Analysis of a Pendulum's Motion with Reference to Angle, Period, Length, and Q-factor

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December 6, 2023

1 Introduction

The structure of a pendulum is quite simple, consisting of a small but heavy mass hung from a protruding arm or ledge via a light string. This physics lab report explores the dynamics of this system, referencing the expected damped harmonic motion mathematical model, $\theta(t) = \theta_0 e^{-t/\tau} cos(2\pi(t/T) + \phi_0)$ [1]. The goal of this report is to examine how the pendulum's length impacts both the period and Q factor, including the results from Report 1, which analyzed the period versus drop angle, the Q factor, and the amplitude versus time. The experimental setup of this lab involves constructing a pendulum using commonly available products, avoiding the need for special materials understanding.

To begin, the pendulum's period was compared to its initial drop angle, exhibiting a parabolic relationship that can be represented by the power series equation, which provides a T_0 value:

$$T = T_0(1 + B\theta_0 + C\theta_0^2 + \ldots) \tag{1}$$

The values of T_0, B, and C were established as (1.508 ± 0.003)[s], (-0.001 ± 0.001)[rad^{-1}], and (0.068 ± 0.003)[rad^{-2}], respectively. However, at "small angles", experimentally determined to be around 0.3 rad, this dependence seems to experimentally become linear instead of quadratic. Further, the pendulum's amplitude was examined over time, revealing an exponential decay relationship that took on the equation:

$$a(t) = Ae^{-x/\tau} \tag{2}$$

Next, the Q factor, a parameter that quantifies the extent of damping in our pendulum(the ability to maintain motion) was numerically extracted by using the τ value of 42 ± 3 from the decay trend and the period:

$$Q = \pi\frac{\tau}{T} \tag{3}$$

This Q factor was then taken for multiple pendulum lengths, presenting a positively linear relationship between these two quantities. However, the change in length did seem to influence the period of oscillation of the pendulum, plotted to the relation:

$$T = kL^n \tag{4}$$

The results of this trend were compared to those of the equation, $T(L) = k\sqrt{L}$, confirming a congruence.

The ensuing sections will more thoroughly cover the experimental details, methodology, results, and a final conclusion in a valuable manner to readers involved with a deeper analysis of pendulum qualities.

2 Methods and Procedures

A 26.7 by 16.5 by 73.7-centimeter toilet paper holder[2] was used as the mount for the pendulum string. This was thought to suffice as a strong and sturdy base, able to resist the motion of the swinging object, due to its flat, metal build. A marble radius (1.25±0.05) cm and mass (0.019±0.001) kg was used, attached to the frame by braided jute rope with an adjustable length.

The string was orientated in a unique way to prevent an elliptical motion of the weight and promote a straight back-and-forth oscillation. A double string, which resembled the shape of an isosceles triangle when a mass is hung on it, was used. The plane of reference of this "triangle" was aligned to be parallel to the camera's line of sight. This would reduce the pendulum's movement outside of its ideal plane of oscillation. Further, the string was also able to be adjusted so the isosceles triangle in question could change in dimension, and the marble was tied to it in a manner in which it could freely slide to its lowest point. A protractor was lastly attached to the pendulum's frame so that the drop angle could be kept around 50° and the discrepancies could be minimized.

Figure 1: String and ball setup. The string is attached to the pendulum arm at two ends instead of one, and the ball is tied to the string via a loose knot that allows it to move along the string freely.

A Pixel 7 back camera with 4k resolution and 60fps The pendulum was aligned to the pendulum using the approximation provided by the tiled flooring, and the marble was dropped in line with the tiles in another attempt to maintain two-dimensional oscillation. Data collection was done through the Tracker App, which is a data tracking software that can provide various types of data associated with the pendulum's oscillation. The app was used to find the position and angle of the pendulum mass at its highest points with respect to a set axis. Further, the $fitexample.py$ script was edited so that periods were considered to begin at each maximum; in the same oscillation, the period was tracked starting at a positive θ and again at a negative θ. This way, this one set of data would provide almost twice as many period measurements as it would originally have. This was then plotted into a Period versus Angle graph and an Amplitude versus Time graph via a custom Python code. Finally, since the videos were taken at 60fps, the video capture error is half a frame, or 0.0083s.

