Solution to the AC/DC Power Flow with the Holomorphic Embedding Method

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***Abstract*—The Holomorphic Embedding Load-Flow Method has been regarded as a powerful and complete tool to solve to AC power flow. Its theoretical base makes it stand apart from the traditional iterative methods, which suffer from convergence issues and are not guaranteed to obtain a feasible solution.**

The application of the holomorphic embedding method to the DC power flow as well as circuits with nonlinear devices has been very limited. Despite that, its properties are meant to still be favorable. Some technologies have been growing in popularity, like DC power systems, which are regarded as less costly than AC power systems for large distances.

This paper approaches the solution to the AC/DC power flow by means of the holomorphic embedding method. A simplistic test case is considered in order to show the performance of the algorithm.

***Index Terms*—power flow, holomorphic embedding, AC/DC, convergence, rectifier, inverter**

I. Introduction

It is a well-known fact that the equations that define the load flow of electric power systems are nonlinear. Since they can not be solved directly through a linear system of equations, iterative methods have been to go-to option [14]. Among them, the Newton-Rapshon has been the most popularized choice, since it usually converges with few iterations.

By definition, ill-conditioned systems operate under conditions that increase the difficulty to solve them. Two problems arise with iterative algorithms: first, the method may diverge even though there is a feasible solution, and in second place, the method could converge to a mathematically possible but physically infeasible result [15].

In contrast, the holomorphic embedding method consists in a recursive algorithm that builds the solution step by step. It is based on complex analysis, algebraic curves and approximation theory [9]. Its major upside is that it can tell when the solution is infeasible and find out the result when the power flow is feasible.

HVDC systems have been proven to be a reliable and cost-saving choice [2] with increasing popularity [13]. The interconnection between the AC and the DC system is due to a converter, which acts as a rectifier or an inverter. Its equations are nonlinear. Thus, several works have employed the Newton-Raphson method to solve them [1], [2], [3] and [4].

This paper focuses on formulating the equations that define the AC/DC converter in a way that makes them solvable with the holomorphic embedding. A basic system has been selected to show the derivation of the equations. More complex systems could also be solved by following a similar procedure. One of its advantages is found in that the typical AC formulation could add without much complexity the presence of the AC/DC converter. Furthermore, Sigma approximants, which are used as a diagnostic tool [9], would remain valid for the AC side.

The paper is structured as follows: Section II presents the simple test case and its equations; Section III adapts the equations to the holomorphic embedding method; Section IV details the algorithm to be followed at each step of the process; Section V shows the most interesting results that have been generated; finally, Section VI concludes the paper and mentions what the future work could be.

II. AC/DC systems equations

AC/DC converters station consists of a structure that contains a transformer used to adjust voltages as well as a converter. That last item is the main responsible of the transformation from AC to DC or vice-versa. In the first case, it would act as a rectifier, while in the second, as an inverter. Its functionality can be reversed since it can be understood as a three-phase bridge with thyristors. The behavior is dependent upon the chosen delay angle for the thyristors. If the delay angle $0 < \alpha < 90°$ it operates as a rectifier; if $180° > \alpha > 90°$ it acts as an inverter.

The aforementioned stations could also incorporate filters, reactive power compensators, a smoothing reactor, etc. All these auxiliary components do not play a critical role during the fundamental analysis of the converter, so they will be omitted.

A simple system such as the one in Figure 1 will be employed to present the equations involved and the corresponding embedding. It has already been converted to a per-unit base system. The transformer stands as an off-nominal transformer due to the fact that base voltages may not match their nominal voltages.

