SET readout using RF reflectometry and kinetic inductance nonlinearity

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Abstract

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$$\Gamma = \frac{Z - Z_0}{Z + Z_0} \tag{2.1}$$

2.4 Kinetic inductance and his nonlinearity

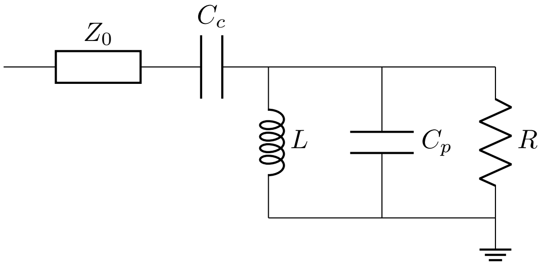

Figure 1: Topology of the resonator that we are going to use. C_p and R_p are a virtual capacitor and resistance used to model losses in the circuit, while R_{SET} is the resistance of the SET in any state. That leaves C_c and L as the degrees of freedom in our system.

3 The parallel RLC resonator

Now that we have a good theoretical context of all the parts of the problem, we will start by analyzing the resonator with a non-kinetic inductance.

3.1 Resonant frequency and effective impedance

Our analysis begins with obtaining expressions for the resonant frequency and the effective impedance of our resonator. It's easy to see that the impedance of our resonator in figure 1 is

$$Z = \frac{1}{j\omega C_p + \frac{1}{j\omega L} + \frac{1}{R}} + \frac{1}{j\omega C_c} \tag{3.1}$$

Which after a little massaging turns into

$$Z = \frac{\omega^2 L^2 R}{R^2(1-\omega^2 C_p L)^2 + \omega^2 L^2} + j\left(\frac{\omega L R^2 (1-\omega^2 C_p L)}{R^2(1-\omega^2 C_p L)^2 + \omega^2 L^2} - \frac{1}{\omega C_c}\right) \tag{3.2}$$

The resonant frequency ω_r that makes $\operatorname{Im} Z = 0$ is

$$\omega_r^2 = \frac{1}{L(C_c + C_p)}\left(1 + \frac{C_c}{2C_p} - \frac{L}{2R^2 C_p} \pm \sqrt{\left(1 + \frac{C_c}{2C_p} - \frac{L}{2R^2 C_p}\right)^2 - 1 - \frac{C_c}{C_p}}\right) \tag{3.3}$$

Choosing C_c and L such that $\frac{C_c}{C_p}, \frac{L}{R^2 C_p} \ll 1$, leaves us with the approximate expression for the resonant frequency

$$\omega_r \approx \frac{1}{\sqrt{L(C_c + C_p)}} \tag{3.4}$$

Finally, to obtain the effective impedance we use this expression in $\operatorname{Re} Z$

$$Z_{eff} = \operatorname{Re} Z(\omega_r) = \frac{\omega_r^2 L^2 R}{R^2(1-\omega_r^2 C_p L)^2 + \omega_r^2 L^2} \approx \frac{L(C_c+C_p)}{RC_c^2}\left(1+\frac{L(C_c+C_p)}{R^2C_c^2}\right)^{-1} \tag{3.5}$$

And by, again, choosing L and C_c such that $\frac{L(C_c+C_p)}{R^2C_c^2} \ll 1$ we arrive to our expression for the effective impedance

$$Z_{\text{eff}} \approx \frac{L(C_c+C_p)}{RC_c^2} \tag{3.6}$$

In future sections we will be using quite a lot of expressions obtained via approximations in non-approximated systems, only to do more approximations with them. Due to this, it is really important to have a clear picture of the regimes we are working in to ensure that our results work in the state-of-the-art technology, and that's why after each result we are going to recontextualize our approximations.