3 Results

Figure 2 depicts the relationship between period and drop angle. In an ideal system, the line of best fit should be a horizontal line, but this graph does not follow the predicted model and instead plots a parabola. The experimental trend suggests that a greater absolute angle correlates to a larger period, which also disagrees with the prediction that $T = 2\sqrt{L}$ as the period should be constant along with a constant length. Possible explanations for this could be the inherent resisting forces, including drag and friction that are present in this non-perfect system, or the results of pendulum oscillation due to gravity. Further, another factor that could have influenced the accuracy of the trendline is the impreciseness of the maxima plotting as the phone camera only captures 60fps, meaning the uncertainty is 0.0083s. Uncertainties that could not be defined by numerical values are the fluctuation in the pendulum's oscillation path and the effects of the camera's viewing perspective. The data points are also scattered with large differentials, and the trendline does not fall within the uncertainty of many data points.

Variable B can be perceived as experimentally zero as its value is within twice its uncertainty. This value being experimentally zero represents that there is no asymmetry within the period versus angle trend, further depicted by the graph. It is also important to answer the question of whether there is a range of amplitudes that is "small enough" that the value of C can be considered to be zero, where the trendline becomes horizontal. In this case, $C\theta^2$ can be ignored when its value is smaller than its uncertainty. This was reevaluated through another dataset as the original dataset did not contain angles of that extent. The values in which C can be experimentally represented as zero is when the angles range from -0.344 rad to 0.333 rad(-19.7° to 19.1°).

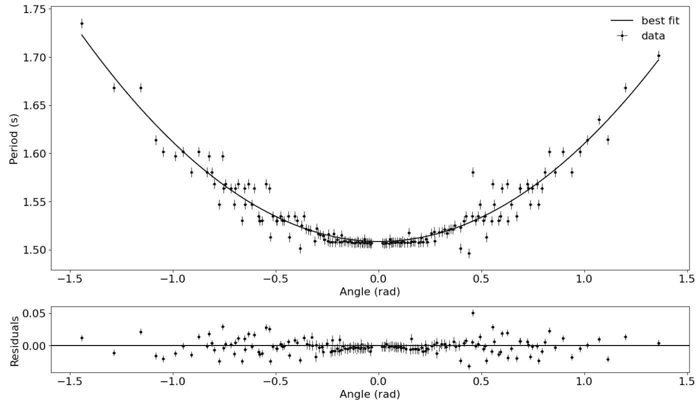

Figure 2: Data comparison between the period of motion and the angle of the pendulum at its highest position. The angular displacement to the right of the equilibrium position has been considered positive, and the left, negative. The best-fit line is a quadratic, represented by $T = (0.102 \pm 0.004)[s(rad)^{-2}]\theta^2 + (-0.001 \pm 0.002)[s(rad)^{-1}]\theta + (1.508 \pm 0.003)[s]$. Therefore, in power series form, T_0, B, and C were established as 1.508 ± 0.003 $[s]$, -0.001 ± 0.001 $[rad^{-1}]$, and 0.068 ± 0.003 $[rad^{-2}]$. trendline.

