Fundamentos Lógicos y Algebraicos

Interpretation of First-Order Formulas

DEPARTAMENTO DE SISTEMAS INFORMÁTICOS Y COMPUTACIÓN (DSIC)
UNIVERSIDAD POLITÉCNICA DE VALENCIA (UPV)

Salvador Lucas
http://slucas.webs.upv.es/

MÁSTER EN INGENIERÍA Y TECNOLOGÍA DE SISTEMAS SOFTWARE (MITSS)

What is the *meaning* of the following expressions?

- 5+5
- zusatz(V,V)
- f(a,b)
- x+0
- philosopher(plato)
- philosopher(david)
- q(c)
- $(\forall x)\ \text{isPhilosopher}(x) \Rightarrow \text{isClever}(x)$
- $(\forall m)(\forall n)\ m :: \mathit{Int} \wedge n :: \mathit{Int} \Rightarrow m + n :: \mathit{Int}$

1. Interpreting function symbols. Algebras
2. Interpreting terms
3. Interpreting predicate symbols. First-Order Interpretations
4. Interpreting atoms
5. Interpreting formulas

Given a signature of function symbols $\mathcal{F}$, an $\mathcal{F}$-algebra is a pair $\mathcal{A} = (\text{dom}(\mathcal{A}), \mathcal{F}_\mathcal{A})$, where

- $\text{dom}(\mathcal{A})$ is a *non-empty set* (*the domain* or *carrier*), often written $\mathcal{A}$

$\mathcal{F} = \{\texttt{homer}, \texttt{plato}, \texttt{socrates}\}$, all of them constant symbols

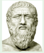
$\mathcal{A}_1$ $\{$ , ,  $\}$

$\mathcal{A}_2$ $\{$ , ,  $\}$

$\mathcal{A}_3$ $\mathbb{N} = \{0, 1, 2, \ldots\}$

$\mathcal{A}_4$ $\{$  $\}$

- $\mathcal{F}_\mathcal{A}$ is a set of *mappings* $f^\mathcal{A} : \mathcal{A}^k \to \mathcal{A}$ for each $f \in \mathcal{F}$ with $k = ar(f)$.

$\mathcal{F} = \{\texttt{homer}, \texttt{plato}, \texttt{socrates}\}$	$f \in \mathcal{F}$	$f^{\mathcal{A}_1}$	$f^{\mathcal{A}_2}$	$f^{\mathcal{A}_3}$	$f^{\mathcal{A}_4}$	$\ldots$
	homer			5		
	plato			5		
	socrates			8		

A *valuation mapping* $\alpha : \mathcal{X} \to \mathcal{A}$ gives values to the variable symbols.

The *evaluation* mapping $[\_]^{\mathcal{A}}_{\alpha} : \mathcal{T}(\mathcal{F},\mathcal{X}) \to \mathcal{A}$ is defined as follows: for all terms $t \in \mathcal{T}(\mathcal{F},\mathcal{X})$:

t	$[t]^{\mathcal{A}}_{\alpha}$	Comment
variable x	$\alpha(x)$	Use the valuation mapping α
constant a	$a^{\mathcal{A}}$	a interpreted using $\mathcal{F}$-algebra $\mathcal{A}$
$f(t_1,\ldots,t_k)$	$f^{\mathcal{A}}([t_1]^{\mathcal{A}}_{\alpha},\ldots,[t_k]^{\mathcal{A}}_{\alpha})$	Recursion + interpretation $f^{\mathcal{A}}$ of f

Example (Interpreting terms without variables - valuation unneeded!)

