Spherically and static solutions of Polynomial Affine Gravity in the torsion-free sector

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In the following subsections, I present a brief introduction to the polynomial affine model of gravity, how to build the ansatz compatible with the spherical symmetry and the field equations.

2.1 Model review

The Polynomial Affine Gravity uses the affine connection to mediated gravitational interactions, instead of the metric tensor. To build the action, we decompose the affine connection into its irreducible fields

$$\begin{aligned}\hat{\Gamma}_\alpha{}^\beta{}_\gamma &= \hat{\Gamma}_{(\alpha}{}^\beta{}_{\gamma)} + \hat{\Gamma}_{[\alpha}{}^\beta{}_{\gamma]},\\ &= \Gamma_\alpha{}^\beta{}_\gamma + \mathcal{B}_\alpha{}^\beta{}_\gamma + \delta^\beta_{[\gamma}\mathcal{A}_{\alpha]},\end{aligned} \tag{1}$$

where the first term Γ stands for the symmetric part and $\mathcal{A}$, $\mathcal{B}$ fields are related to the skew symmetric part of the affine connection.

Additionally, is necessary to introduce a volume form without the use of the metric tensor. In order to achieve this, we used the volume element defined as $\mathrm{d}V^{\alpha\beta\gamma\delta} = \mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \wedge \mathrm{d}x^\gamma \wedge \mathrm{d}x^\delta$, which is completely antisymmetric.

Moreover, we want to preserve the invariance under diffeomorphism, which is why the symmetric part of the affine connection, goes indirectly through the covariant derivative ∇^Γ.

The action is built up using a sort of *dimensional analysis* technique. This strategy allows us to generate every scalar density composed by powers of the irreducible fields of the affine connection. This method has been using to build the action in four dimensions, see Refs. and in three dimensions Ref [].

The most general action (up to topological invariants and boundary terms) in four dimensions is given by

$$\begin{aligned}S = \int \mathrm{d}V^{\alpha\beta\gamma\delta}\Big[& B_1\mathcal{R}_{\mu\nu}{}^\mu{}_\rho\mathcal{B}_\alpha{}^\nu{}_\beta\mathcal{B}_\gamma{}^\rho{}_\delta + B_2\mathcal{R}_{\alpha\beta}{}^\mu{}_\rho\mathcal{B}_\gamma{}^\nu{}_\delta\mathcal{B}_\mu{}^\rho{}_\nu + B_3\mathcal{R}_{\mu\nu}{}^\mu{}_\alpha\mathcal{B}_\beta{}^\nu{}_\gamma\mathcal{A}_\delta\\ & + B_4\mathcal{R}_{\alpha\beta}{}^\sigma{}_\rho\mathcal{B}_\gamma{}^\rho{}_\delta\mathcal{A}_\sigma + B_5\mathcal{R}_{\alpha\beta}{}^\rho{}_\rho\mathcal{B}_\gamma{}^\sigma{}_\delta\mathcal{A}_\sigma + C_1\mathcal{R}_{\mu\alpha}{}^\mu{}_\nu\nabla_\beta\mathcal{B}_\gamma{}^\nu{}_\delta\\ & + C_2\mathcal{R}_{\alpha\beta}{}^\rho{}_\rho\nabla_\sigma\mathcal{B}_\gamma{}^\sigma{}_\delta + D_1\mathcal{B}_\nu{}^\mu{}_\lambda\mathcal{B}_\mu{}^\nu{}_\alpha\nabla_\beta\mathcal{R}_\gamma{}^\lambda{}_\delta + D_2\mathcal{B}_\alpha{}^\mu{}_\beta\mathcal{B}_\mu{}^\lambda{}_\nu\nabla_\lambda\mathcal{B}_\gamma{}^\nu{}_\delta\\ & + D_3\mathcal{B}_\alpha{}^\mu{}_\nu\mathcal{B}_\beta{}^\lambda{}_\gamma\nabla_\lambda\mathcal{B}_\mu{}^\nu{}_\delta + D_4\mathcal{B}_\alpha{}^\lambda{}_\beta\mathcal{B}_\gamma{}^\sigma{}_\delta\nabla_\lambda\mathcal{A}_\sigma + D_5\mathcal{B}_\alpha{}^\lambda{}_\beta\mathcal{A}_\sigma\nabla_\lambda\mathcal{B}_\gamma{}^\sigma{}_\delta\\ & + D_6\mathcal{B}_\alpha{}^\lambda{}_\beta\mathcal{A}_\gamma\nabla_\lambda A_\delta + D_7\mathcal{B}_\alpha{}^\lambda{}_\beta\mathcal{A}_\lambda\nabla_\gamma A_\delta + E_1\nabla_\rho\mathcal{B}_\alpha{}^\rho{}_\beta\nabla_\sigma\mathcal{B}_\gamma{}^\sigma{}_\delta\\ & + E_2\nabla_\rho\mathcal{B}_\alpha{}^\rho{}_\beta\nabla_\gamma\mathcal{A}_\delta + F_1\mathcal{B}_\alpha{}^\mu{}_\beta\mathcal{B}_\gamma{}^\sigma{}_\delta\mathcal{B}_\mu{}^\lambda{}_\rho\mathcal{B}_\sigma{}^\rho{}_\lambda + F_2\mathcal{B}_\alpha{}^\mu{}_\beta\mathcal{B}_\gamma{}^\nu{}_\lambda\mathcal{B}_\delta{}^\lambda{}_\rho\mathcal{B}_\mu{}^\rho{}_\nu\\ & + F_3\mathcal{B}_\nu{}^\mu{}_\lambda\mathcal{B}_\mu{}^\nu{}_\alpha\mathcal{B}_\beta{}^\lambda{}_\gamma\mathcal{A}_\delta + F_4\mathcal{B}_\alpha{}^\mu{}_\beta\mathcal{B}_\gamma{}^\nu{}_\delta\mathcal{A}_\mu\mathcal{A}_\nu\Big].\end{aligned} \tag{2}$$

