Part A Question 1

Part A

$$\frac{d^2y}{dt^2} + \frac{c}{m}\frac{dy}{dt} + \frac{k}{m}y = \frac{F(t)}{m}$$

y = Vertical position
c = damping co-efficient
k = Spring constant
m = mass
F(t) = external forcing

$c = 4000$
$k = 40000$
$m = 1000$
$F(t) = -200t$

$$\frac{d^2y}{dt^2} + \frac{4000}{1000}\frac{dy}{dt} + \frac{40\,000}{1000}y = \frac{-200t}{1000}$$

$$\boxed{\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 40y = -0.2t}$$

Step 1 - Solve homogeneous solution

let $r = \frac{dy}{dt}$

$r^2 + 4r + 40 = 0$ → characteristic equation

find roots

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-4 \pm \sqrt{4^2-(4\times 1\times 40)}}{2}$$

$$= \frac{-4 \pm \sqrt{-144}}{2}$$

$$= -2 \pm \frac{\sqrt{144}}{2}i$$

$$\boxed{= -2 \pm 6i}$$ ∴ $r_1 = -2+6i$ $r_2 = -2-6i$

Since complex conjugate roots

$r = \alpha + \beta i = (-2+6i = r_1)$ & $(-2-6i = r_2)$

$y_h = e^{\alpha t}(c_1\cos(\beta t) + c_2\sin(\beta t))$

∴ $y_h = e^{-2t}(c_1\cos(6t) + c_2\sin(6t))$

↳ ① homogeneous solution

Part A Question 2

Step 2 - Solve Particular Solution

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 40y = -0.2t$$

from table → $y_p = b_1 t + b_0$

$\therefore y_p' = b_1 + b_0 \quad y_p'' = 0$

$$0 + 4b_1 + 40(b_1 t + b_0) = -0.2t$$
$$4b_1 + 40b_1 t + 40b_0 = -0.2t$$
$$40b_1 t + (4b_1 + 40b_0) = -0.2t$$
$$40b_1 = -0.2$$
$$b_1 = -\frac{1}{200}$$

find c $\therefore 4 \times \left(-\frac{1}{200}\right) + 40c = 0$

$$c = \frac{1}{2000}$$

(Particular Solution) → $\therefore y_p = -0.005t + 0.0005$

Part A Question 3

Step 3 Solve ODE given $y(0)=0$ and $y'(0)=0$

$$y = y_h + y_p$$

$$= e^{-2t}(C_1 \cos(6t) + C_2 \sin(6t)) - 0.005t + 0.0005$$

apply initial conditions

$$y(0)=0 \quad y'(0)=0$$

$$y(0) = e^{-2\times 0}(C_1 \cos(6\times 0) + C_2 \sin(6\times 0)) - 0.005\times 0 + 0.0005$$

$$= 1\times(C_1) + 0.0005$$

$$= C_1 + 0.0005$$

$$\boxed{C_1 = -0.0005}$$

$y'(0) =$

$u = e^{-2t}$ $\quad V = (C_1 \cos(6t) + C_2 \sin(6t))$

$u' = -2e^{-2t}$ $\quad V' = -6C_1 \sin(6t) + 6C_2 \cos(6t)$

$$y' = e^{-2t}(-6C_1 \sin(6t) + 6C_2 \cos(6t) - 2e^{-2t}(C_1 \cos(6t) + C_2 \sin(6t)) - 0.005$$

$$y'(0) = 1(6C_2) - 2(C_1) - 0.005$$

$$0 = 6C_2 - 2C_1 - 0.005$$

$$0 = 6C_2 - (2\times 0.0005) - 0.005$$

$$= 6C_2 + (1\times 10^{-3}) - 0.005$$

$$6C_2 = -4\times 10^{-3}$$

$$C_2 = -6.667\times 10^{-4}$$

$$C_2 = 0.000667$$

$$\therefore y = e^{-2t}(-0.0005 \cos(6t) + 0.000667 \sin(6t) - 0.005t + 0.0005$$