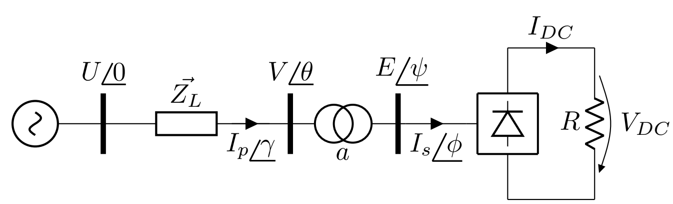

Fig. 1. Simplified system with an AC/DC converter

The combination of AC and DC power flow has traditionally implied a decoupling of references between both types of systems [1], [4]. On the one hand, the DC power flow sets the reference to I_s, while the AC power flow establishes it to the slack bus voltage. A side-effect of that is that even though the absolute value of the variable V remains the same, its phase change according to the chosen system, either the AC or DC part.

Despite not being mandatory to set $I_s\underline{/0}$, it is regarded as a convenient option [3]. Note that I_p coincides in phase to I_s because the transformer is assumed to be ideal. There is no leakage reactance to take into account, only the off-nominal tap ratio a. However, the derivation of equations will consider the reference chosen in Figure 1, i.e. the systems will not be decoupled when solved with the holomorphic embedding method, the reason being that it becomes much more complex when embedding angles.

The equations that define the DC power flow in a given per unit base system [3] are

$$\begin{cases} V_{DC} = & k_1 a V \cos\alpha - 3\frac{X_c}{\pi} I_{DC}, \\ P_{DC} = & \Re[V\underline{/\theta} I_p\underline{/\gamma}], \\ V_{DC} = & I_{DC} R, \end{cases} \tag{1}$$

where X_c is the commutation reactance, P_{DC} the power demand at the DC side, α the delay angle and $k_1 = \frac{3\sqrt{2}}{\pi}$. The commutation overlap, which would slightly modify k_1 will be ignored, but it could be considered without further consequences [3].

On the AC side, the first KCL has to be evaluated in order to find the expression that must equal $I_p\underline{/\gamma}$. Two equations will be derived from that: one for the real part and the other concerning the imaginary part. First, it should be noted that

$$I_p = k_1 a I_{DC}, \tag{2}$$

so I_p will no longer be treated directly as an unknown [1], [2] and [3]. Both a and I_{DC} will have to be found. The balance of currents yields

$$\begin{cases} \Re[I_p\underline{/\gamma}] & = UG - VG\cos\theta + VB\sin\theta, \\ \Im[I_p\underline{/\gamma}] & = UB - VG\sin\theta - VB\cos\theta, \end{cases} \tag{3}$$

where G and B represent the real and imaginary part of $1/\vec{Z_L}$.

At that stage of the formulation, there are five relevant equations and a total of seven variables (V, θ, γ, V_{DC}, I_{DC}, a and $\cos\alpha$). Two control equations have to be added for each converter [3]. Out of a handful of expressions to choose, the selected ones become

$$\begin{cases} a^{sp} & = a, \\ P_{DC}^{sp} & = V_{DC} I_{DC}, \end{cases} \tag{4}$$

where a^{sp} is the specified off-nominal tap ratio and P_{DC}^{sp} corresponds to the specified DC power. In reality, they would depend on the desired operative state. For convenience, in that example it developed from an arbitrary choice.

Since the DC power is known and also the resistive load, V_{DC} and I_{DC} can be found without any other information. Furthermore, because a turns out not to be a variable, from a total of seven variables, four of them have to be calculated (V, θ, γ, $\cos\alpha$).

III. Embedding

The Holomorphic Embedding Method has been proven to also be a fertile ground of development for DC circuits with nonlinear loads, for instance [8]. Just like the AC power flow, the first step consists in embedding the equations. Then, the first coefficients have to be obtained so that the reference state becomes fully defined. From then on the solution is built by calculating more and more coefficients.

The choice of embedding is not unique [9]. In the AC power flow there have been some variations that embed the equations differently [10] and [11] from the most notorious source, which is [9]. When it comes to the DC power flow, the selection is probably even more diverse, due to containing several elements and not possessing a reference like the slack bus.