Notice the Riemann curvature and the Ricci tensor are defined with respect to the symmetric part of the connection.

The action written in Eq. (14) is purely affine and does not required the existence of a metric tensor to be defined. Additionally is polynomial in the connection and its covariant derivative, unlike the Einstein-Hilbert action, where there is the factor $\sqrt{-g}$.

As a consequence of the lack of metric tensor in our model, the numbers of terms that can go to the action, is limited due to the geometrical constraint coming from its formulation. We call this property, the *rigidity* of the model.

Moreover, is possible to coupled a scalar field using the *dimensional analysis* technique. This provide a non-standard procedure to coupled the *kinetic term* of the scalar field to the irreducible fields coming from the antisymmetric part of the affine connection, the covariant derivative and the volume form, without the use of a metric tensor. The effects of a scalar field in polynomial affine gravity has been studied in Ref.[].

Interestingly, all coupling constant are dimensionless, which suggest some sort of conformal symmetry, at least at a classical level, and also indicates that the model is power-counting renormalizable. This, is a necessary condition but not sufficient condition to guarantee that the model is renormalizable.

Finally, in the torsion-free limit it is possible to recover all Einstein vacuum solutions, meaning that it is a subspace of solutions of of polynomial affine gravity.

2.2 Building the ansatz

To build up the ansatz of the affine connection Γ, we compute its Lie derivative $\mathcal{L}_{\xi_j}\Gamma_\alpha{}^\beta{}_\gamma$ along the Killing vectors ξ_j that generate the desired symmetry, in this case a spherical symmetry. The Lie derivative of a a connection is written as

$$\mathcal{L}_{\xi_i}\Gamma_\alpha{}^\beta{}_\gamma = \xi_i^\delta\partial_\delta\Gamma_\alpha{}^\beta{}_\gamma - \Gamma_\alpha{}^\delta{}_\beta\partial_\delta\xi^\beta + \Gamma_\alpha{}^\beta{}_\delta\partial_\gamma\xi^\delta + \Gamma_\delta{}^\beta{}_\gamma\partial_\alpha\xi^\delta + \frac{\partial^2\xi^\beta}{\partial x^\alpha\partial x^\gamma}, \tag{3}$$