Part A Question 4

Step 4 - finding net force

$$F = m \times \frac{d^2 y}{dt^2} \qquad m = 1000\,kg$$

$$F = 1000 \times \frac{d^2 y}{dt^2} \to = y'' = 2^{nd} \text{ derivative of position / displacement} = \text{acceleration}$$

$$\boxed{F = m \times a}$$

Position

$$y = e^{-2t}(-0.0005\cos(6t) + 0.000667\sin(6t)) - 0.005t + 0.0005$$

$u = e^{-2t}$ $\qquad V = (-0.0005\cos(6t) + 0.000667\sin(6t))$

$u' = -2e^{-2t}$ $\qquad V' = (3\times10^{-3}\sin(6t) + 4.002\times10^{-3}\cos(6t))$

Velocity $= y'$

$$y' = e^{-2t}(3\times10^{-3}\sin(6t) + 4.002\times10^{-3}\cos(6t)) + -2e^{-2t}(-0.0005\cos(6t) + 0.000667\sin(6t)) - 0.005$$

y'' = acceleration

①

$u = e^{-2t}$ $\qquad V = (3\times10^{-3}\sin(6t) + 4.002\times10^{-3}\cos(6t))$

$u' = -2e^{-2t}$ $\qquad V' = (0.018\cos(6t) - 0.024012\sin(6t))$

$$① = e^{-2t}(0.018\cos(6t) - 0.024012\sin(6t)) - 2e^{-2t}(3\times10^{-3}\sin(6t) + 4.002\times10^{-3}\cos(6t))$$

---

②

$u = -2e^{-2t}$ $\qquad V = (-0.0005\cos(6t) + 0.000667\sin(6t))$

$u' = 4e^{-2t}$ $\qquad V' = (3\times10^{-3}\sin(6t) + 4.002\times10^{-3}\cos(6t))$

$$② = -2e^{-2t}(3\times10^{-3}\sin(6t) + 4.002\times10^{-3}\cos(6t)) + 4e^{-2t}(-0.0005\cos(6t) + 0.000667\sin(6t))$$

$\boxed{① + ② = y'' = \text{acceleration}}$

$$y'' = e^{-2t}(0.018\cos(6t) - 0.024012\sin(6t)) - 2e^{-2t}(3\times10^{-3}\sin(6t) + 4.002\times10^{-3}\cos(6t))$$
$$- 2e^{-2t}(3\times10^{-3}\sin(6t) + 4.002\times10^{-3}\cos(6t)) + 4e^{-2t}(-0.0005\cos(6t) + 0.000667\sin(6t))$$

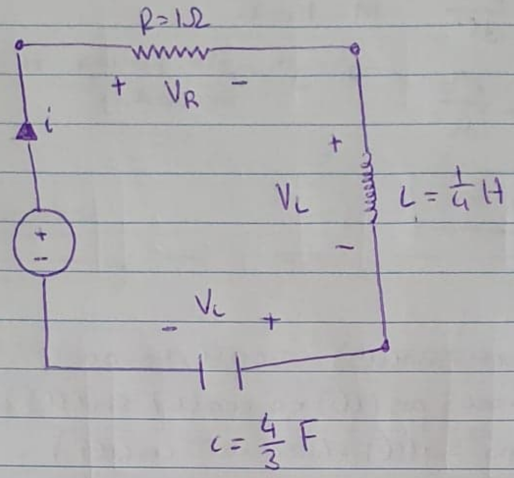
y'' is in m/s

$f = 1000 \times y''$ } in Newtons

Part B Question 1

Part B

$R = 1\Omega$, $L = \frac{1}{4} H$, $C = \frac{4}{3} F$, $V_{in} = 10V$



1) $$\begin{bmatrix} i' \\ V_L' \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{c} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} i \\ V_L \end{bmatrix} + \begin{bmatrix} 0 \\ V_{in} \end{bmatrix}$$

Kirchoff's laws $\Sigma V_{loop} = 0$

$$0 = -V_R - V_L - V_C + V_{in} \quad (1)$$

$\Sigma\, i_{loop} = 0$ → Since series same current in all components.

$$i = i_R = i_C = i_L$$

Component laws

Resistor: $V_R = iR$ .... (2)

Inductor: $V_L = L \frac{di}{dt}$ ... (3)

