Fundamentos Lógicos y Algebraicos

Normalization of First-Order Formulas

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Two formulas F and F' are *equivalent* iff they are satisfied by the same *interpretations* and *valuations* [Mendelson97, pages 67-66]

Normalizing a formula as a set of clauses

Every *sentence* F can be *transformed* as a set $\mathcal{S}_F$ of clauses, see [DP60] and [CL73, Section 4.2]:

1. Put F into a *prenex normal form* F';
2. Put M in *conjunctive normal form* to obtain F'';
3. Remove *existential quantifiers* (*Skolemization*) from F'' to obtain F''';
4. Write F''' as *set of clauses* $\mathcal{S}_F$, a *normal form* of F.

The first two steps *preserve equivalence*

Sentences F, F', and F'' are *equivalent*.

In general, the whole *normalization* transformation *does not* preserve equivalence; but it preserves *unsatisfiability*, i.e.,

A sentence F is *unsatisfiable* iff F''' (or $\mathcal{S}_F$) is [CL73, Theorem 4.1].

Definition (Formula in prenex normal form)

A formula F is said to be in a *prenex normal form* iff F is of the form

$$(Q_1 x_1) \cdots (Q_q x_q) M,$$

where Q_i is either $\forall$ or $\exists$ and M is a formula containing no quantifiers.

- $(Q_1 x_1) \cdots (Q_q x_q)$ is called the *prefix*, and
- M is called the *matrix*.

$(\forall x)\,(\text{isPhilosopher}(x) \Rightarrow \text{isClever}(x))$ is in *prenex normal form*.

$(\forall x) P(x) \Rightarrow (\exists x) Q(x)$ is *not* in prenex normal form.

Definition (Formula in conjunctive normal form)

A formula F is said to be in a *conjunctive normal form* iff F has the form $F_1 \wedge \cdots F_n$ for some $n \geq 1$ and for all $1 \leq i \leq n$, F_i is a disjunction of literals (i.e., F is $\bigwedge_{i=1}^{m} \bigvee_{j=1}^{n_i} L_{ij}$)

Transformation to *prenex* form [CL73, Section 3.3, pp. 37-38], part I

Repeatedly use the following equivalences *from left to right*

1. Eliminate $\Leftrightarrow$ and $\Rightarrow$

$$F \Leftrightarrow F' \quad \text{equiv.} \quad (F \Rightarrow F') \wedge (F' \Rightarrow F) \tag{1}$$
$$F \Rightarrow F' \quad \text{equiv.} \quad \neg F \vee F' \tag{2}$$

2. Bring negation signs *immediately before* atoms

$$\neg(\neg F) \quad \text{equiv.} \quad F \tag{3}$$
$$\neg(F \vee F') \quad \text{equiv.} \quad \neg F \wedge \neg F' \tag{4}$$
$$\neg(F \wedge F') \quad \text{equiv.} \quad \neg F \vee \neg F' \tag{5}$$
$$\neg((\forall x)F[x]) \quad \text{equiv.} \quad (\exists x)(\neg F[x]) \tag{6}$$
$$\neg((\exists x)F[x]) \quad \text{equiv.} \quad (\forall x)(\neg F[x]) \tag{7}$$

3. Rename *bound* variables if necessary, as $(Q\,x)F[x]$ equiv. $(Q\,y)F[y]$ provided that y occurs *nowhere* in the formula under normalization.

Transformation to *prenex* form [CL73, Section 3.3, pp. 37-38], part II

4. Move quantifiers *to the left* of the entire formula by using the laws (where x does *not* occur free in F' unless explicitly indicated as $F'[x]$):

$$
\begin{array}{rcll}
(Qx)F[x] \vee F' & \text{equiv.} & (Qx)(F[x] \vee F') & (8)\\
(Qx)F[x] \wedge F' & \text{equiv.} & (Qx)(F[x] \wedge F') & (9)\\
(\forall x)F[x] \wedge (\forall x)F'[x] & \text{equiv.} & (\forall x)(F[x] \wedge F'[x]) & (10)\\
(\exists x)F[x] \vee (\exists x)F'[x] & \text{equiv.} & (\exists x)(F[x] \vee F'[x]) & (11)\\
(Q_1 x)F[x] \vee (Q_2 x)F'[x] & \text{equiv.} & (Q_1 x)(Q_2 z)(F[x] \vee F'[z]) & (12)\\
(Q_1 x)F[x] \wedge (Q_2 x)F'[x] & \text{equiv.} & (Q_1 x)(Q_2 z)(F[x] \wedge F'[z]) & (13)
\end{array}
$$