notice this is the standard definition of a Lie derivative of a tensor $(1,2)$, however, there is an extra term (non-homogeneous), because the affine connection does not transform as a tensor. The Killing vectors are

$$\xi_1 = \sigma\left(0, \cos\phi\sin\theta, \frac{\cos\phi\cos\theta}{r}, -\frac{\sin\phi}{r\sin\theta}\right), \tag{4}$$

$$\xi_2 = \sigma\left(0, \sin\phi\sin\theta, \frac{\sin\phi\cos\theta}{r}, \frac{\cos\phi}{r\sin\theta}\right), \tag{5}$$

$$\xi_3 = \sigma\left(0, \cos\theta, -\frac{\sin\theta}{r}, 0\right), \tag{6}$$

where σ is defined as

$$\sigma = \sqrt{1-\kappa r^2}. \tag{7}$$

This procedure has been cover in Refs. [], and, an explicit computation of every term can be found in Ref.[]

The affine connection is completely defined by twelve function[1], where each

[1] This is valid in the affine geometry without torsion. If we introduce a non trivial torsion field, the affine connection is defined by twenty time and radial dependent functions.

function has a time and radial dependence as follow

$$
\begin{array}{llll}
\Gamma_t{}^t{}_t = V(t,r) & \Gamma_t{}^r{}_t = B(t,r) & \Gamma_t{}^\theta{}_\theta = Z(t,r) & \Gamma_t{}^\phi{}_\theta = \dfrac{D(t,r)}{\sin\theta} \\
\Gamma_t{}^t{}_r = A(t,r) & \Gamma_t{}^r{}_r = Y(t,r) & \Gamma_t{}^\theta{}_\phi = -D(t,r)\sin\theta & \Gamma_t{}^\phi{}_\phi = Z(t,r) \\
\Gamma_r{}^t{}_r = W(t,r) & \Gamma_r{}^r{}_r = C(t,r) & \Gamma_r{}^\theta{}_\theta = G(t,r) & \Gamma_r{}^\phi{}_\theta = \dfrac{H(t,r)}{\sin\theta} \\
\Gamma_\theta{}^t{}_\theta = X(t,r) & \Gamma_\theta{}^r{}_\theta = F(t,r) & \Gamma_r{}^\theta{}_\phi = -H(t,r)\sin\theta & \Gamma_r{}^\phi{}_\phi = G(t,r) \\
\Gamma_\phi{}^t{}_\phi = X(t,r)\sin^2\theta & \Gamma_\phi{}^r{}_\phi = F(t,r)\sin^2\theta & \Gamma_\phi{}^\theta{}_\phi = -\cos\theta\sin\theta & \Gamma_\theta{}^\phi{}_\phi = \dfrac{\cos\theta}{\sin\theta}
\end{array}
\tag{8}
$$

Notice that, under a parametrization of the time coordinate, the first coefficient can be set equal to zero. This transformation has been extensively use in the frame of cosmology and can also be applied to the spherical case. For more information on this type of transformation, refer to Ref. [1].

The above ansatz can be simplified even further by imposing additional symmetries on the affine connection. First, lets consider time reversal

$$\nabla_t e_t = e_t \Gamma_t{}^t{}_t + e_r \Gamma_t{}^r{}_t \tag{9}$$

$$\nabla_{-t}\left(-e_t\right) = -e_t \Gamma_t{}^t{}_t + e_r \Gamma_t{}^r{}_t \tag{10}$$

the consistent condition requires that $\Gamma_t{}^t{}_t = 0$. Applying the same principle to the other basis vectors, then

$$\Gamma_t{}^j{}_i = 0 \qquad\qquad \Gamma_i{}^t{}_j = 0 \tag{11}$$

where i, j are restricted to space index.