In this case, the approximations to obtain ω_r are clear and straight forward:

$$\frac{C_c}{C_p} \ll 1 \tag{3.7}$$

$$\frac{L}{R^2C_p} \ll 1 \tag{3.8}$$

But the approximation for Z_{eff} needs a little bit of extra work. If we multiply $(C_c/C_p)^2$ in both sides, it turns into

$$\frac{L}{R^2C_p}\left(1+\frac{C_c}{C_p}\right) \ll \left(\frac{C_c}{C_p}\right)^2 \tag{3.9}$$

And since we used equation 3.4 to arrive here, it must hold the approximation 3.7, turning the previous expression into

$$\frac{L}{R^2C_p} \ll \left(\frac{C_c}{C_p}\right)^2 \tag{3.10}$$

While approximations 3.7 and 3.8 impose a **general condition** in our degrees of freedom, approximation 3.10 imposes a **relative condition** between the previous two.

For checking that our results are correct, we can graph the modulus of the reflection coefficient Γ (equation 2.1) as a function of the voltage frequency ω. With the standard impedance for a transmission line $Z_0 = 50\Omega$, a parasitic capacitance $C_p = 500\text{fF}$ and a resistance $R = 50\text{k}\Omega$ (2 times the quantum of resistance), we need to choose a C_c and a L such that the conditions 3.7, 3.8 and 3.10 are met and $Z_{\text{eff}} = Z_0$. With this in mind, we choose $C_c = 100$fF and $L = 41.67$nH (these values are also on the ballpark of real values used in the lab **REFERENCE NEEDED**), which means that we should see $|\Gamma|$ dip to zero at a frequency of 1.00654GHz, which is what we see in the black line figure 2.

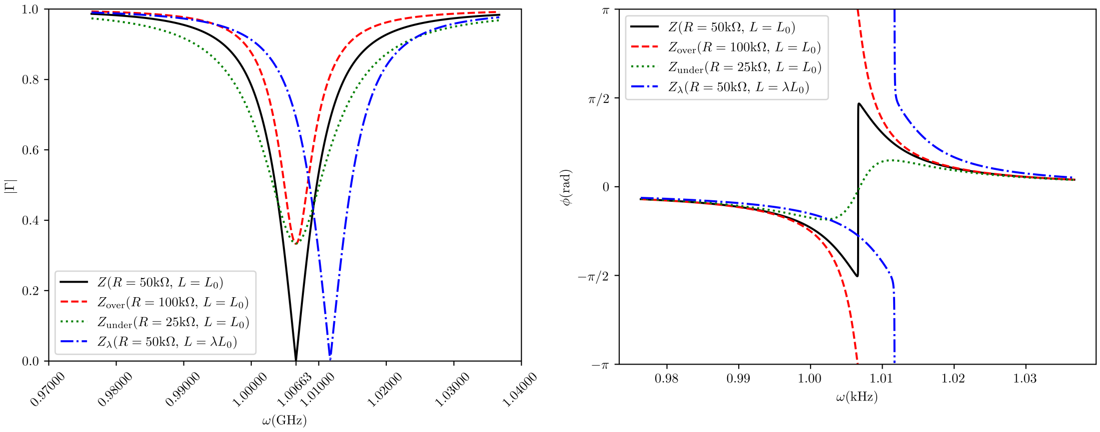

Figure 2: Modulus and phase of Γ in multiple configurations

In addition to this configuration, we have also graphed an over and an under coupled system, and one with a slight variation of the inductance ($\lambda = 0.99$).

3.2 Contrast and it's optimization

With an expression for the effective impedance of the system in resonance and (more importantly) an expression for the resonant frequency, now we ask ourselves the question: What are the values of L and C_c that maximize the contrast $|\Delta\Gamma| = |\Gamma(R_{\text{Off}}) - \Gamma(R_{\text{On}})|$?

Since on the lab the sizes of L and C_p are on the order of the ones used to produce figure 2, is easy to see that ω_r will be a lot more sensible to changes in L than to changes in C_c, and thus we will use C_c to optimize the contrast, while we will use L to ensure that we stay in an acceptable frequency of operation.

