An Introduction to Vectors

January 7, 2018

1 Geometry of vectors

Informally, a *vector* $\vec{v}$ in two dimensions can be thought of as an arrow in the plane, starting at the origin $(0,0)$ and ending at the point (x,y).

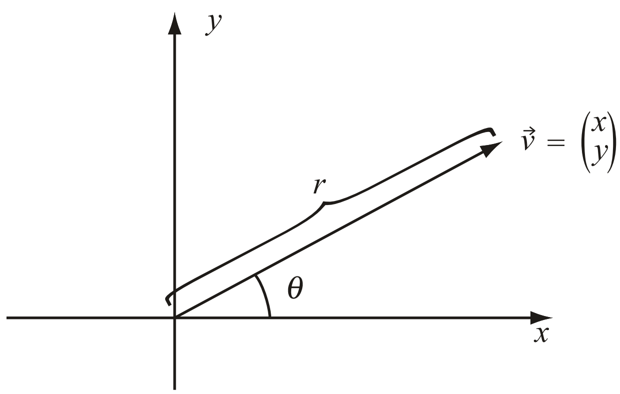

Figure 1:

We denote this vector by $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, writing the coordinates of the head of the vector as a *column*. [1] Since our vectors always begin at the origin, there's no need to record the location of the tail.

While usually the most convenient way to specify a particular vector is to write the column giving the coordinates of its head, one could also specify a vector by its length r (also called the *magnitude* and denoted by $||\vec{v}||$) and its direction. In two dimensions, the direction is specified by the angle θ the vector makes with the positive x-axis. From trigonometry we get the following

[1] While it might seem more convenient now to write a vector as a row, the fact that it is a column will be important later.

relationships:

$$
\begin{aligned}
||\vec{v}|| = r &= \sqrt{x^2 + y^2} \\
\tan\theta &= \frac{y}{x} \\
x &= r\cos\theta \\
y &= r\sin\theta
\end{aligned}
$$

Thus the vector of length r and angle θ is $\vec{v} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$.

2 Algebra of vectors

From the geometry of vectors we are led to several algebraic definitions.

2.1 Scalar Multiplication

Given a number c and a vector $\vec{v}$, we define the product $c\vec{v}$ by

$$c\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} cx \\ cy \end{pmatrix}.$$

Thus we simply multiply each entry of the vector by the number.

Thinking geometrically, multiplying a vector by 2 will simply make it stick out twice as far in the x- and y-directions. As you might guess, this means the angle of the vector is unchanged, while its length is doubled.

To see this more generally, for any positive c we note

$$c\begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = \begin{pmatrix} cr\cos\theta \\ cr\sin\theta \end{pmatrix}.$$

Thus the length has changed from r to cr, while the angle θ remains unaffected.

To understand the effect of multiplication of a vector by a negative number, first consider the special case of $c = -1$. Since $-1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$, from plotting arrows it's clear that $-1\begin{pmatrix} x \\ y \end{pmatrix}$ and $\begin{pmatrix} x \\ y \end{pmatrix}$ point in opposite directions, yet have the same length. Since multiplication by any negative number c can be thought of as first multiplying by -1 and then by the positive number $|c|$, the effect of multiplication by such a c must be to first flip the vector, and then change its length.

Thus the effect of multiplying a vector by any number is to change the length of the vector (and possibly flip the direction to its opposite) – a drawing of the vector is simply 'rescaled.' For this reason, numbers are often referred to as *scalars* and the multiplication of a vector by a number is called *scalar multiplication*.