In the example, first of all, it is interesting to define $V\underline{/\theta}$ as

$$V\underline{/\theta} = V^{re} + jV^{im}, \tag{5}$$

Right now they are not embedded, but they will eventually become dependent on the complex variable s. Let C be a generic variable. When it is embedded, it becomes $C(s)$, which in turn, follows

$$C(s) = \sum_{k=0}^{n} s^k C[k], \tag{6}$$

where $C[k]$ represents one term that along with many others conform $C(s)$ and n symbolizes the arbitrary maximum index. Consequently, $C(s)$ contains $n+1$ terms.

Given that $I_p\underline{/\gamma}$ has been divided between real and imaginary part, it ought to be treated as such in (1). As a result of that along with the chosen embedding, the second expression of (1) transforms into

$$\begin{aligned} sP_{DC} =& GU(s)V^{re}(s) - GV^{re}(s)V^{re}(s) \\ &+ BU(s)V^{im}(s) - GV^{im}(s)V^{im}(s). \end{aligned} \tag{7}$$

The proposed embedding is adequate to force the initialization $V^{re}[0] = 1$, $V^{im}[0] = 0$. That harmonizes with the reference state described in [9]. To achieve that without invalidating the equations, the slack bus is embedded as

$$U(s) = 1 + s(U_w - 1), \tag{8}$$

where U_w is the data of the voltage corresponding to the slack bus.

There are two remaining equations to embed. One comes from the first expression in (1). Postulating the equality of absolute values, it turns out to be

$$\begin{aligned} \left(\frac{V_{DC}}{k_1 a} + \frac{3X_c I_{DC}}{\pi k_1 a}\right)^2 =& V^{re}(s)V^{re}(s)\alpha(s)\alpha(s) \\ &+ V^{im}(s)V^{im}(s)\alpha(s)\alpha(s), \end{aligned} \tag{9}$$

where $\alpha(s)$ represents $\cos\alpha(s)$. A concern could emerge from (9): the product of series at the right hand-side would imply a convolution of four series. That is unusual since the AC power flow is solved with only convolutions of two series [12] and [9]. Nevertheless, that triple convolution will prove to be a solid choice.

The last equation needed to be incorporated is the one that integrates the definition of I_p seen from the DC side and the one the AC side of the system. By establishing the equality of absolute values

$$\begin{aligned} s\frac{(ak_1 I_{DC})^2}{Y^2} - U(s)U(s) = V^{re}(s)V^{re}(s) \\ + V^{im}(s)V^{im}(s) - s2U(s)V^{re}(s), \end{aligned} \tag{10}$$

where $Y^2 = G^2 + B^2$. In both (7) and (10) there are some terms that multiply directly by s and as a consequence of that their usage is delayed. The purpose behind that lies in conditioning the system of equations so that they remain consistent with the reference state as well as not blocking the access to the solutions. For instance, if the delay of P_{DC} in (7) did not take place, the reference state would be invalidated; on the other hand, multiplying $U(s)V^{re}(s)$ by s is required in order to formulate a determined system of equations.

Once all the equations are properly embedded, the next process is concerned in calculating the terms that build the series. Once the series are constructed, the last step to find the final solution evaluates the series at $s = 1$. That can be achieved thanks to Padé approximants with great precision [9]. They are not the only tool available. Recursive approaches like Wynn's ϵ or Bauer's η could also be employed [12], and even the sum of coefficients is a suitable alternative if the radius of convergence is bigger than the unit.

IV. Algorithm

The combination of (7), (9) and (10) must not be straigthforward. At that stage, the unknowns are $V^{re}(s)$, $V^{im}(s)$ and $\alpha(s)$. (7) and (10) do not depend upon $\alpha(s)$, so the coefficients that conform $V^{re}(s)$ and $V^{im}(s)$ must be extracted from these expressions. Then, they have to be employed in (9) to solve for $\alpha(s)$.

Hence, a two-step process is constructed. The first step builds a system of two equations. Its solution generates a further coefficient of $V^{re}(s)$ and $V^{im}(s)$. Then, this information is used in the triple convolutions that revolve around $\alpha(s)$. Because of that, a new coefficient of $\alpha(s)$ is obtained.