Capacitor: $V_c = i = c \frac{dv}{dt}$ or $V_c = \frac{1}{c} \int i\, dt$ (5)

→ (4)

(1) model as system of ODE's

$$V_L = L \frac{di}{dt}$$

$$\boxed{\frac{di}{dt} = i' = \frac{1}{L} V_L}$$

② Sub (2), (4), into ①

$$0 = -V_R - V_L - V_c + V_{in}$$

$$0 = -iR - C\frac{dv}{dt} - V_L + V_{in}$$

$$\boxed{V_L' = \frac{dV_L}{dt} = -\frac{R}{L}V_L - \frac{1}{C}i + V_{in}'}$$

$$\boxed{V_{in} = 10}$$

Putting ① & ② together

$$\begin{bmatrix} i' \\ V_L' \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} i \\ V_L \end{bmatrix} + \begin{bmatrix} 0 \\ 10 \end{bmatrix}$$

V'_{in}

Part B Question 2

2 eigen Values

$$\det |A - \lambda| I = 0$$

$$0 = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{R}{L} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$0 = \left| \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{R}{L} \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right|$$

$$0 = \left| \begin{bmatrix} -\lambda & \frac{1}{L} \\ -\frac{1}{C} & \left(-\frac{R}{L} - \lambda\right) \end{bmatrix} \right|$$

$$\left(-\lambda\left(\frac{-R}{L} - \lambda\right)\right) - \left(\frac{1}{L} - \frac{1}{C}\right)$$ Sub values

$$\left(-\lambda\left(-4 - \lambda\right)\right) - \left(4\left(-\frac{3}{4}\right)\right)$$

$$4\lambda + \lambda^2 + 3 = 0$$

$$\lambda^2 + 4\lambda + 3 = 0$$

3 1

$$(\lambda + 1) = 0 \quad (\lambda + 3) = 0$$

$$\boxed{\lambda_1 = -1} , \boxed{\lambda_2 = -3}$$

eigen values = $\boxed{\lambda_1 = -1}$ and $\boxed{\lambda_2 = -3}$

Part B Question 3

3 eigen vectors

$$\lambda_1 = -1$$

$$(A - \lambda_1 I)c = 0$$

$$\begin{bmatrix} 0 & 4 \\ -\frac{3}{4} & -4 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 4 \\ -\frac{3}{4} & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

$$c_1 + 4c_2 = 0$$

$$-\frac{3}{4}c_1 - 3c_2 = 0$$

---→ row 1 $= c_1 + 4c_2 = 0$

let $c_1 = 1$ $\quad 1 + 4c_2 = 0$

$$4c_2 = -1$$

$$c_2 = \frac{-1}{4}$$

$$c_1 = \begin{bmatrix} 1 \\ -\frac{1}{4} \end{bmatrix}$$ → vector 1

$$\lambda_2 = -3$$

$$(A - \lambda_2 I)c = 0$$

$$\begin{bmatrix} 0 & 4 \\ -\frac{3}{4} & -4 \end{bmatrix} - \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} 3 & 4 \\ -\frac{3}{4} & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = 0$$

① $3c_1 + 4c_2 = 0$

② $-\frac{3}{4}c_1 - c_2 = 0$

----→ row 1 $= 3c_1 + 4c_2 = 0$

let $c_1 = 1$ $\quad 3c_1 = -4c_2$

$$c_2 = -\frac{3}{4}$$

$$c_2 = \begin{bmatrix} 1 \\ -\frac{3}{4} \end{bmatrix}$$ → vector 2

Part B Question 4

```
% MATLAB CODE
clc

A = [0,4;-(3/4),-4]
[eigenvectors, eigenvalues] = eig(A)
vec = eigenvectors;

vec1 = vec(:,1)/vec(1,1)
vec2 = vec(:,2)/vec(1,2)

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

Answer this code produces:

eigenvalues =