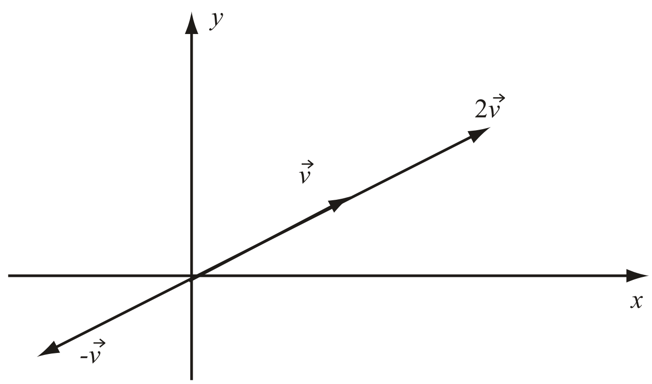

Figure 2:

There is one remaining situation worth pointing out: what if the scalar is 0? For any vector $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, we get $0\vec{v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \vec{0}$, the *zero vector*. This is a perfectly good vector (just as 0 is a perfectly good number). Thought of as an arrow, it begins *and* ends at the origin. Thus it has length 0. However, there is no natural notion of the angle of this vector; you might reasonably claim it has either all angles or no angle. Fortunately, this issue will have no practical importance, though the zero vector itself is quite important.

Finally, we'll occasionally find it useful to rescale a given vector $\vec{v}$ so that its new length is 1 but its direction is unchanged. To do this, we just multiply $\vec{v}$ by the scalar $\frac{1}{||\vec{v}||}$. More formally, given $\vec{v}$, we *normalize* $\vec{v}$ to produce

$$\vec{u} = \frac{1}{||\vec{v}||}\vec{v}.$$

Then $\vec{u}$ is said to be a *unit vector*, since its length is 1. Of course if $\vec{v} = \vec{0}$, so that $||\vec{v}|| = 0$, then we can't normalize $\vec{v}$; there is no natural choice of a unit vector with the same direction as $\vec{0}$.

Unit vectors in two dimensions are particularly easy to describe in terms of the angle they make with the x-axis. If $\vec{v} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$ is any non-zero vector with length r, then

$$\vec{u} = \frac{1}{r}\begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

2.2 Vector Addition

Given two vectors $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} u \\ v \end{pmatrix}$, we define the sum by

$$\vec{v} + \vec{w} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x+u \\ y+v \end{pmatrix}.$$

Thus we add vectors by adding corresponding entries.

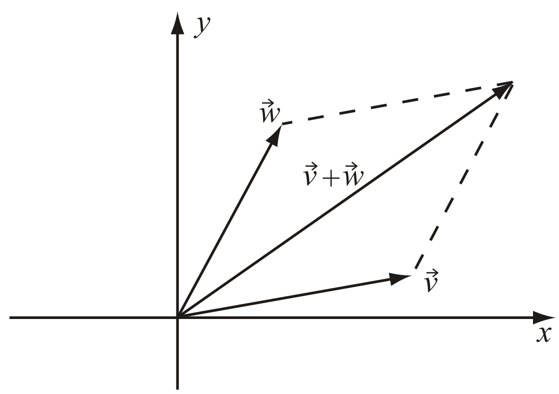

Figure 3:

Thinking about this geometrically, we see the sum $\vec{v} + \vec{w}$ must point in the x-direction an amount that is the sum of how far the individual vectors $\vec{v}$ and $\vec{w}$ point in that direction, and similarly for the y-direction. Thus we can geometrically see the sum of two vectors $\vec{v}$ and $\vec{w}$ by simply sliding $\vec{w}$ over so that its tail is at the head of $\vec{v}$. Then the new location of the head of $\vec{w}$ will be such that its position in the x-direction is the sum of the x-direction positions of the individual vectors, and similarly for the y-direction. This is just the *head-to-tail method* of adding vectors geometrically: slide one vector over without changing its direction so that the two are lined up head-to-tail. The sum then goes from the origin to the head of the vector that was moved.

Alternately, we could use the two unmoved vectors $\vec{v}$ and $\vec{w}$ as two sides of a parallelogram, completing the parallelogram by drawing in two other sides. The sum is just the diagonal of this parallelogram that starts at the origin and goes to the opposite corner. A glance at a drawing shows this *parallelogram method* gives the same result as the head-to-tail method.