Let $\mathcal{F} = \{1, add_{/2}\}$ and $\mathcal{A} = (\mathbb{N}, \mathcal{F}_{\mathbb{N}})$ be given by $1^{\mathcal{A}} = 1$ and $add^{\mathcal{A}}(x, y) = x + y$ for all $x, y \in \mathbb{N}$. For any *valuation function* α,

- $$[1]^{\mathcal{A}}_{\alpha} = 1^{\mathcal{A}} = 1$$

- $$\begin{array}{rcl} [add(1, add(1, 1))]^{\mathcal{A}}_{\alpha} & = & add^{\mathcal{A}}([1]^{\mathcal{A}}_{\alpha}, [add(1, 1)]^{\mathcal{A}}_{\alpha}) \\ & = & add^{\mathcal{A}}(1, add^{\mathcal{A}}([1]^{\mathcal{A}}_{\alpha}, [1]^{\mathcal{A}}_{\alpha})) \\ & = & add^{\mathcal{A}}(1, add^{\mathcal{A}}(1, 1)) \\ & = & add^{\mathcal{A}}(1, 1 + 1) \\ & = & add^{\mathcal{A}}(1, 2) \\ & = & 1 + 2 \\ & = & 3 \end{array}$$

This is an *intended interpretation*, but it is *not* the only one

Example (Interpreting terms without variables - valuation unneeded!)

Let $\mathcal{F} = \{1, add_{/2}\}$ and $\mathcal{A} = (\mathbb{N}, \mathcal{F}_\mathbb{N})$ be given by $1^\mathcal{A} = 2$ and $add^\mathcal{A}(x, y) = x \times y$ for all $x, y \in \mathbb{N}$. For any *valuation function* α,

- $$[1]^\mathcal{A}_\alpha = 1^\mathcal{A} = 2$$

- $$\begin{array}{rcl} [add(1, add(1, 1))]^\mathcal{A}_\alpha & = & add^\mathcal{A}([1]^\mathcal{A}_\alpha, [add(1, 1)]^\mathcal{A}_\alpha) \\ & = & add^\mathcal{A}(2, add^\mathcal{A}([1]^\mathcal{A}_\alpha, [1]^\mathcal{A}_\alpha)) \\ & = & add^\mathcal{A}(2, add^\mathcal{A}(2, 2)) \\ & = & add^\mathcal{A}(2, 2 \times 2) \\ & = & add^\mathcal{A}(2, 4) \\ & = & 2 \times 4 \\ & = & 8 \end{array}$$

This is (perhaps) an *unintended (but still valid) interpretation*

Example (Interpreting terms with variables - valuation required!)

Let $\mathcal{F} = \{0, +_{/2}, \times_{/2}\}$ and $\mathcal{A} = (\mathbb{N}, \mathcal{F}_{\mathbb{N}})$ be given by

$$0^{\mathcal{A}} = 0, \qquad x +^{\mathcal{A}} y = x + y, \qquad \text{and} \qquad x \times^{\mathcal{A}} y = x \text{ (!)}$$

for all $x, y \in \mathbb{N}$. Let α be given by $\alpha(x) = \alpha(y) = 2$ and $\alpha(z) = 1$. Then,

1. $[x + 0]^{\mathcal{A}}_{\alpha} = [x]^{\mathcal{A}}_{\alpha} +^{\mathcal{A}} [0]^{\mathcal{A}}_{\alpha} = \alpha(x) +^{\mathcal{A}} 0^{\mathcal{A}} = 2 +^{\mathcal{A}} 0 = 2 + 0 = 2$
2. $$\begin{array}{rcl}
[(x + y) \times (x + z)]^{\mathcal{A}}_{\alpha} & = & [x + y]^{\mathcal{A}}_{\alpha} \times^{\mathcal{A}} [x + z]^{\mathcal{A}}_{\alpha} \\
& = & ([x]^{\mathcal{A}}_{\alpha} +^{\mathcal{A}} [y]^{\mathcal{A}}_{\alpha}) \times^{\mathcal{A}} ([x]^{\mathcal{A}}_{\alpha} +^{\mathcal{A}} [z]^{\mathcal{A}}_{\alpha}) \\
& = & (\alpha(x) +^{\mathcal{A}} \alpha(y)) \times^{\mathcal{A}} (\alpha(x) +^{\mathcal{A}} \alpha(z)) \\
& = & (2 +^{\mathcal{A}} 2) \times^{\mathcal{A}} (2 +^{\mathcal{A}} 1) \\
& = & 4 \times^{\mathcal{A}} 3 \\
& = & 4
\end{array}$$