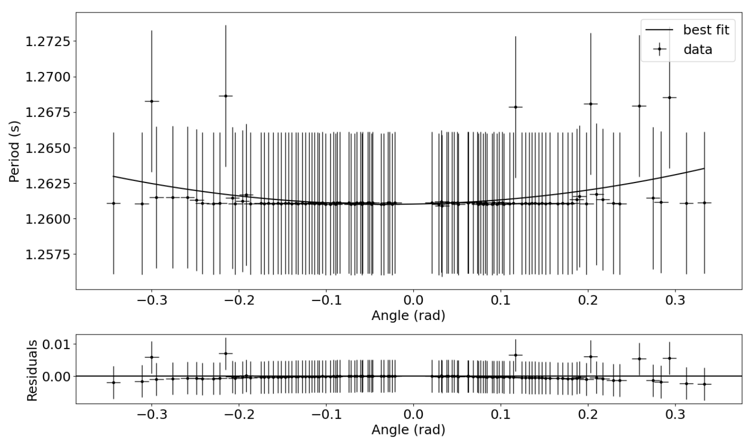

Figure 3: Small angles extracted from the dataset shows no correspondence between angle and period. Here, C has the value of 0.010 ± 0.007, thus can be considered experimentally zero. $T = 1.26(1 + (0)\theta + (0)\theta^2)$
$T = 1.26$

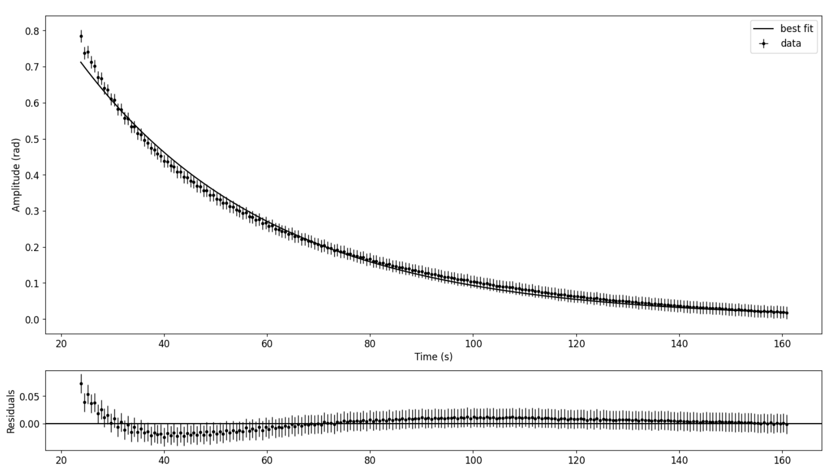

Figure 4: Time series data of the amplitude of the pendulum. The best fit takes on the equation of $(1.05 \pm 0.04)e^{-t/(37.4 \pm 0.3)}$. The residual graph displays that the data points generally follow the trendline, except at greater angles.

As shown in Figure 4, the amplitude decays at a much faster rate than the theoretical model at greater angles, but coincides with the best-fit line as time progresses. An explanation for this variance at the beginning of the oscillation may be due to the greater degrees of movement during the earlier stages of oscillation. As the pendulum swings over a greater distance in the beginning, it is more susceptible to the effects of friction and drag, meaning more energy is dissipated than predicted by the theoretical model. The Tracker App's ability to measure angles was also a consideration for uncertainty, with a measure of 0.02 rad. Further, the Q factor was extracted from this plot via two methods: counting oscillations for Q/4 by finding when the amplitude becomes around 46% and utilizing the small angle approximation to find a more accurate Q-factor approximation. The former provided a highly inaccurate value of 83 ± 3, and the latter, a more accurate approach, found the Q-factor as 90 ± 1, a 1.08 multiple of the result from the former method.