Next, we demand an azimuthal angle symmetry, in which case the

$$\Gamma_\phi{}^\phi{}_\phi = 0 \qquad\qquad \Gamma_t{}^j{}_\phi = 0 \qquad\qquad \Gamma_i{}^\phi{}_j = 0 \tag{12}$$

Finally, we restrict the affine coefficient to be time independent (static). The final form of the ansatz is written as

$$
\begin{array}{lll}
\Gamma_t{}^t{}_r = A(r) & \Gamma_t{}^r{}_t = B(r) & \Gamma_r{}^r{}_r = C(r) \\
\Gamma_\theta{}^r{}_\theta = F(r) & \Gamma_\phi{}^r{}_\phi = F(r)\sin^2\theta & \Gamma_r{}^\theta{}_\theta = G(r) \\
\Gamma_\phi{}^\theta{}_\phi = -\cos\theta\sin\theta & \Gamma_r{}^\phi{}_\phi = G(r) & \Gamma_\theta{}^\phi{}_\phi = \dfrac{\cos\theta}{\sin\theta}
\end{array}
\tag{13}
$$

Notice that, originally we have twelve time and radial dependence, which was reduced to only five radial dependent functions.

2.3 Field equations

In the torsion-free limit, the only non trivial contribution are coming from the terms that are liner in the irreducible fields of the torsion tensor $\mathcal{A}$ and $\mathcal{B}$. In this case, the effective action is written as

$$S = \int \mathrm{d}V^{\alpha\beta\gamma\delta}\left[C_1 \mathcal{R}_{\mu\alpha}{}^{\mu}{}_{\nu} \nabla_\beta \mathcal{B}_\gamma{}^\nu{}_\delta\right], \tag{14}$$

whose variation with respect to the $\mathcal{B}$ leads to the field equation

$$\nabla_{[\sigma}\mathcal{R}_{\mu]\nu} = 0, \tag{15}$$

where is said that the Ricci tensor is a Codazzi tensor. From the field equation written in Eq. (15) we distinguish three branches of solutions: the first type of solutions requires the Ricci tensor to vanish, meaning $\mathcal{R}_{\mu\nu} = 0$, which written using Eq. (13) leads to

$$\frac{\partial B}{\partial r} + B\left(2G + C - A\right) = 0, \tag{16}$$

$$\frac{\partial A}{\partial r} + 2\frac{\partial G}{\partial r} + A^2 + 2G^2 - AC - 2CG = 0, \tag{17}$$

$$\frac{\partial F}{\partial r} + F\left(A + C\right) + 1 = 0, \tag{18}$$

where we have three first order differential equations for five unknown functions. Therefore, the system is underdetermined.

The second type of solution is the subspace of parallel Ricci, which implies that $\nabla_\sigma\mathcal{R}_{\mu\nu} = 0$, whose field equation under the ansatz in Eq. (13) can be written as follow

$$0 = \frac{\partial^2 B}{\partial r^2} - \frac{\partial B}{\partial r}\left(3A - C - 2G\right) + B\left(\frac{\partial C}{\partial r} + 2\frac{\partial G}{\partial r} - \frac{\partial A}{\partial r} + 2A\left(A - C - 2G\right)\right), \tag{19}$$

$$0 = 2B\frac{\partial G}{\partial r} - A\frac{\partial B}{\partial r} + B\frac{\partial A}{\partial r} - 2GB\left(A + C\right) + 2BG^2 - 2ABC + 2A^2B, \tag{20}$$

$$0 = 2\frac{\partial^2 G}{\partial r^2} + \frac{\partial^2 A}{\partial r^2} - 2\frac{\partial G}{\partial r}\left(3C - 2G\right) - \frac{\partial C}{\partial r}\left(A + 2G\right) + \frac{\partial A}{\partial r}\left(2A - 3C\right) - 2AC\left(A - C\right) - 4CG\left(G - C\right), \tag{21}$$

$$0 = 2F\frac{\partial G}{\partial r} + F\frac{\partial A}{\partial r} - G\frac{\partial F}{\partial r} + 2FG^2 + FA\left(A - C\right) - G\left(F\left(A + 3C\right) + 1\right), \tag{22}$$

$$0 = \frac{\partial^2 F}{\partial r^2} + \frac{\partial F}{\partial r}\left(A + C - 2G\right) + F\left(\frac{\partial A}{\partial r} + \frac{\partial C}{\partial r} - 2G\left(A + C\right)\right) - 2G, \tag{23}$$

on this particular branch, we have five differential equation for five unknown functions.