An *$\mathcal{F},\Pi$-interpretation* (or *structure*) $\mathcal{I}$ on a *domain $\mathcal{A}$* is a triple $\mathcal{I} = (\mathcal{A}, \mathcal{F}_\mathcal{A}, \Pi_\mathcal{A})$ where

- $(\mathcal{A}, \mathcal{F}_\mathcal{A})$ is an $\mathcal{F}$-algebra,
- The equality symbol $=$ is interpreted (if used) as the *equality on $\mathcal{A}$*,
- for each n-ary $P \in \Pi$, $P^\mathcal{I} \in \Pi_\mathcal{A}$ is an n-ary *relation* on $\mathcal{A}$:

$$P^\mathcal{I} \subseteq \mathcal{A}^n = \mathcal{A} \underbrace{\times \cdots \times}_{n \text{ times}} \mathcal{A}$$

i.e., $P^\mathcal{I}$ is a (possibly empty) set of n-tuples $(x_1, \ldots, x_n)$, where $x_i \in \mathcal{A}$ for all $1 \leq i \leq n$.

$\Pi = \{\texttt{philosopher}_{/1}, \texttt{teacherOf}_{/2}\}$

Interpretation	Domain	$\texttt{philosopher}^{\mathcal{I}_i}$	$\texttt{teacherOf}^{\mathcal{I}_i}$
$\mathcal{I}_1$	$\mathcal{A}_1$	{ ,  }	{(, )}
$\mathcal{I}_2$	$\mathcal{A}_1$	{ ,  }	{(, ), (, ), (, )}
$\mathcal{I}_3$	$\mathcal{A}_2$	{  }	{(, )}
$\mathcal{I}_4$	$\mathcal{A}_3$	$\{5\}$	$\emptyset$

Given an *$\mathcal{F},\Pi$-interpretation* $\mathcal{I}$ on a domain $\mathcal{A}$ and a *valuation mapping* $\alpha : \mathcal{X} \to \mathcal{A}$, atoms $P(t_1, \ldots, t_n)$ are interpreted as follows:

$$[P(t_1, \ldots, t_n)]^{\mathcal{I}}_{\alpha} = \text{true iff } ([t_1]^{\mathcal{I}}_{\alpha}, \ldots, [t_n]^{\mathcal{I}}_{\alpha}) \in P^{\mathcal{I}}$$

where $[\_]^{\mathcal{I}}_{\alpha} : \mathit{Form}(\Sigma, \mathcal{X}) \to \mathit{Bool}$ is called the *interpretation* mapping.

Interp.	Domain	$\texttt{homer}^{\mathcal{A}_i}$	$[\texttt{philosopher(homer)}]^{\mathcal{I}_i}_{\alpha}$
$\mathcal{I}_1$	$\mathcal{A}_1$		false ( $\notin \{$ ,  $\} = \texttt{philosopher}^{\mathcal{I}_1}$)
$\mathcal{I}_2$	$\mathcal{A}_1$		false ( $\notin \{$ ,  $\} = \texttt{philosopher}^{\mathcal{I}_2}$)
$\mathcal{I}_3$	$\mathcal{A}_2$		true ( $\in \{$  $\} = \texttt{philosopher}^{\mathcal{I}_3}$)
$\mathcal{I}_4$	$\mathcal{A}_3$	5	true ($5 \in \{5\} = \texttt{philosopher}^{\mathcal{I}_4}$)

Interp.	Domain	$\texttt{homer}^{\mathcal{A}_i}$	$\alpha(x)$	$[\texttt{teacherOf(homer,x)}]^{\mathcal{I}_i}_{\alpha}$
$\mathcal{I}_1$	$\mathcal{A}_1$			false ((, ) $\notin \{$(, )$\}$)
$\mathcal{I}_2$	$\mathcal{A}_1$			true ((, ) $\in \{$(, ), (, ), (, )$\}$)
$\mathcal{I}_3$	$\mathcal{A}_2$			true ((, ) $\in \{$(, )$\}$)

The interpretation mapping is extended to deal with arbitrary (well-formed) formulas. Connectives and quantifiers have a *fixed* interpretation, though.