2.3 Linear Combinations of Vectors and Bases

The most general way to combine vectors with one another to produce a new vector is to use both the operations discussed above: scalar multiplication and vector addition. For instance, if we begin with the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then we can build up a new vector by an expression like

$$2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + -3\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}.$$

An expression of the form $c_1\vec{v}_1 + c_2\vec{v}_2$ is said to be a *linear combination*[2] or *superposition* of the vectors $\vec{v}_1$ and $\vec{v}_2$.

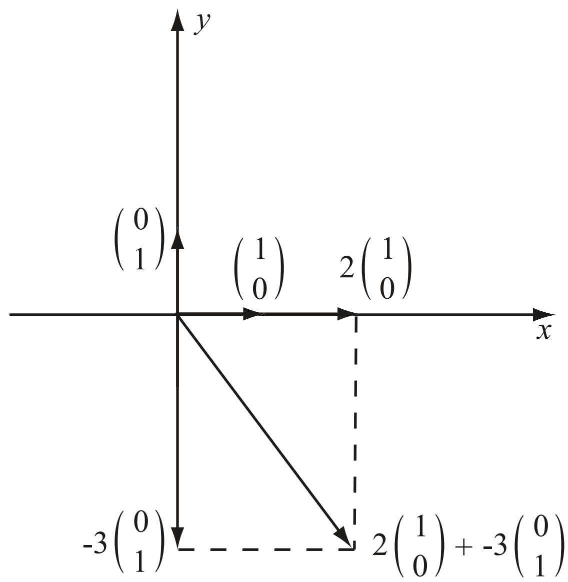

Figure 4:

It should be obvious that by taking linear combinations of the vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we can build up all vectors in two dimensions: To get any vector $\begin{pmatrix} x \\ y \end{pmatrix}$, we simply consider

$$x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Geometrically this means that if we begin with the two unit arrows pointing along the positive x- and y-axes, by lengthening (and perhaps flipping them),

[2]The term 'linear' is used since in this combination we combine vectors by multiplying by scalars and then adding, just as the variables are multiplied by scalars and added to get an equation of a line (e.g., $2x - y = 0$).

and then adding them head-to-tail, we can reach any point in the plane. These two vectors are enough to build *any* vector. Furthermore, if we had only one of these vectors, we would *not* be able to build up every vector. In fact, we'd only be able to build up a line of vectors all pointing in the same (or opposite) direction. This set of two vectors is big enough to build up every vector, and if we made it smaller, we'd lose that capability.

Such a set of vectors is called a *basis*. To summarize, a basis is a set of vectors such that 1) by taking linear combinations of the basis vectors, all vectors can be produced, and 2) any smaller collection of the basis vectors cannot produce all vectors. [3] The particular basis $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is called the *standard basis* for two dimensions.

However, there are other bases we could consider as well. For instance, suppose we consider $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Thinking geometrically, multiples of the first vector give us everything on a line at angle $\frac{\pi}{4}$, while multiples of the second give us everything on the $\frac{3\pi}{4}$ line. By adding these things, we can reach any point in the plane. Thus, linear combinations of the two produce all vectors, but we need both vectors to have that capability.

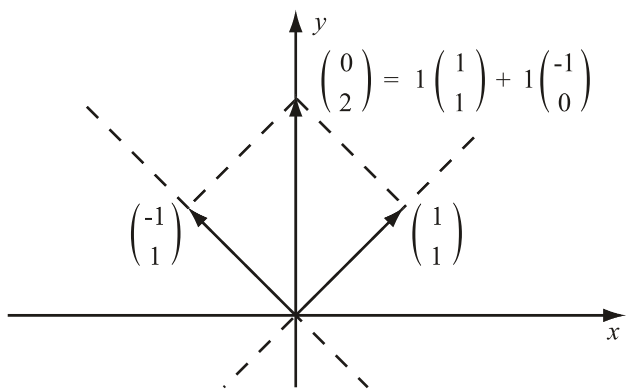

Figure 5:

Algebraically we can see this as well. To get any vector $\begin{pmatrix} x \\ y \end{pmatrix}$, we need to find scalars c_1 and c_2 so that