The algorithm has to be split into the calculation of the terms [0], [1], [2] and $[c \geq 3]$ since it can only be generalized once the first steps are completed. As mentioned, $V^{re}[0] = 1$ and $V^{im}[0] = 0$. On the other hand, it is convenient not to force $\alpha[0]$. Its initial value my be distant from the final result and there is hardly any intuition on what it may become. Thus, using (9) yields

$$\alpha[0] = \pm\sqrt{\left(\frac{V_{DC}}{k_1 a} + \frac{3X_c I_{DC}}{\pi k_1 a}\right)^2}. \tag{11}$$

Only the positive root should be selected, as it can be deduced from the original expression captured in (1).

The next step consists in obtaining the first order coefficients. Consequently, a system of equations is formed from (7) and (10). Fortunately, just like it happens during the AC power flow implementation of the holomorphic embedding, the matrix turns out to be constant at all steps. It becomes an upside in contrast to the Newton-Raphson, where the linear system has to be reconstructed at each iteration. The generalized system comes to be

$$\begin{pmatrix} R1[c] \\ R2[c] \end{pmatrix} = \begin{pmatrix} -G & B \\ 2 & 0 \end{pmatrix} \begin{pmatrix} V^{re}[c] \\ V^{im}[c] \end{pmatrix}, \tag{12}$$

where $R1[c]$ and $R2[c]$ are the right hand-side expressions that have to be computed for every step. For the first order coefficients

$$\begin{cases} R1[1] = & P_{DC} - GV^{re}[0](U-1), \\ R2[1] = & \frac{ak_1 I_{DC}}{Y^2} - 2(U-1). \end{cases} \tag{13}$$

Subsequently, $V^{re}[1]$ and $V^{im}[1]$ become known. The last calculation to perform at this step is

$$\alpha[1] = -\frac{V^{re}[1]\alpha[0]}{V^{re}[0]}. \tag{14}$$

At this stage all the first order cofficients have been obtained. Next, the algorithm is explicitly defined for the [2] terms, and then, generalized.

The following terms start involving trivial convolutions, which will be more and more prominent at each step. When it comes to the system of equations, the right hand-side expressions are derived as

$$\begin{cases} R1[2] = & -GV^{re}[1](U-1) - BV^{im}[1](U-1) \\ & +G\sum_{k=1}^{1} V^{re}[k]V^{re}[2-k] \\ & +G\sum_{k=1}^{1} V^{im}[k]V^{im}[2-k], \\ R2[2] = & -(U-1)(U-1) + 2V^{re}[1] + 2(U-1) \\ & -\sum_{k=1}^{1} V^{re}[k]V^{re}[2-k] \\ & -\sum_{k=1}^{1} V^{im}[k]V^{im}[2-k]. \end{cases} \tag{15}$$

Employing (12) and (15) provides the desired voltage coefficients. Then, the expression to compute $\alpha[2]$ also increases its complexity, transforming into

$$\begin{aligned} \alpha[2] =\Bigg(& -\sum_{k=1}^{1} \alpha[k]\alpha[2-k] \\ & -\sum_{k=1}^{2} (V^{re})^2[k] \sum_{j=0}^{2-k} \alpha[j]\alpha[2-k-j] \\ & -\sum_{k=1}^{2} (V^{im})^2[k] \sum_{j=0}^{2-k} \alpha[j]\alpha[2-k-j] \Bigg) \frac{1}{2\alpha[0]}, \end{aligned} \tag{16}$$

where $(V^{re})^2(s)$ corresponds to the squared $V^{re}(s)$. It must no be confused with squaring a given term. From now on the algorithm can be fully generalized. Given an arbitrary depth