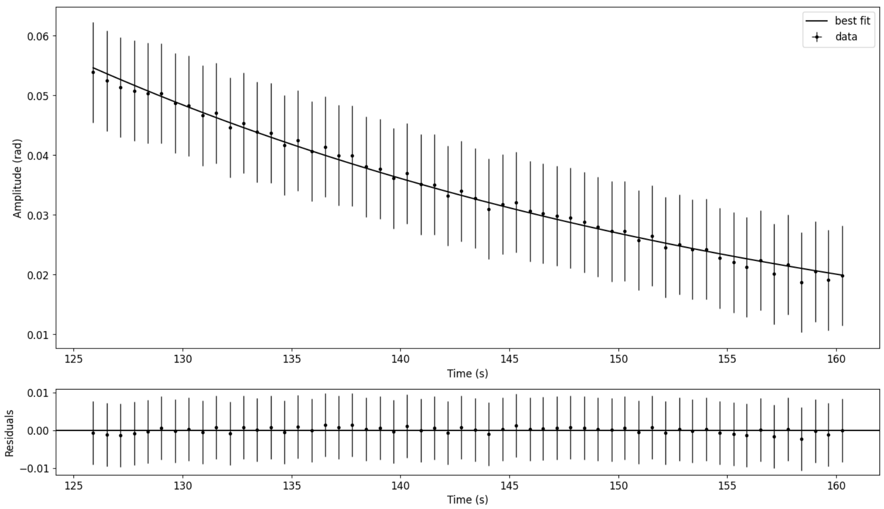

Figure 5: Small angle analysis of amplitude[rad] compared to time[s], which instead depicts a near trend even though plotted through an exponential decay function. Here, $\tau = 36.0 \pm 0.5$. Data points almost perfectly overlap with the trendline.

Using the equation $Q = (\pi * \tau)/T$, where $\tau = 36.0 \pm 0.5$ and T was the average period value of the data points in Figure 4, or 1.25 ± 0.03, the Q-factor was found to be $Q = \pi * 36/1.26 = 90 \pm 1$.

To compare string length and period, 8 lengths were analyzed: (39.4 ± 0.1) cm, (32.0 ± 0.1) cm, (26.7 ± 0.1) cm, (21.6 ± 0.1) cm, (19.1 ± 0.1) cm, (0.159 ± 0.1) cm, (8.8 ± 0.1) cm, (5.5 ± 0.1) cm. The video data taken from these string lengths were summarized into a period versus length graph and a Q-factor versus length graph.

Figure 6 compares the period data to the length of the pendulum string. The period was obtained by taking the average of measured periods produced throughout the oscillation of a single string length, and then changing the length to obtain a trend. This fit is intuitively sound when observing the period as length approaches zero as a string length of zero would allow for no possible oscillation. Further, changes within string lengths would increase the gravitational potential energy within the system, so an initial increase in length would result in an immediate increase in the period. However, the resistance forces within the pendulum's motion change differently. As the pendulum mass was consistently dropped at approximately the same angle, the friction force between the string and pendulum arm remained consistent, but the drag changed in proportion to the arc motion traveled by the pendulum. With this constant friction but changing drag forces, the percentage change in work done on the pendulum will be smaller with changes between shorter lengths and greater between changes of longer lengths. Though the period-length graph was fit to a relationship much like $T = 2\sqrt{L}$, the data points do not accurately follow the best-fit line possibly due to damping within the pendulum's system. However, it is acceptable to say that the trend follows the trend of $T = 2\sqrt{L}$.