With this in mind, we begin obtaining a workable expression of the contrast by plugging expression 3.4 into 3.1 without considering any of the approximations related to 3.4:

$$
\begin{aligned}
Z &= \frac{1}{j\omega C_p + \frac{1}{j\omega L} + \frac{1}{R}} + \frac{1}{j\omega C_c} \\
&= \frac{\omega RL}{R(1-\omega^2 LC_p) + j\omega L} + \frac{1}{j\omega C_c} \\
&= \frac{\frac{jRL}{\sqrt{L(C_c+C_p)}}}{R\left(1 - \frac{\cancel{L}C_p}{\cancel{L}(C_c+C_p)}\right) + \frac{jL}{\sqrt{L(C_c+C_p)}}} + \frac{\sqrt{L(C_c+C_p)}}{jC_c} \\
&= \frac{jRL}{R\sqrt{L(C_c+C_p)}\left(\frac{C_c}{C_c+C_p}\right) + jL} + \frac{\sqrt{L(C_c+C_p)}}{jC_c} \\
&= \frac{jR\cancel{L}}{R\frac{\cancel{L(C_c+C_p)}}{\sqrt{L(C_c+C_p)}}\left(\frac{C_c}{\cancel{C_c+C_p}}\right) + j\cancel{L}} + \frac{\sqrt{L(C_c+C_p)}}{jC_c} \\
&= \frac{jR}{RS+j} + \frac{1}{jS} = \frac{\cancel{-RS} + \cancel{RS} + j}{jRS^2 - S} = \frac{1}{RS^2 + jS} \text{ with } S = \frac{C_c}{\sqrt{L(C_c+C_p)}}
\end{aligned}
\tag{3.11}
$$

Then we use this expression of the impedance to obtain the reflection coefficient, but using the admittance of the transmission line instead of the impedance ($Y_0 = 1/Z_0$)

$$
\begin{aligned}
\Gamma &= \frac{Z - Z_0}{Z + Z_0} = \frac{Y_0 - 1/Z}{Y_0 + 1/Z} \\
&= \frac{2Y_0}{Y_0 + 1/Z} - 1 = \frac{2Y_0}{RS^2 + Y_0 + jS} - 1 \\
&= 2Y_0 \frac{RS^2 + Y_0 - jS}{(RS^2 + Y_0)^2 + S^2} - 1
\end{aligned}
\tag{3.12}
$$

Since $\frac{1}{\sqrt{L(C_c+C_p)}} \approx \omega_r$ then $\operatorname{Im} Z \approx 0$ and by extension $\operatorname{Im} \Gamma \approx 0$, so

$$
\Gamma \approx 2Y_0 \frac{RS^2 + Y_0}{(RS^2 + Y_0)^2 + S^2} - 1 \tag{3.13}
$$

Next, using the parameters utilized for figure 2 to get a sense of the scale, it is safe to assume that the following approximation is correct

$$
(RS^2 + Y_0)^2 \gg S^2 \tag{3.14}
$$

Which leaves us with the following expression for the reflection coefficient

$$
\Gamma \approx \frac{2Y_0}{RS^2 + Y_0} - 1 \tag{3.15}
$$

And this one for the contrast

$$|\Delta\Gamma| = |\Gamma(R = R_{\text{Off}}) - \Gamma(R = R_{\text{On}})| \approx 2Y_0 \left| \frac{1}{R_{\text{Off}} S^2 + Y_0} - \frac{1}{R_{\text{On}} S^2 + Y_0} \right| \tag{3.16}$$

Now, thanks to this simplified form of the contrast, to obtain the optimum value for C_c we don't need any fancy tricks, just to derive with respect to C_c and equate to 0. Doing this we arrive at the equation

$$S^2 = \frac{Y_0}{\sqrt{R_{\text{Off}} R_{\text{On}}}} \tag{3.17}$$

And solving it for C_c, we get the single solution (for $R_{\text{On}}, R_{\text{Off}}, Y_0, L, C_p, C_c \geq 0$)

$$C_{c\text{Max}} = \frac{LY_0}{2\sqrt{R_{\text{Off}} R_{\text{On}}}} \left(1 + \sqrt{1 + 4C_p \frac{\sqrt{R_{\text{Off}} R_{\text{On}}}}{LY_0}} \right) \tag{3.18}$$

We could simply plug this result into a simulation and call it a day, but with a little bit more digging we can extract some interesting results.