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

This means

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 \\ c_1 \end{pmatrix} + \begin{pmatrix} -c_2 \\ c_1 \end{pmatrix} = \begin{pmatrix} c_1 - c_2 \\ c_1 + c_2 \end{pmatrix},$$

[3]In Linear Algebra, a basis is defined as a linearly independent set of vectors spanning the vector space. The definition above is equivalent.

so c_1 and c_2 must satisfy

$$x = c_1 - c_2$$
$$y = c_1 + c_2.$$

By adding and subtracting these equations to one another we find

$$c_1 = \frac{x+y}{2}$$
$$c_2 = \frac{y-x}{2}.$$

This proves any vector $\begin{pmatrix} x \\ y \end{pmatrix}$ can be built from $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ by just choosing the appropriate coefficients c_1 and c_2 for a linear combination. For example, to build $\begin{pmatrix} 2 \\ -3 \end{pmatrix}$ from this basis, we choose $c_1 = \frac{2-3}{2} = \frac{1}{2}$ and $c_2 = \frac{-3-2}{2} = -\frac{5}{2}$.

What sets of vectors could give us a basis in two-dimensions? Well, we could begin picking a basis by choosing any vector (other than $\vec{0}$). Multiples of it, however, only produce a line. We then have to choose a second vector. It's important this second vector not be a multiple of the first, though, since otherwise we won't be able to build up anything new. Any vector that isn't a multiple of the first will point off that line, and thus give us a way of building up all other vectors. So we see any choice of two non-zero vectors that have 'different' directions will form a basis. Though in our examples above we chose the basis vectors to be perpendicular to each other, that isn't really needed; all that we need is that there be two vectors and they have different directions.

The whole notion of a basis may seem a little pedantic in two dimensions, but it is ultimately a crucial one for understanding quantum mechanics. When we have a space (like the two-dimensional plane), we will try to understand everything in that space by understanding a basis. Unlike in the plane where the standard basis is a natural choice, there isn't always an obvious 'right' basis to use – in fact, that is one reason why quantum mechanics can be so confusing at first. If we think of basis vectors as the framework for analyzing all vectors, then choosing the right framework can give different insights.

3 More on lengths and angles

3.1 Orthogonality and the inner product

In quantum mechanics, the most important bases are made of vectors orthogonal (ortho (right) + gon (angle) = perpendicular) to one another, and so it's convenient to have a simple test for that.

Let's begin with two vectors $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} u \\ v \end{pmatrix}$ which are orthogonal. Then adding them head-to-tail, as in the figure below, we see a right triangle.

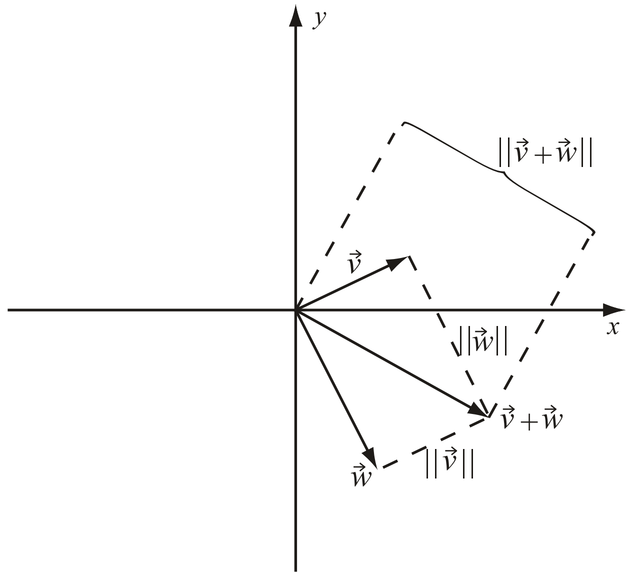

Figure 6:

Therefore, the Pythagorean theorem will apply to the triangle, and the lengths of the sides will be related by

$$||\vec{v}+\vec{w}||^2 = ||\vec{v}||^2 + ||\vec{w}||^2.$$

In terms of the entries of the vectors, this means

$$(x+u)^2 + (y+v)^2 = (x^2+y^2) + (u^2+v^2).$$

After a little algebra, this simplifies to

$$xu + yv = 0.$$

In fact, this argument can also be followed backwards, so we see that two vectors being orthogonal is mathematically equivalent to the statement that $xu+yv=0$. The expression on the left hand side here will turn out to be so important that we give it a name:

Definition: If $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} u \\ v \end{pmatrix}$, then the *inner product* of $\vec{v}$ and $\vec{w}$ is

$$\langle \vec{v}, \vec{w} \rangle = xu + yv.$$

The simple way to remember how to compute inner products is: multiply the corresponding entries of the vectors, and then add up the results. Notice that when we compute the inner product of two vectors, we get a *scalar*, not a vector. The inner product is also called the *scalar product.* [4]

From our argument above we have the important fact that makes the inner product so useful: Two vectors are orthogonal exactly when their inner product is zero. For example, we can immediately deduce that $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ are orthogonal:

$$\left\langle \begin{pmatrix} 2 \\ -4 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle = (2)(2) + (-4)(1) = 0.$$

It wasn't necessary to draw any pictures, or use any trigonometry to understand the geometric relationship between these vectors, though we can confirm our deduction with a figure:

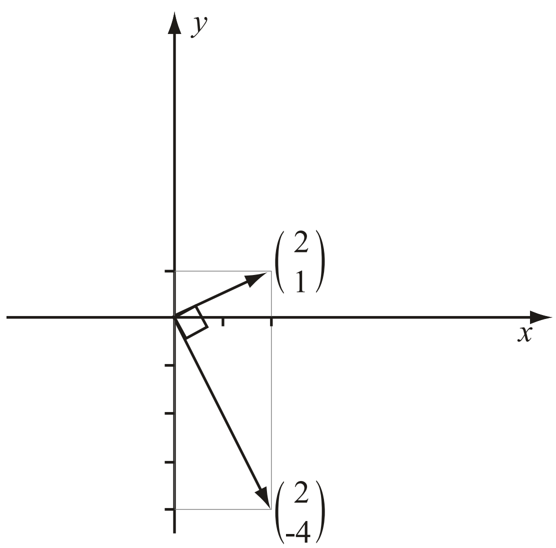

Figure 7:

The inner product is also intimately related to lengths of vectors. If $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, then $||\vec{v}||^2 = x^2 + y^2 = \langle \vec{v}, \vec{v} \rangle$. Thus

$$||\vec{v}|| = \sqrt{\langle \vec{v}, \vec{v} \rangle}.$$

Although we won't need it, it's possible to go further with the inner product and show it always has a geometric meaning. More precisely, one can show

[4]In linear algebra courses where the vectors have only real numbers as entries, it is also sometimes called the *dot product*, since it is written $\vec{v} \cdot \vec{w}$. We won't use that notation, since soon we'll be forced to use complex numbers.

that $\langle \vec{v}, \vec{w} \rangle = ||\vec{v}||\ ||\vec{w}|| \cos\phi$, where ϕ is the angle between $\vec{v}$ and $\vec{w}$. Thus the inner product can be used as the basic means of extracting length and angle information from vectors.

3.2 Orthonormal bases

In quantum mechanics, the most important bases are ones where 1) each vector in the basis has unit length, and 2) each vector in the basis is orthogonal to the other basis vectors. In geometric terms, this second property just means the basis vectors are positioned relative to one another so that they would point along a reasonable set of axes. These axes might be rotated (or flipped) somehow from the usual ones, but they would still be at right angles to one another. The first property means the basis vectors would also naturally encode the unit distances along the axes. Thus a basis with these two properties is essentially just a choice of a new, but reasonable, coordinate system.

More formally, we call such a basis *orthonormal* (ortho (orthogonal) + normal (unit length)). An orthonormal basis in two dimensions is a basis $\vec{b}_1$ and $\vec{b}_2$ such that 1) $||\vec{b}_1|| = ||\vec{b}_2|| = 1$ and 2) $\langle \vec{b}_1, \vec{b}_2 \rangle = 0$.