```
  -1    0
   0   -3
```

vec1 =

```
   1.0000
  -0.2500
```

vec2 =

```
   1.0000
  -0.7500
```

The MATLAB solutions and the hand worked solutions both perfectly agree. The eigen values are the same and the eigenvectors are the same and provide the same relationship therefore are correct.

Part B Question 5

Solve for $i(t)$ and $V_L(t)$

given $i(0) = 10A$ & $V_L(0) = 0$

general solution $= y = k_1 C_1 e^{\lambda_1 t} + k_2 C_2 e^{\lambda_2 t}$

$$\begin{bmatrix} V_L \\ i \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ -\frac{1}{4} \end{bmatrix} e^{-1t} + k_2 \begin{bmatrix} 1 \\ -\frac{4}{3} \end{bmatrix} e^{-3t}$$

$$\boxed{V_L = 0 \, , \quad i(0) = 10}$$

$$\begin{bmatrix} 0 \\ 10 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ -\frac{1}{4} \end{bmatrix} \cancel{e} + k_2 \begin{bmatrix} 1 \\ -\frac{4}{3} \end{bmatrix} \cancel{e}$$

$$\begin{bmatrix} 0 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -\frac{1}{4} & -\frac{4}{3} \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

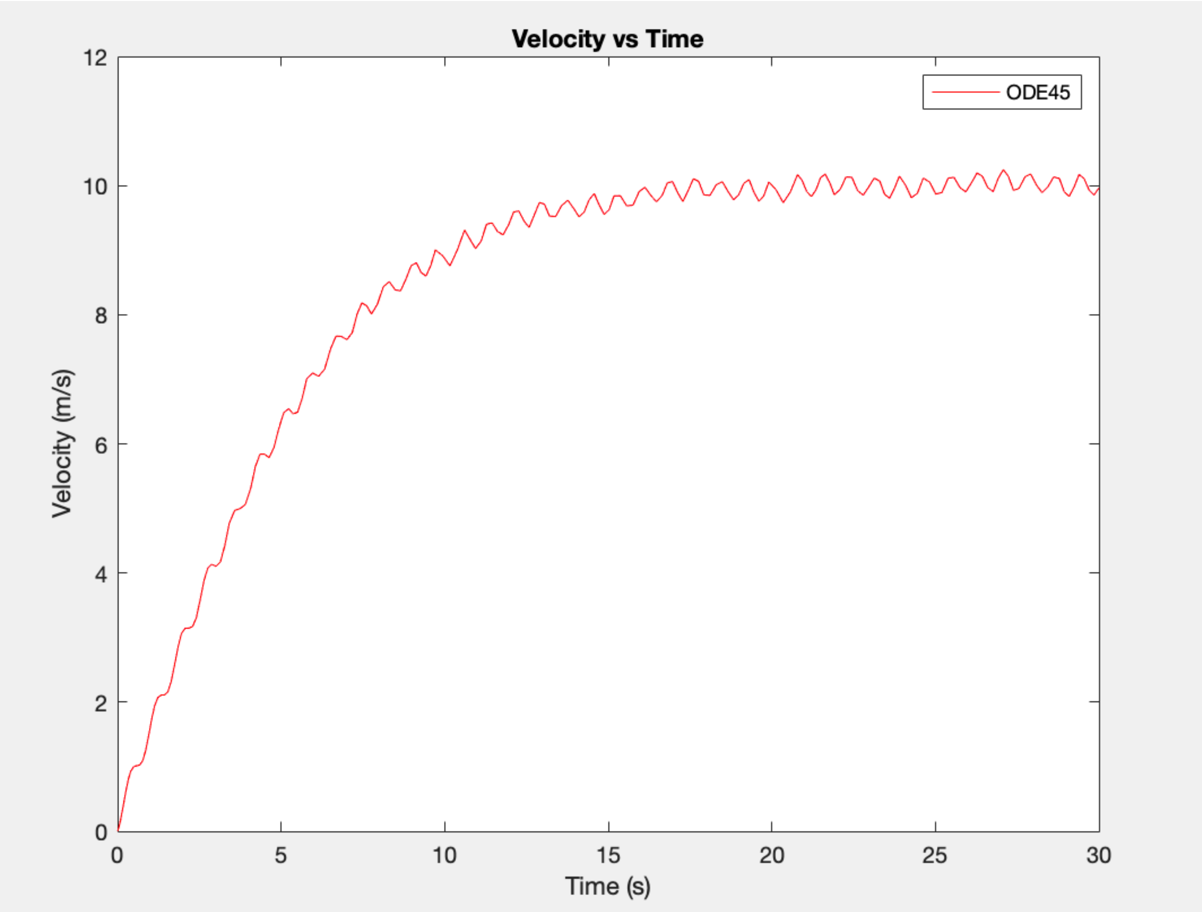
Solving for k_1, k_2 : $k_1 = 12.307$

$k_2 = -2.307$

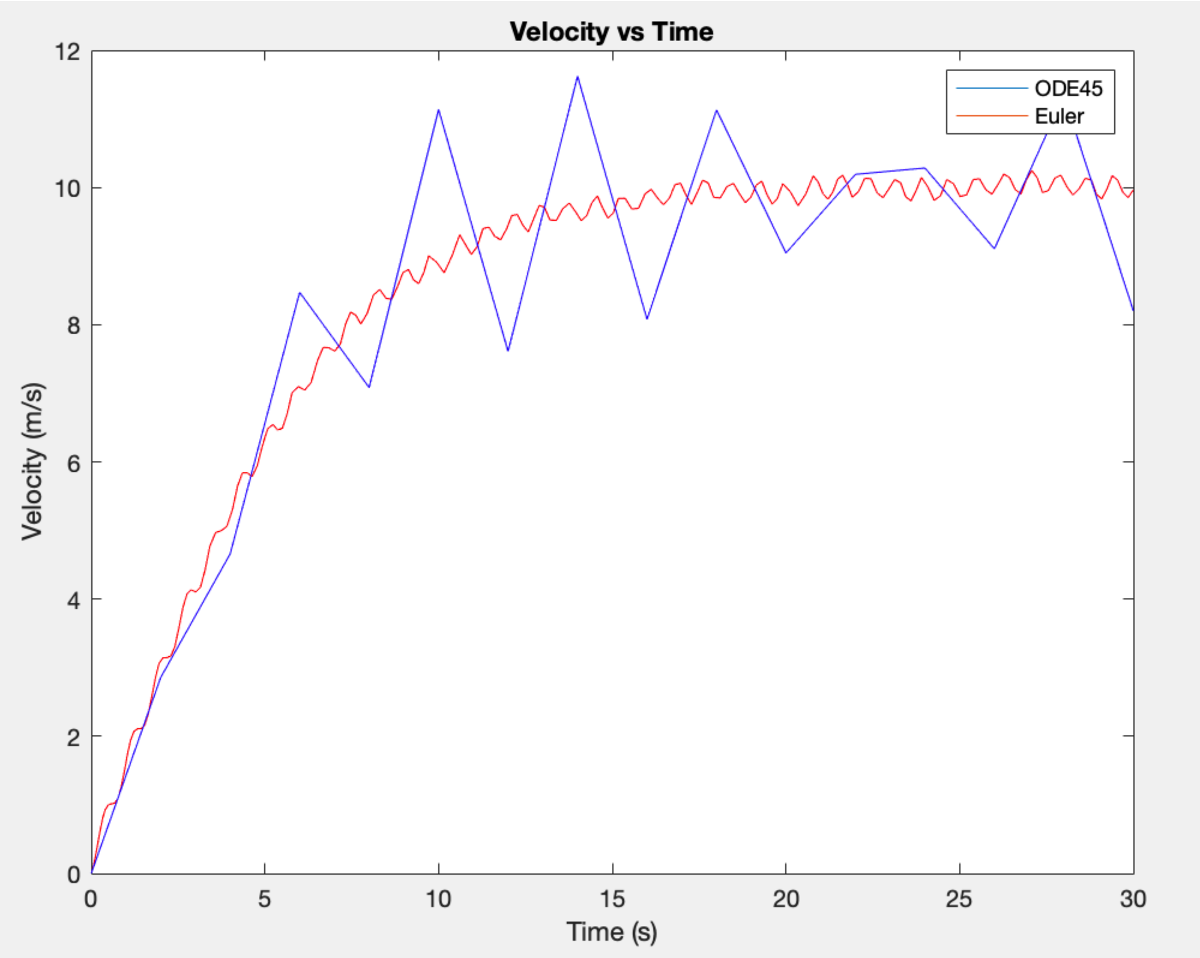
$$\begin{bmatrix} V_L \\ i \end{bmatrix} = 12.307 \begin{bmatrix} 1 \\ -\frac{1}{4} \end{bmatrix} e^{-1t} - 2 \begin{bmatrix} 1 \\ -\frac{4}{3} \end{bmatrix} e^{-3t}$$

Part C Question 1