First off, by the way the resistances appear in 3.17 and 3.18 it leads really naturally to defining a ratio parameter

$$\rho = \frac{R_{\text{Off}}}{R_{\text{On}}} \tag{3.19}$$

With it our expressions 3.17 and 3.18 turn to

$$S^2 = \frac{Y_0}{\sqrt{\rho} R_{\text{On}}} \tag{3.20}$$

$$C_{c\text{Max}} = \frac{LY_0}{2\sqrt{\rho} R_{\text{On}}} \left(1 + \sqrt{1 + 4C_p \frac{\sqrt{\rho} R_{\text{On}}}{LY_0}} \right) \tag{3.21}$$

Then, by using the definition of S from 3.11 and using impedance, we can rearrange 3.20 to

$$Z_0 = \frac{L(C_c + C_p)}{\sqrt{\rho} R_{\text{On}} C_c^2} \tag{3.22}$$

This might remind you of the expression 3.10, and it's easy to see the parallelisms: In the previous case, given a resistance R we can find a L and C_c (in the confines that the restrictions 3.8, 3.7 and 3.9 allow) that will make 3.10 equal to Z_0 and make Γ equal to 0. For the contrast is the exact same, except that for that given R it maximizes it instead of making it 0, and that we have 2 values of R, R_{On} and $R_{\text{Off}} = \rho R_{\text{On}}$, so which one do we use? Well it turns out that neither is the correct choice, it is $\sqrt{R_{\text{Off}} R_{\text{On}}} = \sqrt{\rho} R_{\text{On}}$, the geometric mean of the resistances.

In addition to this insight, we can also use 3.20 in our approximation for the contrast (3.14) to see that, when optimized, it only depends on the ratio of the resistances, and that has a maximum value of 2, which is expected:

$$|\Delta\Gamma| \approx 2Y_0 \left| \frac{1}{R_{\text{Off}} S^2 + Y_0} - \frac{1}{R_{\text{On}} S^2 + Y_0} \right| = 2 \left| \frac{1}{\sqrt{\rho} + 1} - \frac{\sqrt{\rho}}{1 + \sqrt{\rho}} \right| = 2 \left| \frac{1 - \sqrt{\rho}}{1 + \sqrt{\rho}} \right| \tag{3.23}$$

After these results it seems appropriate to analyze with more detail the approximations used, so we can contextualize the regime in which this works. The first approximation done was $\omega \approx \omega_r$, which boils down to 3.8 and 3.7. The second approximation was 3.14, so let's see if with the optimum C_c it holds. Using 3.20 we have

$$(RS^2 + Y_0)^2 \gg S^2 \rightarrow \left(\frac{RY_0}{\sqrt{\rho} R_{\text{On}}} + Y_0 \right)^2 \gg \frac{Y_0}{\sqrt{\rho} R_{\text{On}}} \tag{3.24}$$

Now, considering $R = R_{\text{On}}$ since it's the worst case scenario and returning to the use of impedance instead of admittance, the condition turns to

$$\left(\frac{1}{\sqrt{\rho}} + 1 \right)^2 \gg \frac{Z_0}{\sqrt{\rho} R_{\text{On}}} \tag{3.25}$$

Since by definition $\rho \leq 1$, we can take a stricter version for this approximation but still achievable

$$R_{\text{On}} \gg Z_0 \tag{3.26}$$

This is clearly true in our case. In theory. Because you see, we've been ignoring something up until now for the sake of simplicity, something that was mentioned at the beginning of this section: the parasitic resistance R_p. As was said in the description of figure 1, R_p is a virtual resistance added in parallel with $R_{textSET}$ to model losses in the circuit, and it can make $R < 50\text{k}\Omega$, so it is important that we keep it in mind.

Thankfully is easy to add it back retroactively (that's why we've ignored it up until now): we just need to do the following substitutions

$$
\begin{aligned}
R_{\text{On}} \to R'_{\text{On}} = R_{\text{On}} \parallel R_p &= \frac{R_{\text{On}} R_p}{R_{\text{On}} + R_p} \\
R_{\text{Off}} \to R'_{\text{Off}} = R_{\text{Off}} \parallel R_p &= \frac{R_{\text{Off}} R_p}{R_{\text{Off}} + R_p} \\
\rho \to \rho' = \frac{R'_{\text{Off}}}{R'_{\text{On}}}
\end{aligned}
$$

