# Example - Haar Wavelets

- Suppose we are given a 1D "image" with a resolution of 4 pixels:

$$[9\ 7\ 3\ 5]$$

- The Haar wavelet transform is the following:

$$[6\quad 2\quad 1\quad -1]$$

$L_0$ $D_1$ $D_2$ $D_3$

# Example - Haar Wavelets (cont' d)

- Start by averaging the pixels together (pairwise) to get a new lower resolution image:

$$[8 \quad 4] \text{ (averaged and subsampled)}$$

- To recover the original four pixels from the two averaged pixels, store some *detail coefficients*.

| *Resolution* | *Averages* | *Detail Coefficients* |
|---|---|---|
| 4 | [9 7 3 5] | [] |
| 2 | [8 4] | [1 −1] |

# Example - Haar Wavelets (cont' d)

- Repeating this process on the averages gives the full decomposition:

| *Resolution* | *Averages* | *Detail Coefficients* |
|---|---|---|
| 4 | [9 7 3 5] | [] |
| 2 | [8 4] | [1 −1] |
| 4 | [6] | [2] |

# Example - Haar Wavelets (cont’d)

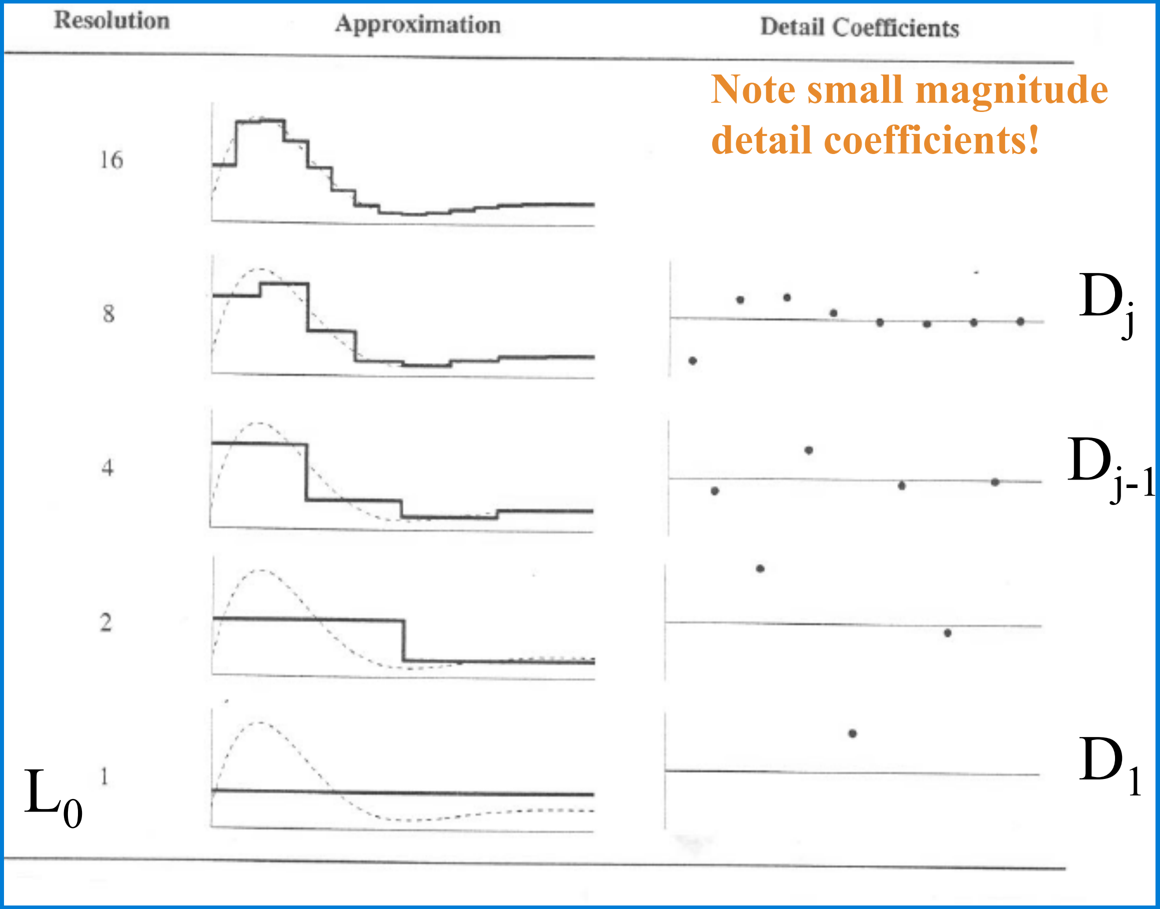
- The Harr decomposition of the original four-pixel image is:

$$[6 \quad 2 \quad 1 \quad -1]$$

- We can reconstruct the original image to a resolution by adding or subtracting the detail coefficients from the lower-resolution versions.

$$[6 \quad 2 \quad 1 \quad -1] \xrightarrow{2} [8 \quad 4] \xrightarrow{1 \ -1} [9 \quad 7 \quad 3 \quad 5]$$

# Example - Haar Wavelets (cont' d)




How to compute $D_i$ ?