```
ODEFUN = @(t,v)  (1/70).*(-v.^2 + ((100*sin(8*t)+100)));
TSPAN = [0 30];
Y0 = 1;

clc

[TOUT,YOUT] = ode45(ODEFUN, TSPAN, Y0);
plot(TOUT,YOUT, 'r')

title('Velocity vs Time')
xlabel('Velocity (m/s)')
ylabel('Velocity (s)')
legend('ODE45')
```



Part C Question 2




```
ODEFUN = @(t,v)  (1/70).*(-v.^2 + ((100*sin(8*t)+100)));
TSPAN = [0 30];
Y0 = 1;
h = 2;

clc

[TOUT,YOUT] = ode45(ODEFUN, TSPAN, Y0);
plot(TOUT,YOUT, 'r')

hold on
[TOUT2,YOUT2] = euler(ODEFUN, TSPAN, Y0, h);
plot(TOUT2,YOUT2, 'b')

title('Velocity vs Time')
xlabel('Time (s)')
ylabel('Velocity (m/s)')
legend('ODE45','Euler')

hold off
```

Part C Question 3

```
function [t,v] = euler(ODEFUN,TSPAN,Y0,h)
%EULER(ODEFUN,TSPAN,Y0,h) solved the ODE YOUT' = ODEFUN using Euler's
%method given TSPAN is a time span vector, Y0 is the initial condition
%and h is the step size.

v = Y0;

t = TSPAN(1);
TEND = TSPAN(2);

i = 1;

while t(i) < TEND
    if h > (TEND - t(i))
        h = TEND - t(i); %makes sure solution stops at exactly TEND.
    end
    v(i+1) = v(i) + h*ODEFUN(t(i),v(i));
    t(i+1) = t(i) + h;
    i = i + 1;
end

end

% Above is the Euler function used. Left unchanged from what was given.
```

```
% Here is the developed code to calculate euler and ode45. Along with their
plots.
ODEFUN = @(t,v)  (1/70).*(-v.^2 + ((100*sin(8*t)+100)));
TSPAN = [0 30];
Y0 = 1;
h = 0.01;

clc

[TOUT,YOUT] = ode45(ODEFUN, TSPAN, Y0);
plot(TOUT,YOUT, 'r')

hold on
[TOUT2,YOUT2] = euler(ODEFUN, TSPAN, Y0, h);
plot(TOUT2,YOUT2, 'b')

title('Velocity vs Time')
xlabel('Time (s)')
ylabel('Velocity (m/s)')
legend('ODE45','Euler')