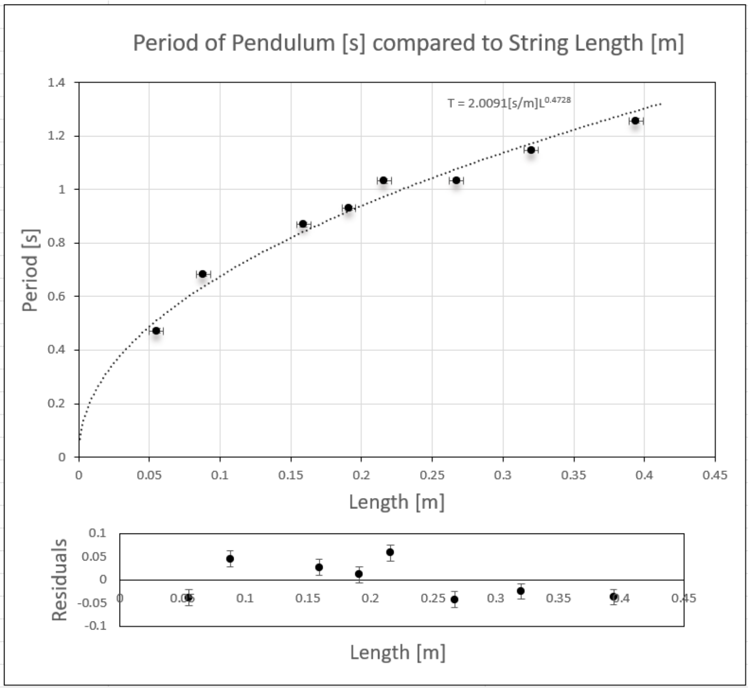

Figure 6: Data comparison between the pendulum's period of oscillation[s] compared to its string length[m]. This depicts a relationship with the equation $T = (2.009 + 0.003)[s/m](L^{0.4728+0.0007})$. Though the data points generally follow the trend, they do not converge with the trendline, even with the uncertainty.

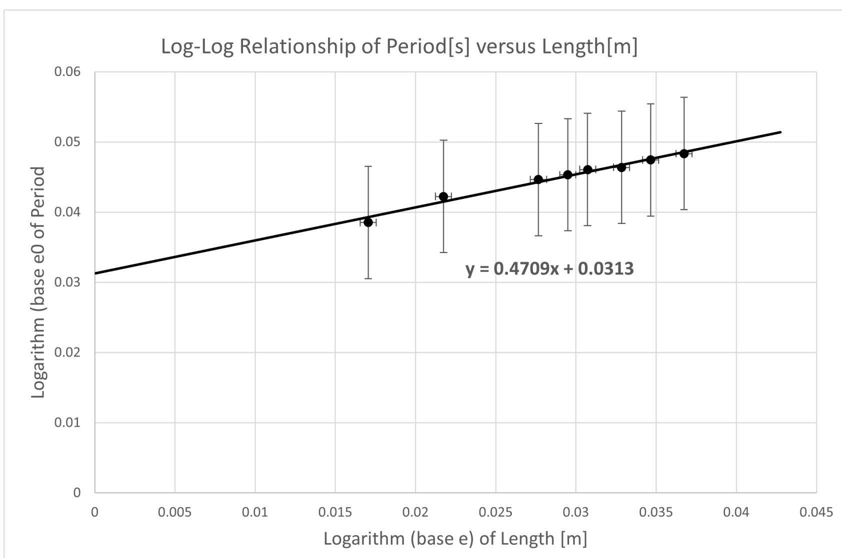

Figure 7: Log-log graph of the period versus string length. $T = 0.4707[s/m]L + 0.0313[s]$

The linear trend of this log-log graph further validates the exponential(square root) trend of the period versus length as the logarithm is the inverse of exponential. The reason why this plot does not converge to the origin may be a result of the measurement uncertainties and the effects of energy dissipation. Further exploration and experimentation will have to be done on this section to enhance the understanding of this predicament.

Figure 8 depicts the possible relationships between the Q-factor and the length of this experiment. The data was plotted to both a linear relationship(black line) and a quadratic relationship(blue line). Both these relationships could resemble the general trend of the data points; however, theoretically, there should be no notable relationship between the Q-factor and length. Possible explanations for the Q-factor's apparent dependency on string length may be attributed to a significant frictional force between my pendulum string

