10. Consider a positive frequency solution of the Dirac equation with $\mathbf{p} = \mathbf{0}$,

$$u(\mathbf{p}) = \sqrt{m}\begin{pmatrix}\xi \\ \xi\end{pmatrix} \quad ; \quad \xi = \text{constant}.$$

(a) Show that spatial rotations of $\psi(x) = u(\mathbf{p})e^{-ip\cdot x}$ with angles $\boldsymbol{\varphi}$ act on ξ as

$$\xi \longrightarrow e^{i\boldsymbol{\varphi}\cdot\boldsymbol{\sigma}/2}\xi$$

Solution (a)

If we consider rotations in the vector sense, they are generated by the $\tilde{\mathcal{M}}^{ij}$

$$\tilde{\mathcal{M}}^{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -\tilde{\mathcal{M}}^{21} \quad ; \quad \tilde{\mathcal{M}}^{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = -\tilde{\mathcal{M}}^{32}$$

$$\tilde{\mathcal{M}}^{31} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = -\tilde{\mathcal{M}}^{13},$$

so considering a Lorentz transformation with parameters $\Omega_{ij} = -\varepsilon_{ijk}\varphi^k$, we get that a vector transforms under a rotation of angle $\varphi = \sqrt{\varphi^i\varphi_i}$, along an axis of unit vector $\hat{\mathbf{n}} = \boldsymbol{\varphi}/\sqrt{\varphi^i\varphi_i}$ with the following matrix:

$$\tilde{\Lambda}_{rot} = \exp\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\varphi^3 & \varphi^2 \\ 0 & \varphi^3 & 0 & -\varphi^1 \\ 0 & -\varphi^2 & \varphi^1 & 0 \end{pmatrix} = \mathbb{1} + \sin\varphi(\hat{\mathbf{n}}\cdot\mathbf{J}) + (1-\cos\varphi)(\hat{\mathbf{n}}\cdot\mathbf{J})^2$$

$$J_i = -\frac{1}{2}\varepsilon_{ijk}\tilde{\mathcal{M}}^{jk}$$

On the other hand, spinors transform under other representation of the Lorentz group, generated by the matrices:

$$S^{\rho\sigma} \equiv \frac{1}{4}\left[\gamma^\rho, \gamma^\sigma\right] = \frac{1}{2}\gamma^\rho\gamma^\sigma - \frac{1}{2}\eta^{\rho\sigma}$$

If we now consider the generators of rotations we get that:

$$S^{ij} = \frac{1}{2}\gamma^\rho\gamma^\sigma = \frac{1}{2}\begin{pmatrix} 0 & \sigma^i \\ -\sigma & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} = -\frac{i}{2}\varepsilon_{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \quad : \quad i \neq j$$

Which means that under rotations, spinors transform by the multiplication of the following matrix

$$S[\Lambda]_{rot} = \exp\left[\frac{i}{4}\varepsilon_{ijk}\phi^k\varepsilon^{ijm}\begin{pmatrix} \sigma_m & 0 \\ 0 & \sigma_m \end{pmatrix}\right] = \exp\left[\frac{i}{2}\varphi^k\delta^m_k\begin{pmatrix} \sigma_m & 0 \\ 0 & \sigma_m \end{pmatrix}\right] = \begin{pmatrix} e^{i\boldsymbol{\varphi}\cdot\boldsymbol{\sigma}/2} & 0 \\ 0 & e^{i\boldsymbol{\varphi}\cdot\boldsymbol{\sigma}/2} \end{pmatrix}$$

□

(b) By taking $\varphi_1 = \varphi_2 = 0$ -i.e., performing rotations parallel to the axis x_3 - show that the ± 1 eigenstates of σ^3, that we commonly dub **spin up** and **spin down**, correspond, in terms of ψ, to taking

$$\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

respectively.

Solution (b)

If we consider the $\varphi_1 = \varphi_2 = 0$ case, considering $\varphi = \varphi_3$ then the rotation matrix would be:

$$S[\Lambda]_{rot} = \begin{pmatrix} e^{i\varphi/2} & 0 & 0 & 0 \\ 0 & e^{-i\varphi/2} & 0 & 0 \\ 0 & 0 & e^{i\varphi/2} & 0 \\ 0 & 0 & 0 & e^{-i\varphi/2} \end{pmatrix}$$

Then if we consider

$$S[\Lambda]_{rot}u(\mathbf{p}) = \begin{pmatrix} e^{i\varphi/2}\xi^1 \\ e^{-i\varphi/2}\xi^2 \\ e^{i\varphi/2}\xi^1 \\ e^{-i\varphi/2}\xi^2 \end{pmatrix} = \lambda u(\mathbf{p}),$$

we can easily see there exists the eigenstates and corresponding eigenvalue are:

$$\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad \lambda = e^{i\varphi/2} \quad ; \quad \xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad , \quad \lambda = e^{-i\varphi/2}$$

which are directly related (due to the block structure of the rotation matrix) to the eigenstates and eigenvalues of σ^3.