# How to compute $D_i$ ? (cont' d)

- If f(t) ϵ $V_{j+1}$, then f(t) can be represented using basis functions φ(t) from $V_{j+1}$:

$V_{j+1}$

$$f(t) = \sum_k c_k \varphi(2^{j+1} t - k)$$

Alternatively, f(t) can be represented using **two** basis functions, φ(t) from $V_j$ and ψ(t) from $W_j$:

$$V_{j+1} = V_j + W_j$$

$$f(t) = \sum_k c_k \varphi(2^j t - k) + \sum_k d_{jk} \psi(2^j t - k)$$

# How to compute $D_i$ ? (cont’ d)

Think of $W_j$ as a means to represent the parts of a function in $V_{j+1}$ that cannot be represented in $V_j$

$$f(t) = \sum_k c_k \varphi(2^{j+1} t - k)$$

$$\downarrow$$

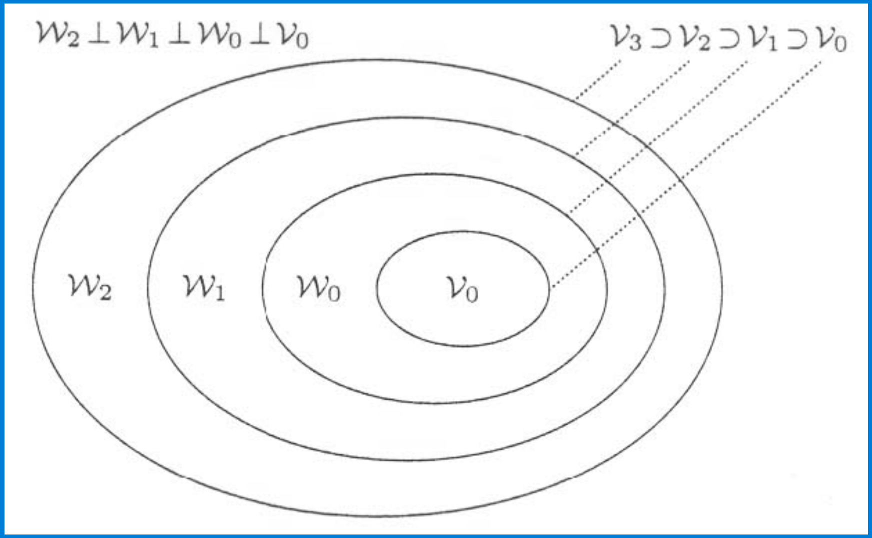
$$f(t) = \underbrace{\sum_k c_k \varphi(2^j t - k)}_{V_j,} + \underbrace{\sum_k d_{j,k} \psi(2^j t - k)}_{W_j}$$

differences between $V_j$ and $V_{j+1}$

# How to compute $D_i$ ? (cont' d)

- $V_{j+1} = V_j + W_j$ → using recursion on $V_j$:

$$V_{j+1} = V_{j-1} + W_{j-1} + W_j = \ldots = V_0 + W_0 + W_1 + W_2 + \ldots + W_j$$




if $f(t) \in V_{j+1}$ , then:

$$f(t) = \sum_k c_k \varphi(t-k) + \sum_k \sum_j d_{j,k} \psi(2^j t - k)$$

$V_0$ basis functions

$W_0, W_1, W_2, \ldots$ basis functions

# Wavelet expansion (cont’d)

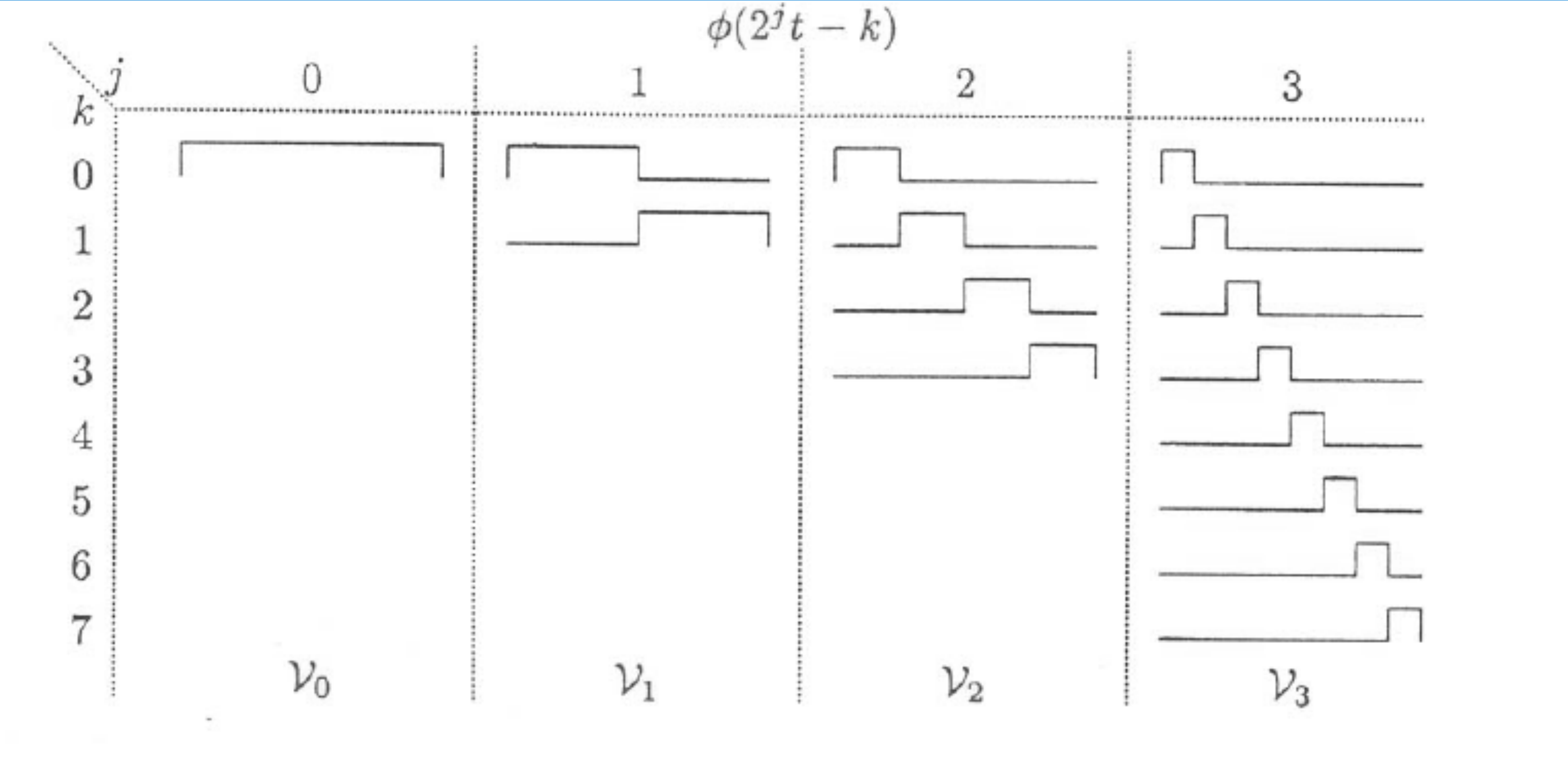
- f(t) is written as a linear combination of $\varphi(t-k)$ and $\psi(2^j t-k)$ :

$$f(t) = \sum_k c_k \varphi(t-k) + \sum_k \sum_j d_{jk} \psi(2^j t - k)$$

scaling function wavelet function

*Note:* in Fourier analysis, there are only two possible values of k ( i.e., 0 and $\pi/2$); the values j correspond to different scales (i.e., frequencies).