The standard basis $(1, 0)$ and $(0, 1)$ is of course orthonormal.[5] However, the $\frac{\pi}{4}$-$\frac{3\pi}{4}$ basis $(1, 1)$ and $(-1, 1)$ we dealt with above is not. Though these vectors are orthogonal since $\langle (1, 1), (-1, 1) \rangle = 0$, they each have length $\sqrt{2}$ since $(\pm 1)^2 + 1^2 = 2$. However, we can create an orthonormal basis by just normalizing them by dividing by their length, yielding $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

3.3 Expressing vectors through an orthonormal basis

Often in quantum mechanics, and even in classical physical science, it is necessary to express some vector in terms of an orthonormal basis. For instance, in classical physics, if $(-1, 3)$ represents a force vector,we might want to write it in term of its horizontal and vertical components, since we can understand its effect on particles by considering these components separately. This simply means we write $(-1, 3) = -1(1, 0) + 3(0, 1)$, denoting a force of -1 in the x-direction and 3 in the y-direction. If the particle were constrained to move along a $\frac{\pi}{4}$ incline, however, we would need to separate the force into orthogonal components at $\frac{\pi}{4}$ and $\frac{3\pi}{4}$. This means we would consider the orthonormal basis $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and need to find c_1 and c_2 so that

$$(-1, 3) = c_1 \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) + c_2 \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right).$$

While c_1 and c_2 must exist (after all, we are dealing with a basis), how can we find them easily?

[5] Though technically all vectors should be written as columns, for typographical convenience we will sometimes right them as rows, with commas between entries. Whenever you see this, you should mentally convert to a column.

A good way of thinking of this is that by choosing an orthonormal basis, we are essentially choosing a new set of axes. Expressing a vector in terms of this basis simply means finding its coordinates with respect to the new axes.

But enough vague generalities; the mathematics turns out to be simple. Let's suppose we have chosen an orthonormal basis for the plane. That means we have two vectors $\vec{b}_1$ and $\vec{b}_2$ such that $||\vec{b}_1|| = ||\vec{b}_2|| = 1$ and $\langle \vec{b}_1, \vec{b}_2 \rangle = 0$. Now we have some other vector $\vec{v}$ that we want to express in terms of our basis. We have to solve:

$$\vec{v} = c_1\vec{b}_1 + c_2\vec{b}_2$$

for the c_i's. We simply take the inner product of each side of the last equation with $\vec{b}_1$ to get

$$\begin{aligned}
\langle \vec{b}_1, \vec{v} \rangle &= \langle \vec{b}_1, (c_1\vec{b}_1 + c_2\vec{b}_2) \rangle \\
&= c_1\langle \vec{b}_1, \vec{b}_1 \rangle + c_2\langle \vec{b}_1, \vec{b}_2 \rangle \quad \text{(by distributing)} \\
&= c_1 1 + c_2 0 \quad \text{since } \langle \vec{b}_1, \vec{b}_1 \rangle = ||\vec{b}_1||^2 = 1 \text{ and } \langle \vec{b}_1, \vec{b}_2 \rangle = 0 \\
&= c_1
\end{aligned}$$

Thus c_1 is easily computed as the inner product $c_1 = \langle \vec{b}_1, \vec{v} \rangle$. Similarly, by taking the inner product with $\vec{b}_2$, we find $c_2 = \langle \vec{b}_2, \vec{v} \rangle$.

Let's return to our example above: we want to express $(-1, 3)$ in terms of the orthonormal basis $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. We need only compute

$$\begin{aligned}
c_1 &= \langle (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (-1, 3) \rangle = \frac{-1}{\sqrt{2}} + \frac{3}{\sqrt{2}} = \sqrt{2} \\
c_2 &= \langle (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (-1, 3) \rangle = \frac{1}{\sqrt{2}} + \frac{3}{\sqrt{2}} = 2\sqrt{2}
\end{aligned}$$

Thus we've found

$$(-1, 3) = \sqrt{2}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) + 2\sqrt{2}\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),$$

as is easy to check.