□

(c) Solutions at $\mathbf{p} \neq \mathbf{0}$ can be obtained by **boosting** the $\mathbf{p} = \mathbf{0}$ solutions above. In particular, boosts along the x^3 direction are generate by S^{03}, which leads to solutions with $\mathbf{p}^T = (0, 0, p^3)$. Prove that the boosted positive frequency solutions with spin up and down, respectively, have the form

$$u(\mathbf{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \sqrt{p\cdot\bar{\sigma}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \sqrt{E - p^3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \sqrt{E + p^3}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

$$u(\mathbf{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \sqrt{p\cdot\bar{\sigma}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \sqrt{E + p^3}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \sqrt{E - p^3}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

Solution (c)

If we consider the boost matrix in the x^3, in the spinor representation we get that, considering that the parameter for this boost must be $\Omega_{03} = -\Omega_{30} = \zeta$, and that $S^{03} = \frac{1}{2}\gamma^0\gamma^3$:

$$S[\Lambda]_{boost} = \exp\left[\frac{1}{2}p^3/E\begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}\right] = \sum_{n=0}^{\infty}\frac{(\zeta/2)^n}{n!}\begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}^n =$$

$$\sum_{k=0}^{\infty}\frac{(\zeta/2)^{2k}}{(2k)!}\begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}^{2k} + \sum_{j=0}^{\infty}\frac{(\zeta/2)^{2j+1}}{(2j+1)!}\begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}^{2j+1}$$

$$= \mathbb{1}\sum_{k=0}^{\infty}\frac{(\zeta/2)^{2k}}{(2k)!} + \begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}\sum_{j=0}^{\infty}\frac{(\zeta/2)^{2j+1}}{(2j+1)!}$$

$$= \mathbb{1}\cosh\frac{\zeta}{2} + \begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}\sinh\frac{\zeta}{2}$$

Now, if we recall from normal vectors we know that this parameter theta ζ fulfills the following relations:

$$E = m\cosh\zeta$$

$$p^3 = m\sinh\zeta$$

Additionally, considering the relations:

$$\cosh\frac{\zeta}{2} = \sqrt{\frac{\cosh\zeta + 1}{2}}$$

$$\tanh\frac{\zeta}{2} = \frac{\sinh\zeta}{\cosh\zeta + 1}$$

we can obtain that:

$$S[\Lambda]_{boost} = \cosh\frac{\zeta}{2}\left[\mathbb{1} + \tanh\frac{\zeta}{2}\begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}\right] = \sqrt{\frac{\cosh\zeta + 1}{2}}\left[\mathbb{1} + \frac{\sinh\zeta}{\cosh\zeta + 1}\begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}\right] =$$

$$= \sqrt{\frac{E+m}{2m}}\left[\mathbb{1} + \frac{p^3}{E+m}\begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}\right] = \sqrt{\frac{E+m}{2m}}\begin{pmatrix} 1-\frac{p^3}{E+m} & 0 & 0 & 0 \\ 0 & 1+\frac{p^3}{E+m} & 0 & 0 \\ 0 & 0 & 1+\frac{p^3}{E+m} & 0 \\ 0 & 0 & 0 & 1-\frac{p^3}{E+m} \end{pmatrix}$$

If we now apply it to a spin up solution:

$$S[\Lambda]_{boost}u(\mathbf{p})^{\uparrow} = \sqrt{\frac{E+m}{2}}\begin{pmatrix} 1-\frac{p^3}{E+m} \\ 0 \\ 1+\frac{p^3}{E+m} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{E+m-p^3}{\sqrt{2(E+m)}} \\ 0 \\ \frac{E+m+p^3}{\sqrt{2(E+m)}} \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{(E+m-p^3)^2}{2(E+m)}} \\ 0 \\ \sqrt{\frac{(E+m+p^3)^2}{2(E+m)}} \\ 0 \end{pmatrix}$$

Let's calculate the following:

$$(E+m+p^3)^2 = E^2 + m^2 + (p^3)^2 + 2Em + 2Ep^3 + 2mp^3 = 2E^2 + 2Em + 2Ep^3 + 2mp^3$$
$$= 2(E+m)(E+p^3)$$
$$(E+m-p^3)^2 = E^2 + m^2 + (p^3)^2 + 2Em - 2Ep^3 - 2mp^3 = 2E^2 + 2Em - 2Ep^3 - 2mp^3$$
$$= 2(E+m)(E-p^3)$$

Which means that, indeed:

$$S[\Lambda]_{boost}u(\mathbf{p})^{\uparrow} = \begin{pmatrix} \sqrt{E-p^3} \\ 0 \\ \sqrt{E+p^3} \\ 0 \end{pmatrix}$$

Conversely, if we apply the boost matrix to the spin down solution we get:

$$S[\Lambda]_{boost}u(\mathbf{p})^{\downarrow} = \begin{pmatrix} \sqrt{E+p^3} \\ 0 \\ \sqrt{E-p^3} \\ 0 \end{pmatrix}$$

$\square$

(a) Show that the boosted solutions have a well-defined massless limit $m \to 0$ (where $E = p^3$), and that

$$u(\mathbf{p}) = \begin{pmatrix} \sqrt{p \cdot \sigma} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \sqrt{p \cdot \bar{\sigma}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \to \sqrt{2E} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$u(\mathbf{p}) = \begin{pmatrix} \sqrt{p \cdot \sigma} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \sqrt{p \cdot \bar{\sigma}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \to \sqrt{2E} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

The existance of a well-defined $m \to 0$ limite is actually one of the reasons behind our choice of normalization $\xi^\dagger \xi = 1$.

Solution (d)

Considering the results obtained in section (c), this result is immediate. Indeed, we need a massless state to be boosted to be well defined, as there is no frame which defines a massless, zero-momentum particle -i.e a massless particle as a photon cannot be at rest.

□

10. Let $\psi^{(c)} = C\psi^*$ be the charge conjugate of the spin $1/2$ Dirac field ψ, where C is the charge conjugation matrix.