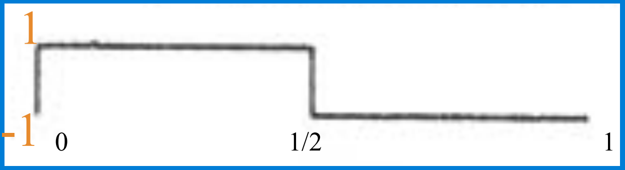
# 1D Haar Wavelets (cont’d)



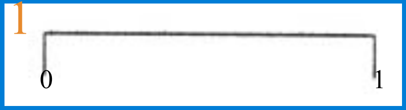
# 1D Haar Wavelets (cont’d)

- Mother wavelet function:

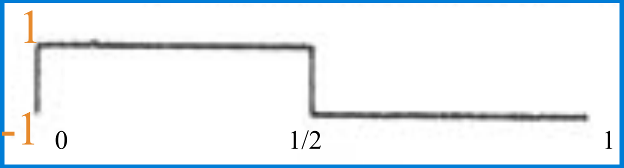
$$\psi(x) = \begin{cases} 1 & \text{if } 0 \le x < 1/2 \\ -1 & \text{if } 1/2 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$




- Note that φ(x) . ψ(x) = 0 (i.e., orthogonal)



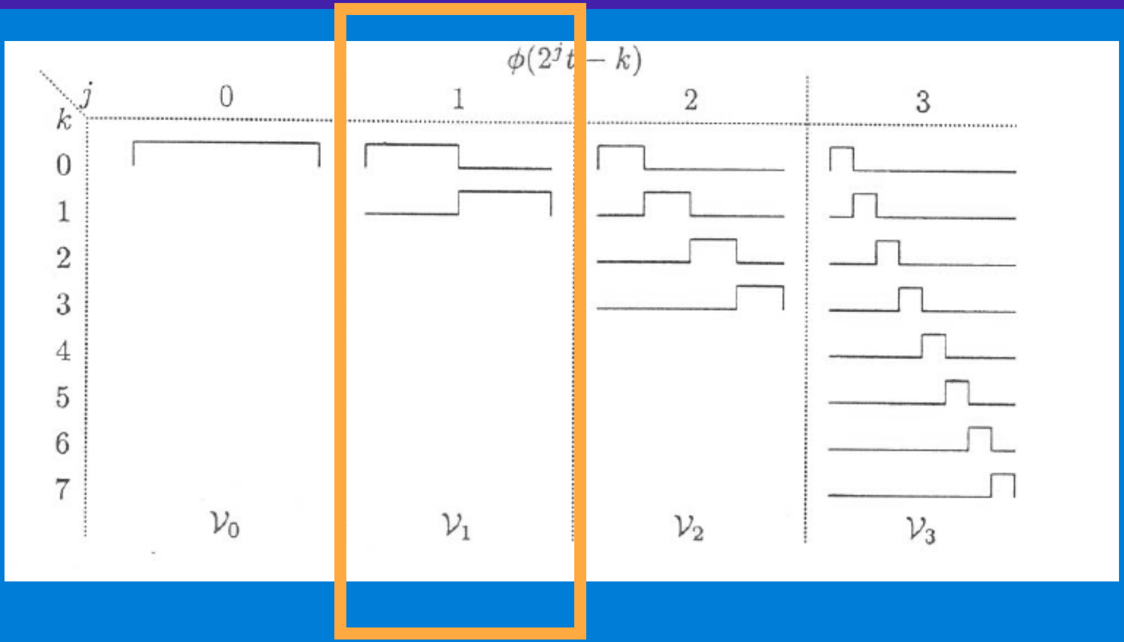

.



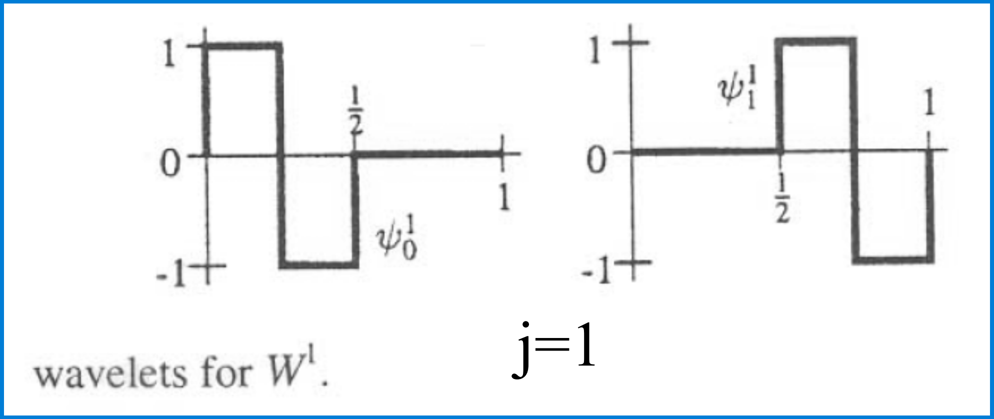

= 0

# 1D Haar Wavelets (cont' d)

basis for $V_1$:



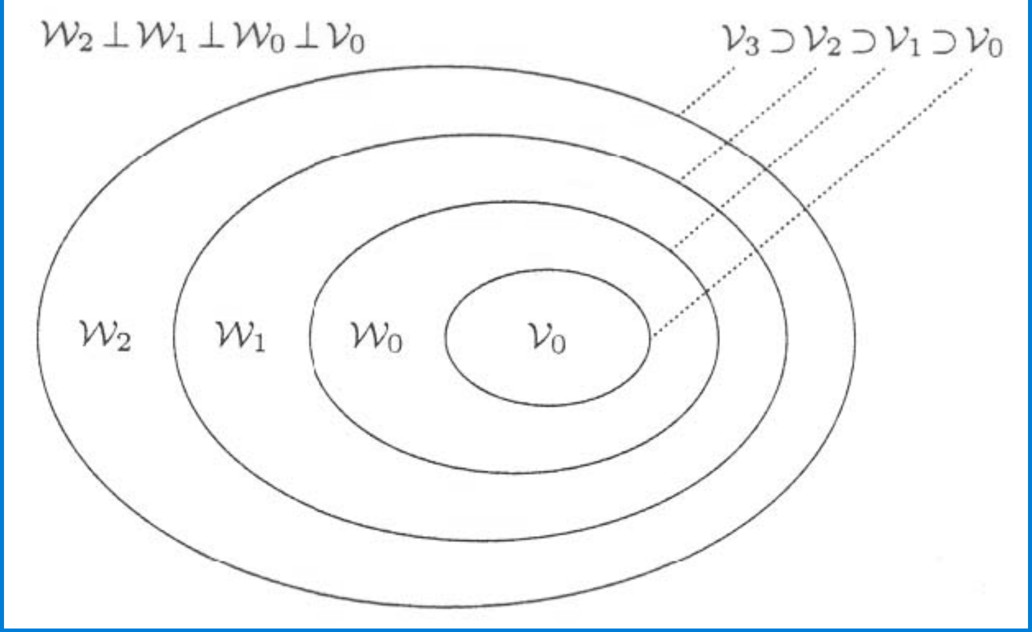

basis $W_1$:



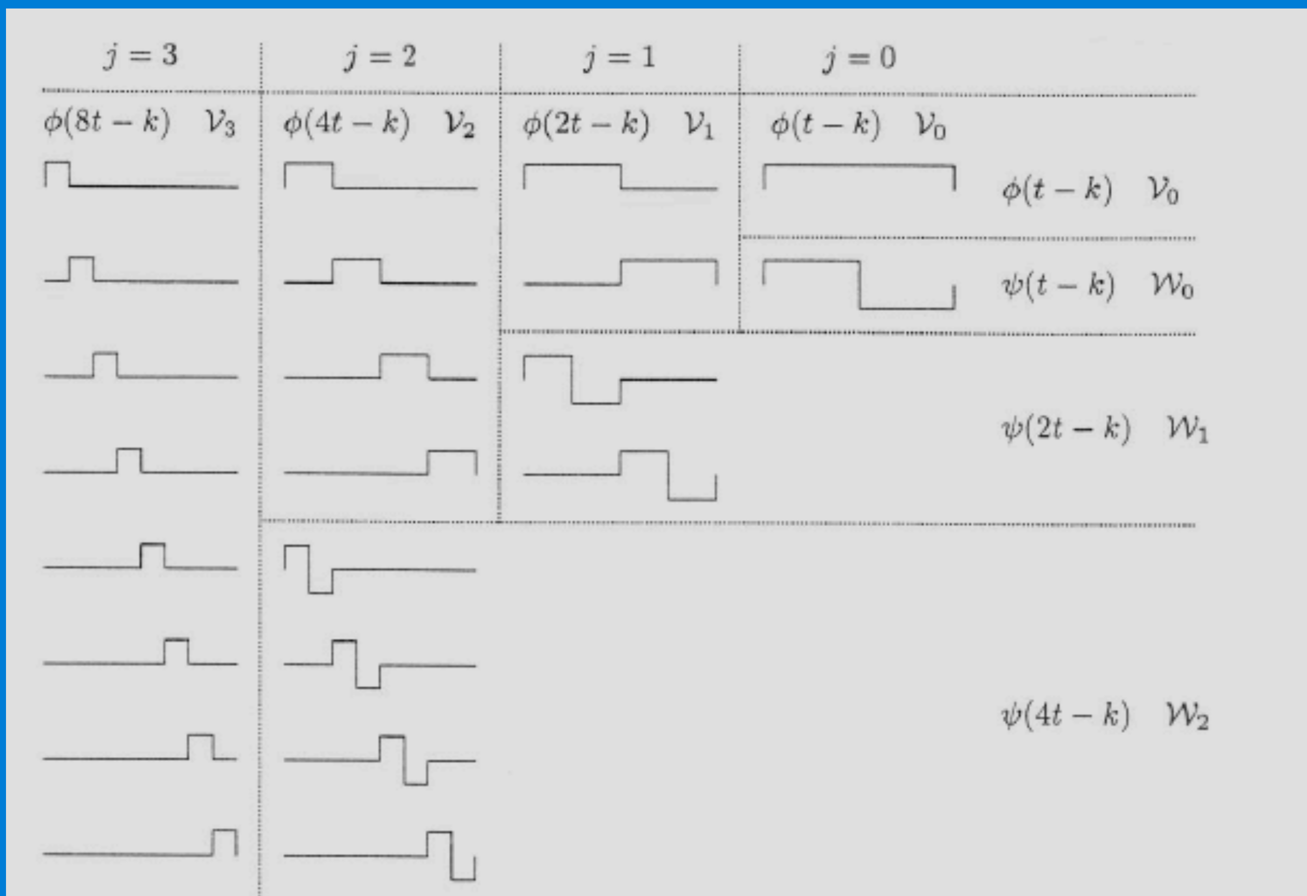

Note that inner product is zero!

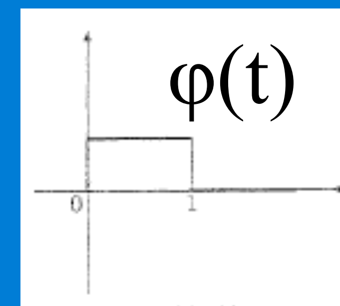
# 1D Haar Wavelets (cont'd)

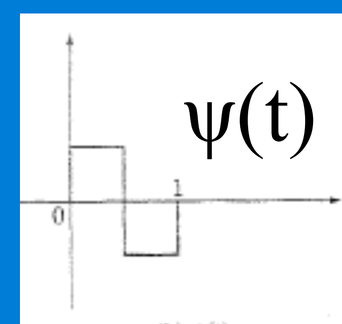
$$\left.\begin{array}{l}\text{Basis functions } \psi^{j}_{i} \text{ of } W_{j} \\ \text{Basis functions } \varphi^{j}_{i} \text{ of } V_{j}\end{array}\right\} \text{ form a basis in } V_{j+1}$$



# 1D Haar Wavelets (cont' d)



# Example - Haar basis (revisited)

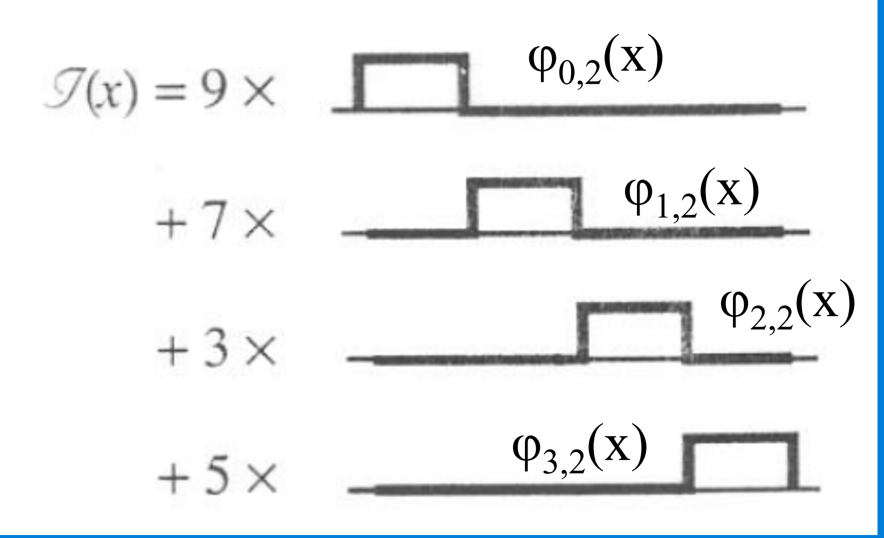
| *Resolution* | *Averages* | *Detail Coefficients* |
|---|---|---|
| 4 | $[9\ \ 7\ \ 3\ \ 5]$ | $[]$ |
| 2 | $[8\ \ 4]$ | $[1\ \ -1]$ |
| 4 | $[6]$ | $[2]$ |