The third type is to solve directly $\nabla_{[\sigma}\mathcal{R}_{\mu]\nu} = 0$, which is known as *harmonic curvature*. Using Eq. (13), the *harmonic curvature* is defined as

$$\frac{\partial^2 B}{\partial r^2} + B\left(\frac{\partial C}{\partial r} - 2\frac{\partial A}{\partial r}\right) - \frac{\partial B}{\partial r}\left(2A - C - 2G\right) - 2GB\left(A - C + G\right) = 0, \tag{24}$$

$$\frac{\partial^2 F}{\partial r^2} + F\left(\frac{\partial C}{\partial r} - 2\frac{\partial G}{\partial r}\right) + \frac{\partial F}{\partial r}\left(A + C - G\right) + F\left(G\left(C - A - 2G\right) - A\left(A - C\right)\right) - G = 0, \tag{25}$$

where there are only two independent equations for five unknown functions. Therefore, the system is underdetermined.

3 Spherical solutions

In this section we explore the space and subspaces of solutions to the field equation in the torsion-free limit.

3.1 Ricci flat solutions

3.2 Parallel Ricci solutions

The subspace of solutions knowns as parallel Ricci solutions, requires a vanishing covariant derivative of the Ricci tensor

$$\nabla_\alpha \mathcal{R}_{\mu\nu} = 0. \tag{26}$$

The above differential equation accepts a non trivial solution in the form of

$$\mathcal{R}_{\mu\nu} = \Lambda k_{\mu\nu}, \tag{27}$$

where Λ is an integration constant and $k_{\mu\nu}$ is a second rank two tensor which is symmetric in its lower index. The most general symmetric rank two tensor compatible with the spherical symmetry can be built using the same method which was used to build the affine connection

$$k_{tt} = a(r), \qquad k_{rr} = b(r), \qquad k_{\theta\theta} = c(r), \tag{28}$$

or as a line element

$$a(r)\mathrm{d}t^2 + b(r)\mathrm{d}r^2 + c(r)\mathrm{d}\Omega^2, \tag{29}$$

where $\mathrm{d}\Omega^2$ is the standar area of the 2-sphere.

In order to have a consistent solution, the covariant derivative of $k_{\mu\nu}$ must be trivial.

$$-2Aa + a' = 0, \qquad -Aa - Bb = 0, \qquad -2Cb + b' = 0, \tag{30}$$

$$-Fb - Gc = 0, \qquad -2Gc + c' = 0. \tag{31}$$

The above differential equations can be solved simultaneously allowing us to find a relation between the connection coefficientes $A(r)$, $B(r)$, $C(r)$, $F(r)$ and $G(r)$ in terms of the components of the metric tensor as follow

$$A(r) = \frac{a'}{2a}, \qquad B(r) = -\frac{a'}{2b}, \qquad C(r) = \frac{b'}{2b}, \tag{32}$$

$$G(r) = \frac{-c'}{2c}, \qquad F(r) = -\frac{c'}{2b}. \tag{33}$$

The above relations ensures that $k_{\mu\nu}$ is also parallel. Now, it is necessary to solve Eq. (27),

$$\frac{bc(a')^2 - 2abca'' + aca'b' - 2aba'c'}{4ab^2c} = \Lambda a, \tag{34}$$

$$\frac{\left(bc^2(a')^2 - 2abc^2a'' + ac^2a'b' + 2a^2cb'c' + 2a^2b(c')^2 - 4a^2bcc''\right)}{4a^2bc^2} = \Lambda b, \tag{35}$$

$$\frac{4ab^2 - 2abc'' - c'\left(ba' - ab'\right)}{4ab^2} = \Lambda c. \tag{36}$$