$c \geq 3$ the right hand-side expressions that participate in the system of equations turn out to be

$$\begin{cases} R1[c] = & -GV^{re}[c-1](U-1) - BV^{im}[c-1](U-1) \\ & G\sum_{k=1}^{c-1} V^{re}[k]V^{re}[c-k] \\ & G\sum_{k=1}^{c-1} V^{im}[k]V^{im}[c-k], \\ R2[c] = & 2(U-1)V^{re}[c-2] + 2V^{re}[c-1] \\ & -\sum_{k=1}^{c-1} V^{re}[k]V^{re}[c-k] \\ & -\sum_{k=1}^{c-1} V^{im}[k]V^{im}[c-k]. \end{cases} \tag{17}$$

Finally, the terms that conform $\alpha[c]$ ressemble $\alpha[2]$. In fact, (16) is the particular case of

$$\begin{aligned} \alpha[c] = \Bigg(& -\sum_{k=1}^{1} \alpha[k]\alpha[c-k] \\ & -\sum_{k=1}^{c} (V^{re})^2[k] \sum_{j=0}^{c-k} \alpha[j]\alpha[c-k-j] \\ & -\sum_{k=1}^{c} (V^{im})^2[k] \sum_{j=0}^{c-k} \alpha[j]\alpha[c-k-j] \Bigg) \frac{1}{2\alpha[0]}. \end{aligned} \tag{18}$$

Other embeddings could also be feasible. However, the advantage of opting for the presented one is that it is compatible with the AC power flow, in the sense that voltages are divided in real and imaginary part. Furthermore, $\alpha(s)$ does not intervene in the AC power flow. Accordingly, the AC power flow could be modified to just integrate the bus that connects with the DC system, and then, compute the delay angle α from that.

Another step further, $alpha$ is obtainable from the final voltage values. So, it is not mandatory to embed it and decompose it with terms to then calculate the resulting value. The reason behind the adopted choice was to show that it is possible to adapt an equation that contains the absolute value of a variable.

The AC bus that links the AC system with the DC side could interconnect several buses, and not just the slack bus. In that case, the formulation would just need to express the current I_p as a function of voltages and admittances. The same procedure would follow from that, and an algorithm of a similar fashion would be obtained.

V. Results

The covered formulation is applied to a simple system like the one in 1. Table I captures the initial data of the problem. All values are expressed per unit according to their respective base.

This section focuses on evaluating up to which point the solution given by the holomorphic embedding method remains correct, finding out its convergence properties and discovering the evolution of the error given diverse input values.

The solution corresponding to the data from Table I is shown in Table II.

At first sight, the results seem coherent. The voltage magnitude is slightly below U since the converter acts as a rectifier, while $\alpha = 66.15°$. For normal operation the delay angle tends

TABLE I
Data of simple AC/DC system

Magnitude	Value
P_{DC}	0.1
R	10
X_c	0.05
U	1.1
a	2
k_1	$3\sqrt{2}/\pi$
G	1
B	-1

TABLE II
Solution to the simple AC/DC system

Magnitude	Value
V^{re}	0.9180
V^{im}	0.0579
$\cos\alpha$	0.4044

to remain close to $10°$ [13], which is clearly not the case. The values correspond to a made-up case, so there is hardly any logic behind them. The maximum error is $2.1{\cdot}10^{-11}$, sufficiently small.

To obtain these results, a depth of 60 coefficients has been used. The study of an optimal depth still poses a question without a definite answer. AC power systems are usually solvable with around 20 to 40 coefficients [12] and [9]. Using more coefficients yields a smaller error if the series are convergent enough. When that is not the case, increasing the depth could have tragic effects because the solution could become degraded. Figure 2 plots the maximum error depending on the current depth.

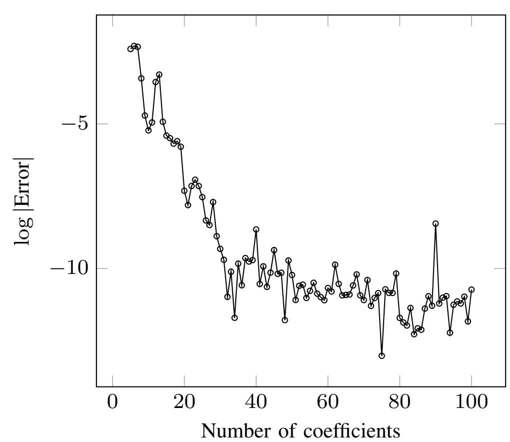

Fig. 2. Maximum errors depending on the number of coefficients

Despite its chaotic nature, the error diminishes quite steadily until it reaches about 10^{-10}. From then on, the usage of more terms does not improve meaningfully the solution. That pattern of behavior is similar to the ones in the AC power flow [12].