Prenex normal form of $[(\forall x)P(x)] \Rightarrow [(\exists x)Q(x)]$

$$
\begin{array}{rcll}
\underline{[(\forall x)P(x)] \Rightarrow [(\exists x)Q(x)]} & \text{equiv.} & [\underline{\neg(\forall x)P(x)}] \vee [(\exists x)Q(x)] & \text{by (2)}\\
 & \text{equiv.} & \underline{[(\exists x)\neg P(x)] \vee [(\exists x)Q(x)]} & \text{by (6)}\\
 & \text{equiv.} & (\exists x)(\neg P(x) \vee Q(x)) & \text{by (11)}
\end{array}
$$

Transformation to *conjunctive normal form* [CL73, Section 2.4, p. 14]

Follow items 1 and 2 above. Then, repeatedly use the other laws and also the *distributive law* (from *left to right*)

$$F \vee (F' \wedge F'') \quad \text{equiv.} \quad (F \vee F') \wedge (F \vee F'') \tag{14}$$

Conjunctive normal form of $(P \vee \neg Q) \Rightarrow R$

$$\begin{array}{llll}
\underline{(P \vee \neg Q) \Rightarrow R} & \text{equiv.} & \underline{\neg(P \vee \neg Q)} \vee R & \text{by (2)} \\
 & \text{equiv.} & (\neg P \wedge \underline{\neg\neg Q}) \vee R & \text{by (4)} \\
 & \text{equiv.} & (\neg P \wedge Q) \vee R & \text{by (3)} \\
 & \text{equiv.} & R \vee (\neg P \wedge Q) & \text{commutativity of } \vee \\
 & \text{equiv.} & (R \vee \neg P) \wedge (R \vee Q) & \text{by (14)}
\end{array}$$

Consider the *prenex normal form* $(Q_1 x_1)\cdots(Q_q x_q)M$, where M is in *CNF*.

Transformation to *Skolem normal form* [CL73, Section 4.2, p. 47]

Repeat the following as much as possible: Let $Q_r = \exists$ for some $1 \leq r \leq q$.

1. If no universal quantifiers appears before Q_r,
   1. we choose a *new* constant c and
   2. replace all occurrences of x_r in M by c, and
2. If $Q_{s_1}, \ldots, Q_{s_m}$ are all the universal quantifiers appearing before Q_r, $1 \leq s_1 < s_2 < \cdots < s_m < r$,
   1. we choose a *new* m-ary function symbol f,
   2. replace all occurrences of x_r in M by $f(x_{s_1}, \ldots, x_{s_m})$

In both cases, *delete* $(Q_r x_r)$.

Skolem normal form of $(\exists x)(\neg P(x) \vee Q(x))$

We obtain $\neg P(c) \vee Q(c)$.

From the three previous steps applied on a sentence F, a *universally quantified conjunction of literals*

$$(\forall x_1)\cdots(\forall x_p)\bigwedge_{i=1}^{m}\bigvee_{j=1}^{n_i} L'_{ij}$$

is obtained. This sentence can be seen as a *set of clauses*

$$\mathcal{S}_F = \{\bigvee_{j=1}^{n_i} L'_{ij} \mid 1 \leq i \leq m\}$$

$[(\forall x)P(x)] \Rightarrow [(\exists x)Q(x)]$ and $(P \vee \neg Q) \Rightarrow R$ as sets of clauses

Sentence	Normalized	Clauses	Equiv.?
$[(\forall x)P(x)] \Rightarrow [(\exists x)Q(x)]$	$\neg P(c) \vee Q(c)$	$\{\{\neg P(c), Q(c)\}\}$	*UnsatEq*
$(P \vee \neg Q) \Rightarrow R$	$(R \vee \neg P) \wedge (R \vee Q)$	$\{\{R, \neg P\}, \{R, Q\}\}$	*Yes*

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