The above system of differential equations admits a parametrized solution in terms of the $c(r)$ function. To see this, we find an expression for a'' from Eqs. (34) and (35) respectively

$$a'' = \frac{1}{2}\left(-4\Lambda ab + \frac{(a')^2}{a} + a'\left(\frac{b'}{b} - \frac{2c'}{c}\right)\right), \tag{37}$$

$$a'' = \frac{1}{2}\left(\frac{(a')^2}{a} + \frac{a'b'}{b} + 2a\left(-2\Lambda b + \frac{b'c'}{bc} + \frac{(c')^2 - 2cc''}{c^2}\right)\right), \tag{38}$$

by comparison, one can obtain the following expression

$$a'c' + a\left(\frac{b'c'}{b} + \frac{(c')^2}{c} - 2c''\right) = 0, \tag{39}$$

the above expression can be written as a total derivative, from which is possible to find an expression for $b(r)$ as

$$b(r) = \frac{2e^{b_0}(c')^2}{ac}, \tag{40}$$

where b_0 is an integration constant. Now, the system is reduced to

$$8e^{b_0}\Lambda(c')^3 + 2cc'a'' + a'\left(3(c')^2 - 2cc''\right) = 0, \tag{41}$$

$$8\Lambda c' + \frac{e^{b_0}}{c}\left(2ca' + ac'\right) = 8. \tag{42}$$

Notice the second equations is a differential equation of first order which can be solved. We choose to solved the $a(r)$ function

$$a(r) = -\frac{8}{3}e^{b_0}\left(\Lambda c - 3\right) + \frac{a_0}{\sqrt{c}}. \tag{43}$$

For the particular case where $c(r) = r^2$ the solution allow us to recover the well known space of solutions Anti de Sitter and de Sitter. Moreover, by fixing $c(r) = r^2$ and taking a vanishing Λ it is possible to recover the Schwarzschild solution. Therefore, the space of solution of General Relativity in vacuum and with cosmological constant are a subspace of solution of Polynomial Affine Gravity.

3.3 Ricci as a codazzi tensor

Under the spherical ansatz there are two differential equations to be solved for the five unknown affine functions. The system of differential equations can be written in a more convenient form

$$B'' + B'(C + 2G - 2A) + B(2G(C - A - G) - 2A' + C') = 0, \tag{44}$$

$$F'' + F'(A + C - G) + F\left(C' - 2G^2 + (A + G)(C - A)\right) = G, \tag{45}$$

the above system resembles a damped harmonic oscillator subject to an external force, the above systems can be written as

$$B'' + B'\gamma_b + B\omega_b^2 = 0, \tag{46}$$

$$F'' + F'\gamma_f + F\omega_f^2 = G, \tag{47}$$

where γ_j stands for the dissipation factor and ω_j is the frequency, both factors are defined as follow

$$\gamma_b = C + 2G - 2A, \qquad \omega_b^2 = 2G(C - A - G) - 2A' + C', \tag{48}$$

$$\gamma_f = A + C - G, \qquad \omega_f^2 = C' - 2G^2 + (A + G)(C - A). \tag{49}$$

It is worth to emphasis that the mentioned factors are radial dependent. The system can not be solved analytically for the most general case, however, it is possible to find an exact solution if one restrict the factors.

Typically, the factors γ_j and ω_j are constante. In this study, we proposed relax the constant condition subject a constraint

$$\gamma_b(r) = b_i r^i, \qquad \omega_b(r) = \omega_f(r), \qquad \gamma_f(r) = f_j r^j. \tag{50}$$

The above three conditions lead to a system of differential equations

$$C + 2G - 2A = b_i r^i, \tag{51}$$

$$(A - G)(A - X) + 2G' = A', \tag{52}$$

$$A + C - G = f_j r^j. \tag{53}$$

The above system can be solved analytically by

$$A(r) = \frac{1}{3}\left(\frac{6i}{r} + 2r^j f_j + r^i b_i + \frac{6(j - i) f_j}{r\left(f_j - r^{i-j} b_i\right)}\right), \tag{54}$$

$$G(r) = \frac{1}{3}\left(\frac{6i}{r} + r^j f_j + 2r^i b_i + \frac{6(j - i) f_j}{r\left(f_j - r^{i-j} b_i\right)}\right), \tag{55}$$

$$C(r) = \frac{1}{3}\left(2r^j f_j + r^i b_i\right). \tag{56}$$

Although it is possible to solve the field equation for the $B(r)$ function under the assumption of $i = j$, the differential equation for $F(r)$ can only be solved analytically for two specific cases $i = j = 0$ and $i = j = -1$.

