Fundamentos Lógicos y Algebraicos

Satisfaction, Models, Validity

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The *truth value* of logic formulas *depends* on the *interpretation* (and *valuation mapping*)

$\mathcal{A}_1 = \{$, , $\}$

$\mathcal{A}_2 = \{$, , $\}$

$homer^{\mathcal{A}_1}$	$homer^{\mathcal{A}_2}$
	

Int.	Dom.	$\mathtt{philosopher}^{\mathcal{I}_i}$
$\mathcal{I}_1$	$\mathcal{A}_1$	$\{$, $\}$
$\mathcal{I}_2$	$\mathcal{A}_1$	$\{$, $\}$
$\mathcal{I}_3$	$\mathcal{A}_2$	$\{$$\}$

Interp.	Domain	$\mathtt{homer}^{\mathcal{A}_i}$	$[\mathtt{philosopher(homer)}]^{\mathcal{I}_i}_{\alpha}$
$\mathcal{I}_1$	$\mathcal{A}_1$		false ( $\notin \{$, $\} = \mathtt{philosopher}^{\mathcal{I}_1}$)
$\mathcal{I}_2$	$\mathcal{A}_1$		false ( $\notin \{$, $\} = \mathtt{philosopher}^{\mathcal{I}_2}$)
$\mathcal{I}_3$	$\mathcal{A}_2$		true ( $\in \{$$\} = \mathtt{philosopher}^{\mathcal{I}_3}$)

In general, we need to consider different "*degrees of truth*"!

1. Satisfiability
2. True and false formulas. Model
3. True and false sentences in an interpretation
4. Validity
5. A hierarchy of truth
6. Logical consequence

Definition (Satisfaction)

A *valuation mapping* α *satisfies* a formula F in *an interpretation* $\mathcal{I}$ iff $[F]^{\mathcal{I}}_{\alpha} = \text{true}$.

Interp.	$\alpha(x)$	$\alpha(y)$	$[\texttt{teacherOf(x,y)}]^{\mathcal{I}_i}_{\alpha}$
$\mathcal{I}_1$			true ((, ) $\notin$ {(, )})
$\mathcal{I}_1$			false ((, ) $\notin$ {(, )})
$\mathcal{I}_2$			true ((, ) $\in$ {(, ), (, ), (, )})

Accordingly, `teacherOf(x,y)` is

- *satisfied* by $\mathcal{I}_1$ if $\alpha(x) =$  and $\alpha(y) =$ .
- *not satisfied* by $\mathcal{I}_1$ if $\alpha(x) =$  and $\alpha(y) =$ , but
- *satisfied* by $\mathcal{I}_2$ if $\alpha(x) =$  and $\alpha(y) =$ .

Satisfiability

A formula F is *satisfiable* iff there is an *interpretation* $\mathcal{I}$ and a *valuation mapping* α such that $[F]^{\mathcal{I}}_{\alpha} = \text{true}$. Otherwise, it is *unsatisfiable*.

Example

- `teacherOf(x,y)` is *satisfiable* (e.g., use $\mathcal{I}_1$ or $\mathcal{I}_2$)
- $\texttt{philosopher}(x) \wedge \neg\texttt{philosopher}(x)$ is (obviously) *unsatisfiable*.

True and false formulas (for an interpretation $\mathcal{I}$)

A formula F is *true* for $\mathcal{I}$ iff for *all valuations* α, $[F]^{\mathcal{I}}_{\alpha} = \text{true}$.

We say that F is *false* for $\mathcal{I}$ iff for *all valuations* α, $[F]^{\mathcal{I}}_{\alpha} = \text{false}$.

A formula F is true in $\mathcal{I}$ iff its negation $\neg F$ is false in $\mathcal{I}$

Formulas that are *neither true nor false* for a given interpretation

For instance, `teacherOf(x,y)`

- is *not* true in $\mathcal{I}_1$ because $\alpha(x) =$  and $\alpha(y) =$  makes it *false*; but,
- it is *not* false in $\mathcal{I}_1$ because $\alpha(x) =$  and $\alpha(y) =$  makes it *true*.

Model of a formula F

An interpretation $\mathcal{I}$ is a *model* of F (written $\mathcal{I} \models F$) if F is *true* in $\mathcal{I}$

We cannot say $\mathcal{I}_1 \models$ `teacherOf(x,y)` nor $\mathcal{I}_1 \models \neg$`teacherOf(x,y)`

Due to the quantification of *all* variables, valuations α play *no role* in the interpretation of *sentences*

Thus, we often write $F^{\mathcal{I}}$ instead of $[F]^{\mathcal{I}}_{\alpha}$ for sentences F

Given *an interpretation* $\mathcal{I}$ and a *sentence* F, either $F^{\mathcal{I}}$ or $(\neg F)^{\mathcal{I}}$ is *true*.

Equivalently, either $\mathcal{I} \models F$ or $\mathcal{I} \models \neg F$ holds

Such a *truth value* depends on the interpretation.

Consider $F = (\forall x)(\exists y)(\mathtt{philosopher}(x) \Rightarrow \mathtt{teacherOf}(y, x))$, i.e., all philosophers have been taught by somebody.

- $F^{\mathcal{I}_1} =$ *false* (e.g., in $\mathcal{I}_1$ nobody taught Socrates, who is a philosopher).
- $F^{\mathcal{I}_2} =$ *true* (in $\mathcal{I}_2$, Socrates and Plato were taught by Homer).

Validity

A formula F is *logically valid* (written $\models F$) if it is true for *every interpretation* (i.e., $\mathcal{I} \models F$ *holds* for every interpretation $\mathcal{I}$).

A formula F is *contradictory* iff it is false for *every interpretation* $\mathcal{I}$.

Validity by refutation

F is *valid* (resp. contradictory) iff $\neg F$ is unsatisfiable (resp. valid)

Hierarchy of truth in first-order formulas

F is	depends on	Example
Satisfiable	$\mathcal{I}$ and α	$\mathtt{teacherOf}(y,x)$
True	$\mathcal{I}$	$(\forall x)(\exists y)(\mathtt{philosopher}(x) \Rightarrow \mathtt{teacherOf}(y,x))$
Valid	–	$(\forall x)(\mathtt{philosopher}(x) \vee \neg\mathtt{philosopher}(x))$

Definition (Model of a set of formulas)

An interpretation $\mathcal{I}$ is a model of a set of formulas Φ (written $\mathcal{I} \models \Phi$) if $\mathcal{I}$ is a model of each formula F in Φ, i.e., for all $F \in \Phi$, we have $\mathcal{I} \models F$

Definition (Logical consequence)

A formula F is a *logical consequence* of a set of formulas Φ (written $\Phi \models F$) if every model of Φ is also a model of F.

Equivalently, if $\Phi \cup \{\neg F\}$ is *unsatisfiable*, i.e., $\Phi \cup \{\neg F\}$ has *no model*

Example (Logical consequence)

We have

$$\mathrm{isPhilosopher}(\mathit{plato}), (\forall x)\ \mathrm{isPhilosopher}(x) \Rightarrow \mathrm{isClever}(x) \models \mathrm{isClever}(\mathit{plato})$$

Fine! But, how to *check* this *in practice*? (!?)

References

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