Under the proposed conditions the series that define by which the unknowns are defined do not converge, yet they slowly diverge. This is not problematic as long as the resulting error turns out to be sufficiently small. By means of Padé approximants one can obtain a solution that converges while the coefficients that conform the series do not. Figure 3 plots the logarithm of the absolute value of the terms corresponding to $V^{re}(s)$.

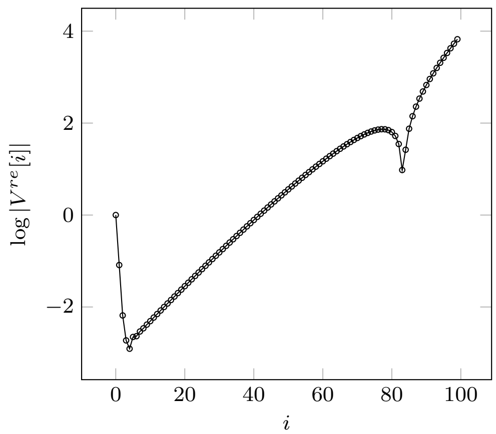

Fig. 3. Logarithm of the $V^{re}(s)$ terms to evaluate the convergence

From what can be derived, right from the start the coefficients diminish. However, around the fifth coefficient, they start increasing until the plot presents a caveat. From that point on, they keep on increasing. As a result of that, the radius of convergence is smaller than 1, but not distant from it. It is approximately 0.95.

Hence, the summation of coefficients is not suitable [9] so some method of analytical continuation must be used, the Padé approximants being the most renowned option. It has been precisely the chosen resource.

Finally, the evolution of the maximum error as a function of the active power demand at the DC bus is displayed in Figure 4. 60 coefficients in each series have been used. The error is

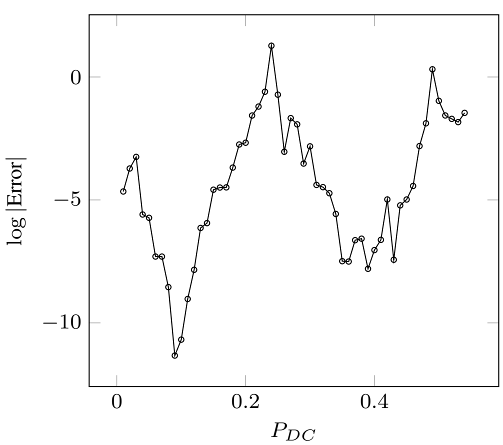

Fig. 4. Maximum errors depending on P_{DC}

at its minimum around $P_{DC} = 0.1$, which matches the value from Table I. Increasing the power implies that the error rises up to a maximum. It then decreases for $P_{DC} \approx 0.4$. From then on, the error keeps more or less constant at around the unit.

One possible hypothesis for that is found in the fact that V_{DC} and I_{DC} depend on both P_{DC} and R. Thus, since R has not changed when P_{DC} is being modified, it is possible that their combination of values causes V_{DC} and I_{DC} to approach values for which the power system analysis is ill-conditioned.

VI. Conclusion

It has been shown that the holomorphic embedding method can also be used to solve the equations involved in the AC/DC power flow with a similar algorithm to the one used for the AC power flow, where rectangular coordinates are also employed. No decoupling of references between systems was needed. Depending on the given control equations the formulation would change, yet the main idea remains the same.

The performance of the algorithm offers a margin of improvement, since the coefficients do not converge. Other embeddings could be chosen and probably the solution would become more well-conditioned. Despite that, Padé approximants have been useful to find a final convergent solution. Its error decays with the number of iterations.