In summary, as long as we are dealing with an orthonormal basis, it's easy to express any vector as a linear combination of the basis vectors. We find the correct scalars to use as coefficients in the linear combination by computing appropriate inner products.

3.4 Projections of vectors

While we used an algebraic approach in the last section, we can use the result above to give a geometric interpretation of the inner product.

Suppose $\vec{v}$ is any vector and $\vec{u}$ is a unit vector. Then to see the meaning of $\langle \vec{u}, \vec{v} \rangle$, first imagine picking a third vector, $\vec{w}$, so that $\vec{u}$ and $\vec{w}$ form on orthonormal basis. We then know

$$\vec{v} = \langle \vec{u}, \vec{v} \rangle \vec{u} + \langle \vec{w}, \vec{v} \rangle \vec{w}.$$

In other words, we've split $\vec{v}$ up into two pieces, one in the direction of $\vec{u}$ and one in the direction of $\vec{w}$. Since these pieces add together to give us $\vec{v}$ back, they must be the *projections* of $\vec{v}$ onto the basis vectors, as shown in the figure.

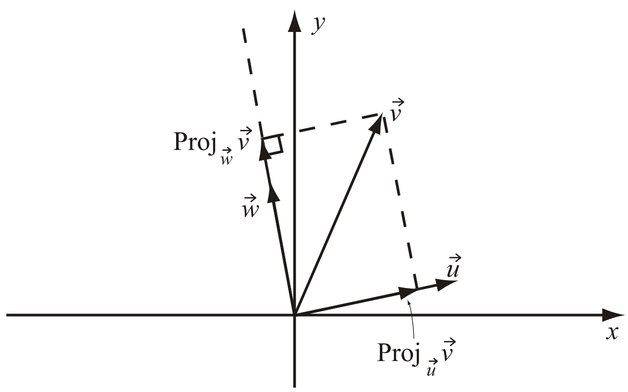

Figure 8:

We summarize this as

$$\text{Proj}_{\vec{u}}\vec{v} = \langle \vec{u}, \vec{v} \rangle \vec{u}.$$

This projection of $\vec{v}$ onto $\vec{u}$ gives us a vector that points in the direction given by $\vec{u}$. It is the 'piece' of $\vec{v}$ that points in the $\vec{u}$ direction. Remember, it is a vector, and not a scalar. [6]

4 Three dimensions and beyond

So far, all the vectors we've dealt with have been in two dimensions. In that setting, we've been able to motivate scalar multiplication, vector addition, bases, the inner product, and projections. We've used a mixture of geometric and algebraic arguments since our intuition of the plane is strong. It is possible to modify these arguments for vectors in three dimensions, and derive analogous results. Ultimately, however, we'll need these concepts in even higher dimensional spaces. To develop them fully and carefully is a matter for a Linear Algebra course – we'll have to limit ourselves here to a summary. The geometric language we use is all motivated by intuition from two and three dimensions.

[6] In this discussion, we've assumed $\vec{u}$ has length 1. For vectors of other lengths, the correct formula is $\text{Proj}_{\vec{b}}\vec{a} = \frac{\langle \vec{b}, \vec{a} \rangle}{\langle \vec{b}, \vec{b} \rangle}\vec{b}$.

By a vector in n-dimensional space, we mean a column of numbers $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$.

Only when n is 2 or 3 can we really visualize this as the location of the head of an arrow, but for larger n we still pretend we can.

We multiply by scalars and add vectors in n-space analogously to when $n = 2$.

$$c \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{pmatrix}, \qquad \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

A linear combination of vectors is anything of the form $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots c_n\vec{v}_n$.

The inner product of two vectors is given by

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \right\rangle = x_1y_1 + x_2y_2 + \cdots x_ny_n.$$

We can use the inner product for computing the length of a vector by the formula

$$||\vec{v}|| = \sqrt{\langle \vec{v}, \vec{v} \rangle}.$$

We say vectors $\vec{v}$ and $\vec{w}$ are orthogonal if $\langle \vec{v}, \vec{w} \rangle = 0$.