The simples case is given by $i = j = 0$, for which, the functions Eqs. (54), (55) and (56) translate into

$$A(r) = \frac{1}{3}(\gamma_b + 2\gamma_f), \quad G(r) = \frac{1}{3}(2\gamma_b + \gamma_f), \quad C(r) = \frac{1}{3}(\gamma_b + 2\gamma_f). \tag{57}$$

Then the Eqs. (46) and (47) take the form of

$$B'' + B'\gamma_b - \frac{2}{9}B(\gamma_b + \gamma_f)^2 = 0, \tag{58}$$

$$F'' + F'\gamma_f - \frac{2}{9}F(\gamma_b + \gamma_f)^2 = \frac{2\gamma_b + \gamma_f}{3}. \tag{59}$$

The above systems admits an analytical solution

$$B(r) = e^{-\frac{r}{6}(3\gamma_b + \alpha)}\left(B_1 + e^{\frac{r\alpha}{3}}B_2\right) \tag{60}$$

$$F(r) = e^{-\frac{r}{6}(3\gamma_f + \beta)}\left(F_1 + e^{\frac{r\beta}{3}}F_2\right) - \frac{3}{2(2\gamma_b + \gamma_f)} \tag{61}$$

where B_1, B_2, F_1 and F_2 are integration constant and α and β are defined as

$$\alpha = \sqrt{41\gamma_b^2 + 32\gamma_b\gamma_f + 8\gamma_f^2} \tag{62}$$

$$\beta = \sqrt{32\gamma_b^2 + 32\gamma_b\gamma_f + 17\gamma_f^2} \tag{63}$$

The second case where is possible to find an analytical solution is $i = j = -1$ the Eqs. (46) and (47) are converted to

$$B'' + B'\left(\frac{9\gamma_b}{r}\right) - B\left(\frac{8\gamma_b^2 + 2(-15 + \gamma_f)(-6 + \gamma_f) + \gamma_b(-75 + 8\gamma_b)}{r^2}\right) = 0, \tag{64}$$

$$F'' + F'\left(\frac{\gamma_f}{r}\right) - F\left(\frac{8\gamma_b^2 + 2(-15 + \gamma_f)(-6 + \gamma_f) + \gamma_b(-75 + 8\gamma_b)}{9r}\right) = \frac{2\gamma_b - \gamma_f - 6}{3r}. \tag{65}$$

The above differential equations can be solved analytically by the following functions

$$B(r) = r^{\frac{1}{6}(3 - 3\gamma_b - \delta)}B_1 + r^{\frac{1}{6}(3 - 3\gamma_b + \delta)}B_2, \tag{66}$$

$$F(r) = r^{\frac{1}{6}(3 - 3\gamma_f - \delta)}F_1 + r^{\frac{1}{6}(3 - 3\gamma_b + \delta)}F_2 - \frac{3r(2\gamma_b + \gamma_f - 6)}{180 + 8\gamma_b^2 + \gamma_f(2\gamma_f - 51) + \gamma_b(8\gamma_f - 75)} \tag{67}$$

where δ is a constant defined as

$$\delta = \sqrt{729 + 41\gamma_b^2 + 8\gamma_f(\gamma_f - 21) + \gamma_b(32\gamma_f - 318)} \tag{68}$$