If we give the same treatment to R_p as to R_{Off} by introducing a ratio parameter

$$
\pi = \frac{R_p}{R_{\text{On}}} \tag{3.27}
$$

Then the substitutions are

$$
\begin{aligned}
R_{\text{On}} \to R'_{\text{On}} &= \frac{\pi}{1+\pi} R_{\text{On}} \\
R_{\text{Off}} \to R'_{\text{Off}} &= \frac{\rho\pi}{\rho+\pi} R_{\text{On}} \\
\rho \to \rho' &= \frac{\rho(1+\pi)}{\rho+\pi}
\end{aligned}
$$

Taking this even further beyond with the fact that in an SET $\rho \approx \infty$ (in the Off state, no electrons are travelling through), the substitutions are

$$
\begin{aligned}
R_{\text{On}} \to R'_{\text{On}} &= \frac{\pi}{1+\pi} R_{\text{On}} \\
R_{\text{Off}} \to R'_{\text{Off}} &= \pi R_{\text{On}} \\
\rho \to \rho' &= 1+\pi
\end{aligned}
$$

The introduction of the parasitic resistance and an infinite R_{Off} doesn't change much, in the sense that for most of the expressions is better to simply use the prime versions of ρ and R_{On} for clarity. Most. Because for two results in specific it helps: in 3.23

$$
|\Delta\Gamma| \approx 2 \left| \frac{1-\sqrt{1+\pi}}{1+\sqrt{1+\pi}} \right| \tag{3.28}
$$

And in 3.26

$$
\frac{\pi}{1+\pi} R_{\text{On}} \gg Z_0 \to \pi \gg \frac{Z_0}{R_{\text{On}} - Z_0} \tag{3.29}
$$

Using the values of Z_0 and R_{On} that we have been considering up until now (50Ω and $50\text{k}\Omega$ respectively) we can see that for 3.26 to work in a worse case scenario, π must be a lot greater than 10^{-3}. It probably would be, given that for a π 100 times greater, $|\Delta\Gamma| \approx 0.0477$, which isn't a good contrast to aim for.

Finally, we check our results by comparing them against the numerically calculated optimum contrast via a simulation that searches the optimum value of C_c for a given value of π.

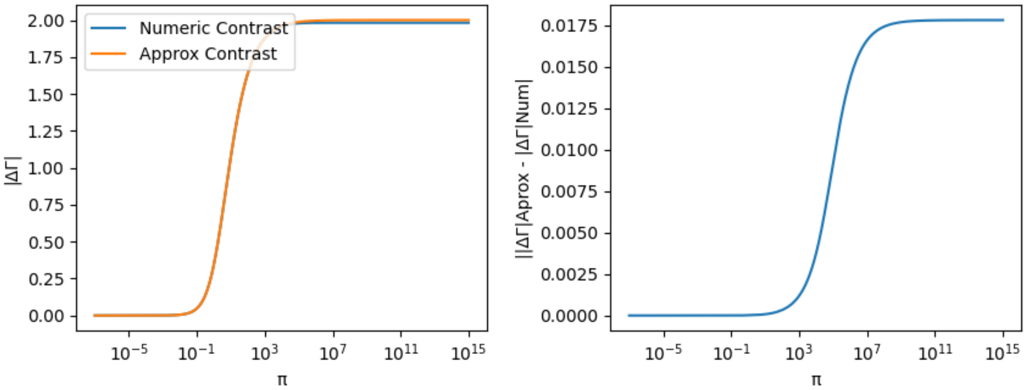

Figure 3: Numerical optimum contrast and our formula (3.28) with the difference between them. $L = 180\text{nH}$, $C_p = 500\text{fF}$, $Z_0 = 50\Omega$, $R_{\text{On}} = 50\text{k}\Omega$, $\rho = 2 \cdot 10^{-6}$.

As we can see in figure 3, even though the approximation gets worse for greater π, it caps off to a difference of 0.0175, which still makes it a pretty good approximation.

Next, we'll try to do the same analysis to a system with a variable inductance via a kinetic inductor.

4 The parallel kinetic RLC resonator

4.1 Impact of the nonlinearity in the contrast

5 Simulations of the models

6 Conclusions