# Decomposition of f(x)

f(x)= [9 7 3 5]

using the basis functions in $V_2$

$$f(x) = c_0^2\phi_0^2(x) + c_1^2\phi_1^2(x) + c_2^2\phi_2^2(x) + c_3^2\phi_3^2(x)$$

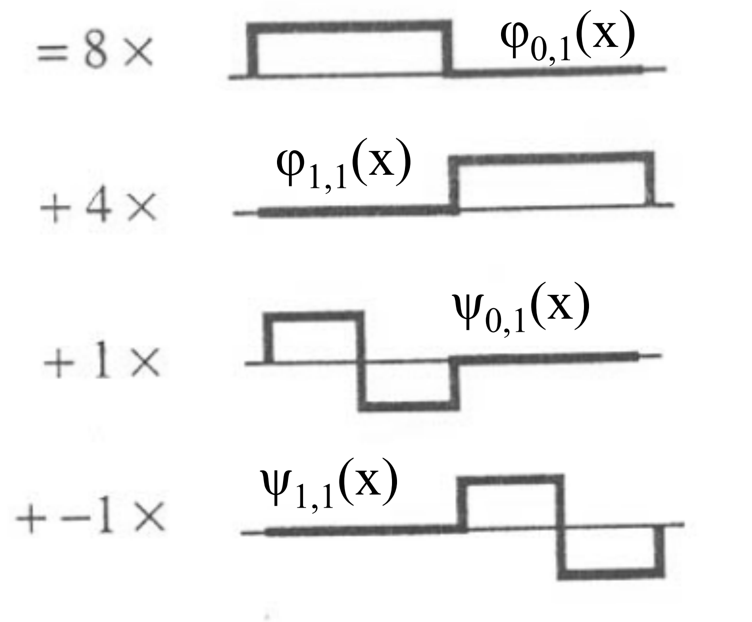


# Decomposition of f(x) (cont’d)

using the basis functions in $V_1$ and $W_1$

$$V_2 = V_1 + W_1$$

$$f(x) = c_0^1 \phi_0^1(x) + c_1^1 \phi_1^1(x) + d_0^1 \psi_0^1(x) + d_1^1 \psi_1^1(x)$$



# Example - Haar basis (revisited)

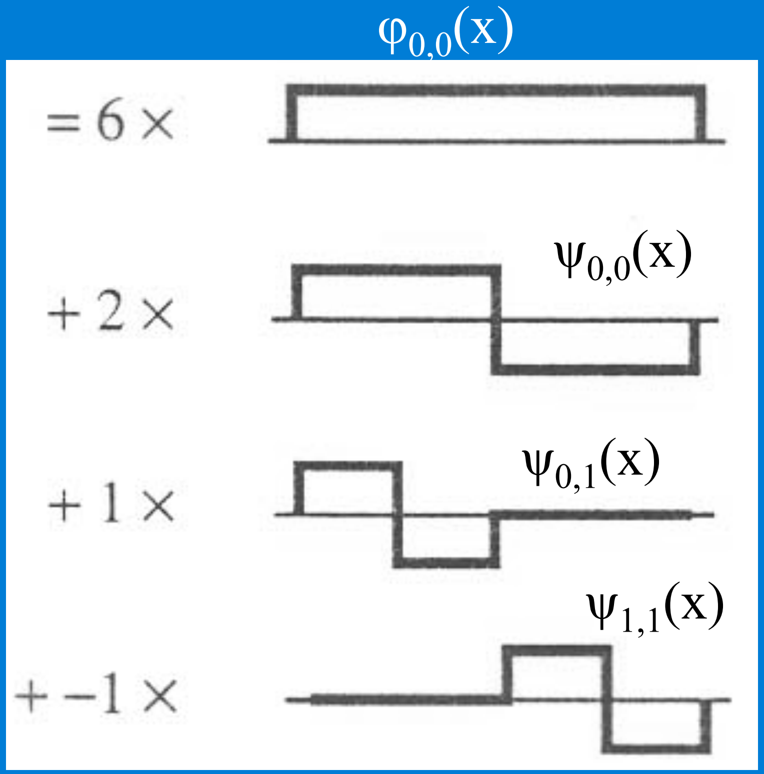
| *Resolution* | *Averages* | *Detail Coefficients* |
|---|---|---|
| 4 | [9 7 3 5] | [] |
| 2 | [8 4] | [1 −1] |

