\left(A-C+G\right)=0, \tag{19}$$

$$\frac{\partial^{2}F}{\partial r^{2}}+F\left(\frac{\partial C}{\partial r}-2\frac{\partial G}{\partial r}\right)+\frac{\partial F}{\partial r}\left(A+C-G\right)+F\left(G\left(C-A-2G\right)-A\left(A-C\right)\right)-G=0, \tag{20}$$

where there are only two independent equations for five unknown functions. Therefore, the system is underdetermined.

3 Solutions

4 Analysis of the solutions

Even though the manifold is only endowed with an affine connection as its fundamental field, is possible to still obtain descendent metric structures, the first comes from its symmetric part, which is the Ricci tensor $\mathcal{R}_{\mu\nu}$, coming from the natural contraction of the Riemann curvature

$$\mathcal{R}_{\mu\nu}\left(\Gamma\right)=\mathcal{R}_{\mu\rho}{}^{\rho}{}_{\nu}\left(\Gamma\right), \tag{21}$$

whereas, the second comes from the antisymmetric part of the connection, specifically, from the contraction of two torsion tensors, defined as

$$\mathcal{P}_{\mu\nu}=\left(\mathcal{B}_{\alpha}{}^{\beta}{}_{\mu}+\delta^{\beta}_{[\alpha]\mathcal{A}_{\mu}}\right)\left(\mathcal{B}_{\beta}{}^{\alpha}{}_{\nu}+\delta^{\alpha}_{[\beta]\mathcal{A}_{\nu}}\right). \tag{22}$$

Therefore, we can use Eq. (21) or Eq. (22) to define the notion of distance, and allowing a classification of vectors into time-lie, space-lie or null-like.[2]

[2]This is only valid when the tensors are well behaved, meaning that, they can not be degenerate, they must be invertible.

5 Final remarks

References

[1] Castillo-Felisola, Oscar, Orellana, Oscar, Perdiguero, José, Ramírez, Francisca, Skirzewski, Aureliano, and Zerwekh, Alfonso R. Aspects of the polynomial affine model of gravity in three dimensions - with focus in the cosmological solutions. *Eur. Phys. J. C*, 82(1):8, 2022.