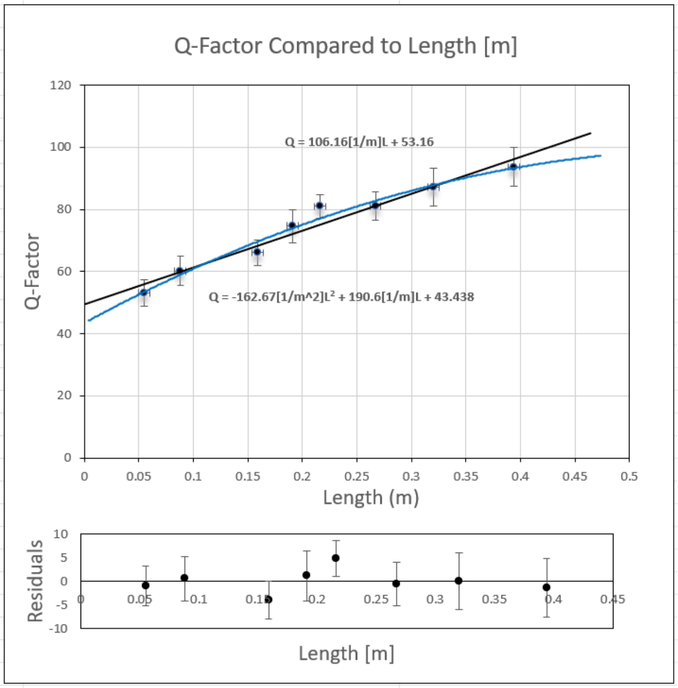

Figure 8: Q-factor compared to Length, which presents a linearly increasing relationship that conforms to the equation $Q = (106 \pm 4)[1/m]L + (53.2 \pm 0.4)$. The residual graph displays that the data points generally converge with the trendline.

and arm. With smaller string lengths, the period is shorter, meaning the string will undergo many more oscillations, and experience a greater effect of energy dissipation. The greater degree of frictional interaction between the string and arm during smaller string lengths could likely be the source of this experimental finding. Further, only eight data points were accounted for, so a greater accuracy in the trend could be discovered through more plots.

4 Conclusion

In conclusion, the period of oscillation was not shown to be independent of the angle. This may be because many factors are involved in the period of motion in a real, non-perfect system, such as frictional components. In terms of amplitude versus time, it was revealed that there were perhaps external damping forces that aligned with the model $Ae^{-T/\tau}$, and the greatest source of uncertainty was likely to be the angle of 0.02 rad. This plot further allowed the Q factor to be extracted and finalized as 90 ± 1. The Q-factor was also not shown to be independent of the string length though it should theoretically be. This is also likely due to the resistance forces that were discovered to be more relevant during shorter lengths and smaller periods.

Overall, the period versus angle analysis, the amplitude versus angle analysis, the period versus length comparisons, and the Q-factor versus length examination revealed, respectively, a quadratic relationship, an exponential decay relationship, a root function trend, and a possible linear or quadratic relationship. In terms of the amplitude versus time, the data was plotted to a power series, with T_0, B, and C as (1.508 ± 0.003)[s], (-0.001 $\pm$ 0.001)[rad^{-1}], and (0.068 ± 0.003)[rad^{-2}]. The amplitude versus time plot heavily followed a decay relationship of $a(t) = Ae^{-x/\tau}$, with $A = 1.05 \pm 0.04$ and $\tau = 37.4 \pm 0.3$. Further, the new period versus time analysis followed the $T = kL^n$ relationship, with $k = 2$ and $n = 0.4728$. Finally, the Q-factor and Length comparison was plotted to 2 possible outcomes, with trendlines as $Q = (106 \pm 4)[1/m]L + (53.2 \pm 0.4)$ or $Q = (-162 \pm 2)[1/m^2]L2 + (190.6 \pm 0.4)[1/m]L + (43.4 \pm 0.8)$.

This lab investigation sheds light on a damped harmonic oscillator and prompts further exploration to gain a better understanding of pendulum motion. To adapt to Report One, the oscillation path was improved by using two strings instead of one. However, with this new analysis, some newfound sources of improvements could include deploying the Tracker Application's auto-tracking ability, increasing the string length, and decreasing the initial drop angle to minimize frictional forces within the system.

4.0.1 references

[1] B. Wilson, “Phy180 pendulum project guidelines,” 2023.

[2] “Umbra loft eclipse toilet paper holder, nickel finish, 10.5 x 6.5 x 29-in,” https://www.canadiantire.ca/en/pdp/umbra-loft-eclipse-toilet-paper-holder-nickel-finish-10-5-x-6-5-x-29-in-0632482p.0632482.html?rq=nickel+finish+toilet+paper+eclipse+holder#srp

[3] “Tracker 6.1.5,” https://physlets.org/tracker/, 2023.