hold off

% Conduction of error testing by doing error testing
Error = YOUT(end) - YOUT2(end)
```

Through the addition of an Error calculation which takes the difference between the y-values of each graph. The step size where high accuracy is provided without wasting too much computer time. Is around 0.01 to 0.03. After this point, by analysing the results the error calculation provides there is no significant improvement in accuracy making it not worth the additional computation time.

Error =

0.0183

This is the error calculated at 0.01 it is fairly small allowing for an accurate model with a step size of 0.01. However, if the step size is changed to 0.001.

Error =

0.0122

As shown here there is no significant improvement I making it smaller. Therefore, values around 0.01 are most efficient and effective.

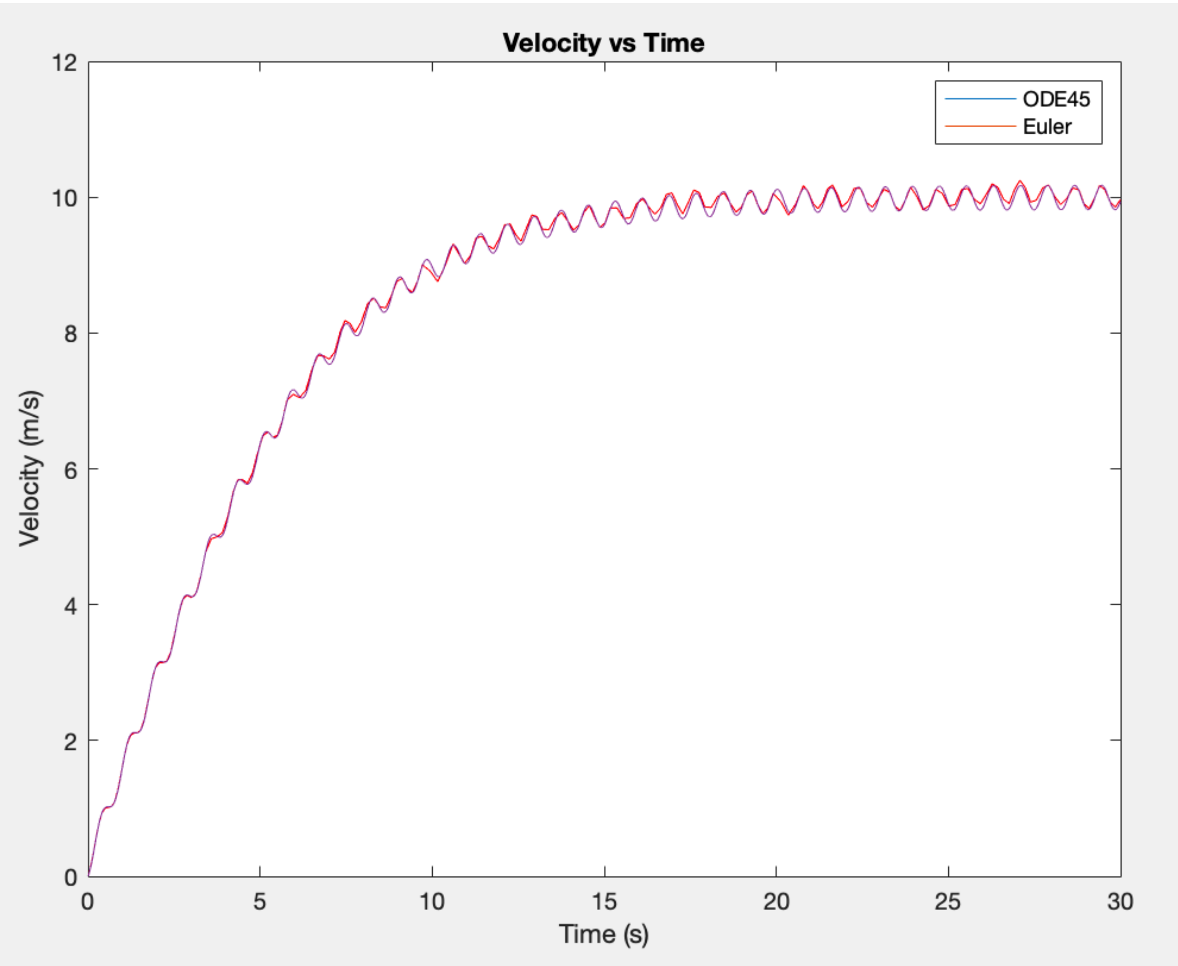


Image above is graph with h = 0.01

Part C Question 4

Let $v' = \frac{dv}{dt}$

$$v' + \frac{1}{70} v^2 = \frac{F(t)}{m}$$

$$v' = \frac{F(t)}{m} - \frac{1}{70} v^2$$

$$x' = V$$

P

Position

```
ODEFUN = @(t,v)  [(1/70).*(-v(1).^2 + ((100*sin(8*t)+100))); v(2)];

TSPAN = [0 30];
Y0 = [0 0];


[TOUT,YOUT] = ode45(ODEFUN, TSPAN, Y0);
plot(TOUT,YOUT, 'r')

title('Velocity vs Position')
xlabel('Position (m)')
ylabel('Velocity (m/s)')
legend('Position vs Velocity')
```