4 Analysis of the solutions

Building the emergent metric in the space of solutions, for the special case $i = j = 0$

$$\mathcal{R}_{tt} = \frac{1}{6} e^{\frac{-r}{6}(3\gamma_b+\alpha)} \left((5\gamma_b + 4\gamma_f) \left(B_1 + e^{\frac{r\alpha}{3}} B_2 \right) + \alpha \left(e^{\frac{r\alpha}{3}} B_2 - B_1 \right) \right) \tag{69}$$

$$\mathcal{R}_{rr} = \frac{2}{9} (\gamma_b - \gamma_f)(2\gamma_b + \gamma_f) \tag{70}$$

$$\mathcal{R}_{\theta\theta} = \frac{1}{6} \left(e^{\frac{-r}{6}(3\gamma_b+\beta)} \left(F_1 + e^{\frac{r\beta}{3}} F_2 \right) (4\gamma_b + 5\gamma_f) - \beta \left(F_1 e^{\frac{-r}{6}(3\gamma_b+\beta)} + F_2 e^{\frac{r}{6}(-3\gamma_b+\beta)} \right) + \frac{18\gamma_b}{2\gamma_b + \gamma_f} - 6 \right) \tag{71}$$

By taking the limit $r \to \infty$ leads to

$$\mathcal{R}_{tt}(r \to \infty) = (\pm)\infty \tag{72}$$

$$\mathcal{R}_{rr}(r \to \infty) = \frac{2}{9} (\gamma_b - \gamma_f)(2\gamma_b + \gamma_f) \tag{73}$$

$$\mathcal{R}_{\theta\theta}(r \to \infty) = (\pm)\infty \tag{74}$$

notice this space is not quit exactly a Anti de Sitter/de Sitter space. The components $\mathcal{R}_{tt}(r \to \infty)$ and $\mathcal{R}_{\theta\theta}$ has the desire structure, and by fixing the constants signs we get what we expect. However, $\mathcal{R}_{rr}(r \to \infty)$ leads to a constant, because of the radial absence dependence.

The second emergent metric tensor comes from the special case $i = j = -1$. The components of the Ricci tensor are

$$\mathcal{R}_{tt} = \frac{1}{6} r^{\frac{1}{6}(-3-3\gamma_b-\delta)} \left(B_2 r^{\frac{\delta}{3}} - B_1 \right) \delta, \tag{75}$$

$$\mathcal{R}_{rr} = \frac{162 + 4\gamma_b^2 - 2\gamma_f (12 + \gamma_f) - \gamma_b (57 + 2\gamma_f)}{9r^2}, \tag{76}$$

$$\mathcal{R}_{\theta\theta} = \frac{162 + 4\gamma_b^2 - 2\gamma_f (12 + \gamma_f) - \gamma_b (57 + 2\gamma_f)}{180 + 8\gamma_b^2 + \gamma_f (2\gamma_f - 51) + \gamma_b (8\gamma_f - 75)} + \frac{\delta}{6} \left(F_2 r^{\frac{1}{6}(-3-3\gamma_f+\delta)} - F_1 r^{\frac{1}{6}(-3-3\gamma_f-\delta)} \right). \tag{77}$$

By taking the limit when $r \to \infty$

$$\mathcal{R}_{tt}(r \to \infty) = 0, (\pm)\infty \tag{78}$$

$$\mathcal{R}_{rr}(r \to \infty) = 0 \tag{79}$$

$$\mathcal{R}_{\theta\theta}(r \to \infty) = 0, (\pm)\infty \tag{80}$$

it is possible to recover the classical space of solutions of Anti de Sistter/de Sitter, however, there is room for a different configurations. Unlike the first case, where is not possible to recover the well known space of solutions, here it is possible.

5 Auto parallel curves and numerical solutions

5.1 Formulation problem

5.2 Numerical solution using PINNs

6 Final remarks

References

[1] Castillo-Felisola, Oscar, Orellana, Oscar, Perdiguero, José, Ramírez, Francisca, Skirzewski, Aureliano, and Zerwekh, Alfonso R. Aspects of the polynomial affine model of gravity in three dimensions - with focus in the cosmological solutions. *Eur. Phys. J. C*, 82(1):8, 2022.