The properties of the algorithm remain to be checked for systems with a multitude of buses and converters. It would also be beneficial to compare the holomorphic embedding to the Newton-Raphson in order to find out which one is more convenient under diverse conditions.

References

[1] J. Arrillaga, P. Bodger. "A.C.-d.c. load flows with realistic representation of the convertor plant", in *Proceedings of the Institution of Electrical Engineers*, vol. 125, no. 1, pp. 41-46, January 1978.

[2] J. Arrillaga, C. P. Arnold, J. R. Camacho, S. Sankar. "AC-DC Load Flow with Unit Connected Generator-Converter Infeeds", in *IEEE Transactions on Power Systems*, vol. 8, no. 2, pp. 701-706, May 1993.

[3] J. Arrillaga, C. P. Arnold. "Computer Analysis of Power Systems". John Wiley and & Sons. 1994.

[4] J. Arrillaga, N. R. Watson, G. N. Bathurst. "Unified Newton Framework For The Steady State Simulation Of Networks With Multiple Ac-Dc Converters" in *Proceedings of the Power Conversion Conference*, Osaka, Japan, 2002, pp. 288-292 vol.1.

[5] G. N. Bathurst, N. R. Watson, J. Arrillaga. "A harmonic domain solution for systems with multiple hig h-power AC/DC converters", in *in IEE Proceedings - Generation, Transmission and Distribution*, vol. 148, no. 4, pp. 312-318, July 2001.

[6] D. J. Tylavsky, "A Simple Approach to the Solution of the ac-dc Power Flow Problem", in *IEEE Transactions on Education*, vol. 27, no. 1, pp. 31-40, February 1984.

[7] B. Yang, L. Chuang, L. Zhu, C. Guo, Z. Gu and Z. Wang, "AC/DC Power Flow Algorithm Considering Various Controls Transformation," in *2018 2nd IEEE Conference on Energy Internet and Energy System Integration (EI2)*, Beijing, 2018, pp. 1-5.

[8] A. Trias and J. L. Marín, "The Holomorphic Embedding Loadflow Method for DC Power Systems and Nonlinear DC Circuits," in *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 63, no. 2, pp. 322-333, Feb. 2016.

[9] A. Trias. HELM: *The Holomorphic Embedding Load-Flow Method*. Foundations and Implementations. Foundations and Trends® in Electric Energy Systems, vol. 3, no. 3-4, pp. 140-370, 2018.

[10] I. Wallace, D. Roberts, A. Grothey and K. I. M. McKinnon. "Alternative PV Bus Modelling with the Holomorphic Embedding Load Flow Method". 2016.

[11] H. Chiang, T. Wang and H. Sheng. "A Novel Fast and Flexible Holomorphic Embedding Power Flow Method". *IEEE Transactions on Power Systems*, vol. 33, no. 3, pp. 2551-2562, May 2018.

[12] S. Rao. "Exploration of a Scalable Holomorphic Embedding Method Formulation for Power System Analysis Applications". Arizona State University. 2017.

[13] D. P. Kothari, I. J. Nagrath. ”Modern Power System Analysis”. Tata McGraw Hill. 2011.

[14] A. Gomez-Exposito and C. Gomez-Quiles. ”Factorized Load Flow”. IEEE Transactions on Power Systems vol. 28, no. 4, pp. 4607-4614. Nov. 2013.

[15] S. C. Tripathy, G. D. Prasad, O. P. Malik and G. S. Hope. ”Load-Flow Solutions for Ill-Conditioned Power Systems by a Newton-Like Method”. IEEE Transactions on Power Apparatus and Systems. vol. PAS-101, no. 10, pp. 3648-3657, Oct. 1982.

[16] A. Trias. ”HELM: The Holomorphic Embedding Load-Flow Method”. 2012 IEEE Power and Energy Society General Meeting, San Diego, CA, 2012, pp. 1-8.