Use of *connectives*: **and** ($\wedge$), **or** ($\vee$), **implies** ($\Rightarrow$), **not** ($\neg$)

F	$[F]^{\mathcal{I}}_{\alpha} =$ *true*	$[F]^{\mathcal{I}}_{\alpha} =$ *false*
$F_1 \wedge F_2$	$[F_1]^{\mathcal{I}}_{\alpha} =$ true and $[F_2]^{\mathcal{I}}_{\alpha} =$ true	otherwise
$F_1 \vee F_2$	$[F_1]^{\mathcal{I}}_{\alpha} =$ true or $[F_2]^{\mathcal{I}}_{\alpha} =$ true	otherwise
$F_1 \Rightarrow F_2$	$[F_1]^{\mathcal{I}}_{\alpha} =$ false or $[F_2]^{\mathcal{I}}_{\alpha} =$ true	otherwise
$\neg F'$	$[F']^{\mathcal{I}}_{\alpha} =$ false	otherwise

Other connectives (e.g., $\Leftrightarrow$, *xor*, ...) handled similarly

$F = \texttt{philosopher(x)} \wedge \texttt{teacherOf(homer,x)}$

Interp.	$\alpha(x)$	$[\texttt{philosopher(x)}]^{\mathcal{I}_i}_{\alpha}$	$[\texttt{teacherOf(homer,x)}]^{\mathcal{I}_i}_{\alpha}$	$[F]^{\mathcal{I}_i}_{\alpha}$
$\mathcal{I}_1$		true	false	false
$\mathcal{I}_2$		true	true	true
$\mathcal{I}_3$		false	true	false

Use of *quantifiers*: **existential** ($\exists$) and **universal** ($\forall$)

F	$[F]^{\mathcal{I}}_{\alpha} = \mathit{true}$	$[F]^{\mathcal{I}}_{\alpha} = \mathit{false}$
$(\exists x)\ F'$	there is $a \in \mathcal{A}$ such that $[F']^{\mathcal{I}}_{\alpha[x \mapsto a]} = \text{true}$	otherwise
$(\forall x)\ F'$	for all $a \in \mathcal{A}$, $[F']^{\mathcal{I}}_{\alpha[x \mapsto a]} = \text{true}$	otherwise

If $(\exists x)\ F'$ holds, then $a \in \mathcal{A}$ as above is called a *witness*.

$F = (\exists x)$ `teacherOf(homer,x)`

Int.	$[F]^{\mathcal{I}_i}_{\alpha}$	$a \in \mathcal{A}$	Int.	$[F]^{\mathcal{I}_i}_{\alpha}$	$a \in \mathcal{A}$	Int.	$[F]^{\mathcal{I}_i}_{\alpha}$	$a \in \mathcal{A}$
$\mathcal{I}_1$	false	–	$\mathcal{I}_2$	true	, 	$\mathcal{I}_3$	true	

$F = (\forall x)($`philosopher(x)` $\Rightarrow$ `teacherOf(y,x)`$)$

Interp.	$\alpha(y)$	$[F]^{\mathcal{I}_i}_{\alpha}$	Comment
$\mathcal{I}_1$		false	Both  and  are philosophers, but not taught by .
$\mathcal{I}_2$		true	Both  and  are philosophers taught by .
$\mathcal{I}_3$		false	 is a philosopher, but he did *not* taught to himself.

References

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