A basis for n-space is a collection of vectors such that 1) any vector in n-space can be expressed as a linear combination of the basis vectors, and 2) no smaller collection has this capability. It is possible to show that any basis for n-space must be a collection of n vectors.

An orthonormal basis is a basis in which each vector has been chosen to 1) have length 1, and 2) be orthogonal to the other basis vectors.

If $\vec{b}_1,\ \vec{b}_2 \ldots \vec{b}_n$ is an orthonormal basis for n-space, and $\vec{v}$ is any vector in n-space, then to find the c_i such that

$$\vec{v} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots c_n\vec{b}_n$$

is easy: $c_i = \langle \vec{b}_i, \vec{v} \rangle$. The projection of $\vec{v}$ onto a unit vector $\vec{u}$ is

$$\mathrm{Proj}_{\vec{u}}\vec{v} = \langle \vec{u}, \vec{v} \rangle \vec{u}.$$

5 Problems

1. Find the length and angle of $(-3, 3)$.
2. Find the vector (x, y) which has length 2 and angle $\frac{\pi}{3}$.
3. Compute the following and draw a picture to illustrate:
   (a) $(1, -1) + (2, 3)$
   (b) $(1, -1) + (2, 3) + (-1, -1)$
   (c) $2(1, 2) - 3(-1, 1)$
4. Normalize $(-3, 4)$.
5. Which of the following give a basis in two dimensions? Explain
   (a) $(1, 1)$ and $(1, 0)$
   (b) $(1, 1)$, $(1, 0)$, and $(-1, 2)$
   (c) $(1, 1)$
   (d) $(3, 2)$ and $(0, 0)$
6. The vectors $(3, 1)$ and $(-1, 2)$ form a basis for the plane. Express the following vectors as a linear combination of these basis vectors.
   (a) $(1, 5)$
   (b) $(3, 4)$
   (c) $(0, 0)$
   (d) (x, y), for all real numbers x, y.
7. Compute inner products to see if the following are orthogonal.
   (a) $(3, -1)$ and $(2, 4)$
   (b) $(3, -1)$ and $(2, 6)$
   (c) $(-1, 7)$ and $(0, 0)$
8. Which of the following are orthonormal bases for the plane? Explain.
   (a) $(2, 1)$ and $(-1, 2)$
   (b) $(\frac{3}{5}, \frac{4}{5})$ and $(-\frac{4}{5}, \frac{3}{5})$
   (c) $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$ and $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$
   (d) $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $(\frac{\sqrt{3}}{2}, \frac{1}{2})$
9. Show the following identities hold by letting $\vec{v}_1 = (x_1, y_1)$, $\vec{v}_2 = (x_2, y_2)$, etc., and simplifying
   (a) $c(\vec{v}_1 + \vec{v}_2) = c\vec{v}_1 + c\vec{v}_2$
   (b) $\langle \vec{v}_1, \vec{v}_2 + \vec{v}_3 \rangle = \langle \vec{v}_1, \vec{v}_2 \rangle + \langle \vec{v}_1, \vec{v}_3 \rangle$
   (c) $\langle \vec{v}_1, c\vec{v}_2 \rangle = c\langle \vec{v}_1, \vec{v}_2 \rangle$

10. Use the inner product to express the following in terms of the basis $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

    (a) $(0, -1)$

    (b) $(1, 1)$

    (c) $(2, 3)$

11. Find the projection of the following onto the unit vector $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. Draw a picture to illustrate.

    (a) $(2, 1)$

    (b) $(-1, 1)$

    (c) $(2, 2)$

12. Do the following calculations in three dimensions:

    (a) Show $(1, 1, 1)$ is orthogonal to $(2, -1, -1)$.

    (b) Find the projection of $(2, 3, 4)$ onto the unit vector $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$.

    (c) Show $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, $(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, and $(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}})$ form an orthonormal basis.

    (d) Express $(1, 2, 0)$ in terms of the basis in part (c), using inner products.