# Decomposition of f(x) (cont' d)

using the basis functions in $V_0$ ,$W_0$ and $W_1$

$$V_2=V_1+W_1=V_0+W_0+W_1$$

$$f(x) = c_0^0\phi_0^0(x) + d_0^0\psi_0^0(x) + d_0^1\psi_0^1(x) + d_1^1\psi_1^1(x)$$

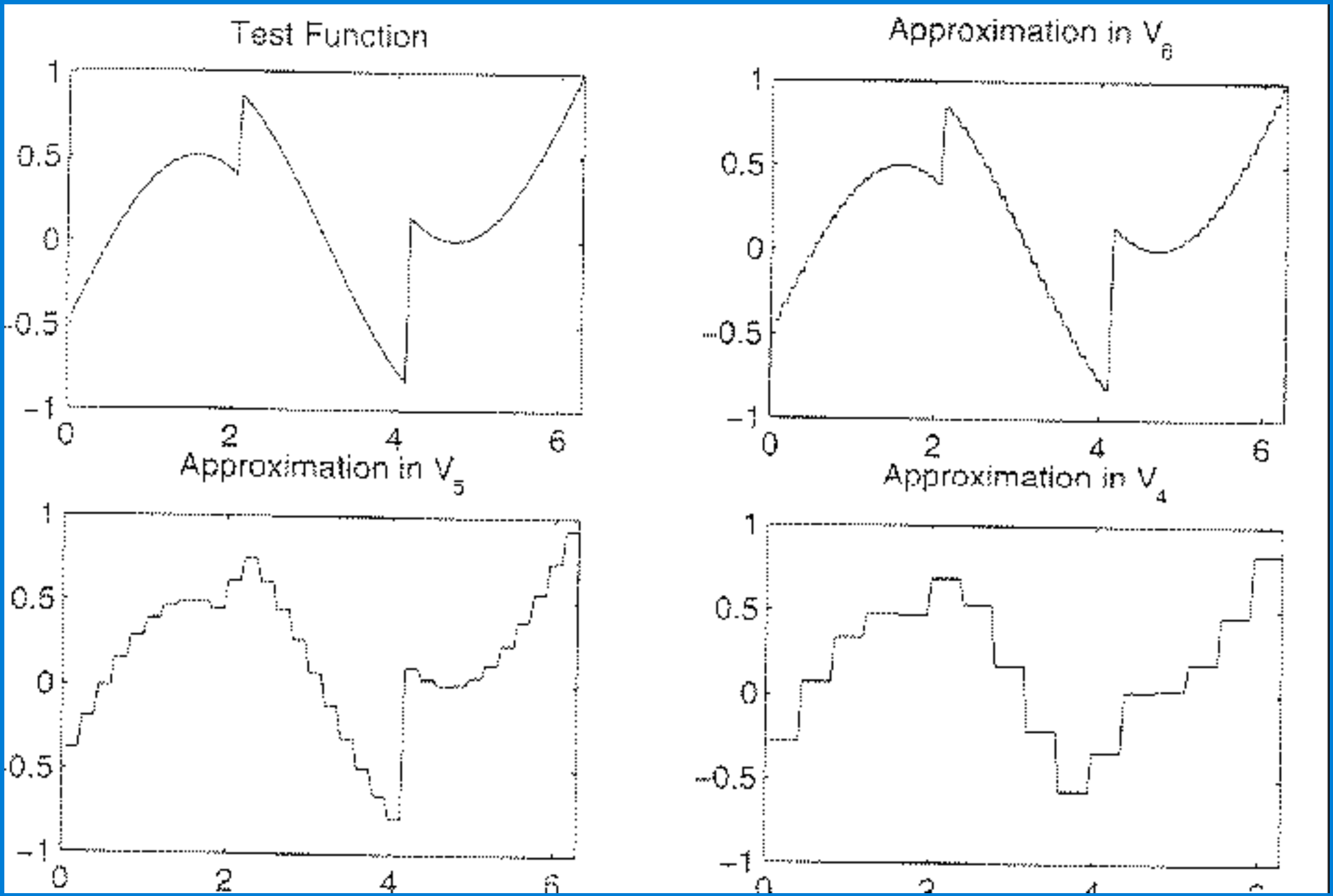


# Example - Haar basis (revisited)

| *Resolution* | *Averages* | *Detail Coefficients* |
|---|---|---|
| 4 | [9 7 3 5] | [] |
| 2 | [8 4] | [1 −1] |
| 4 | [6] | [2] |

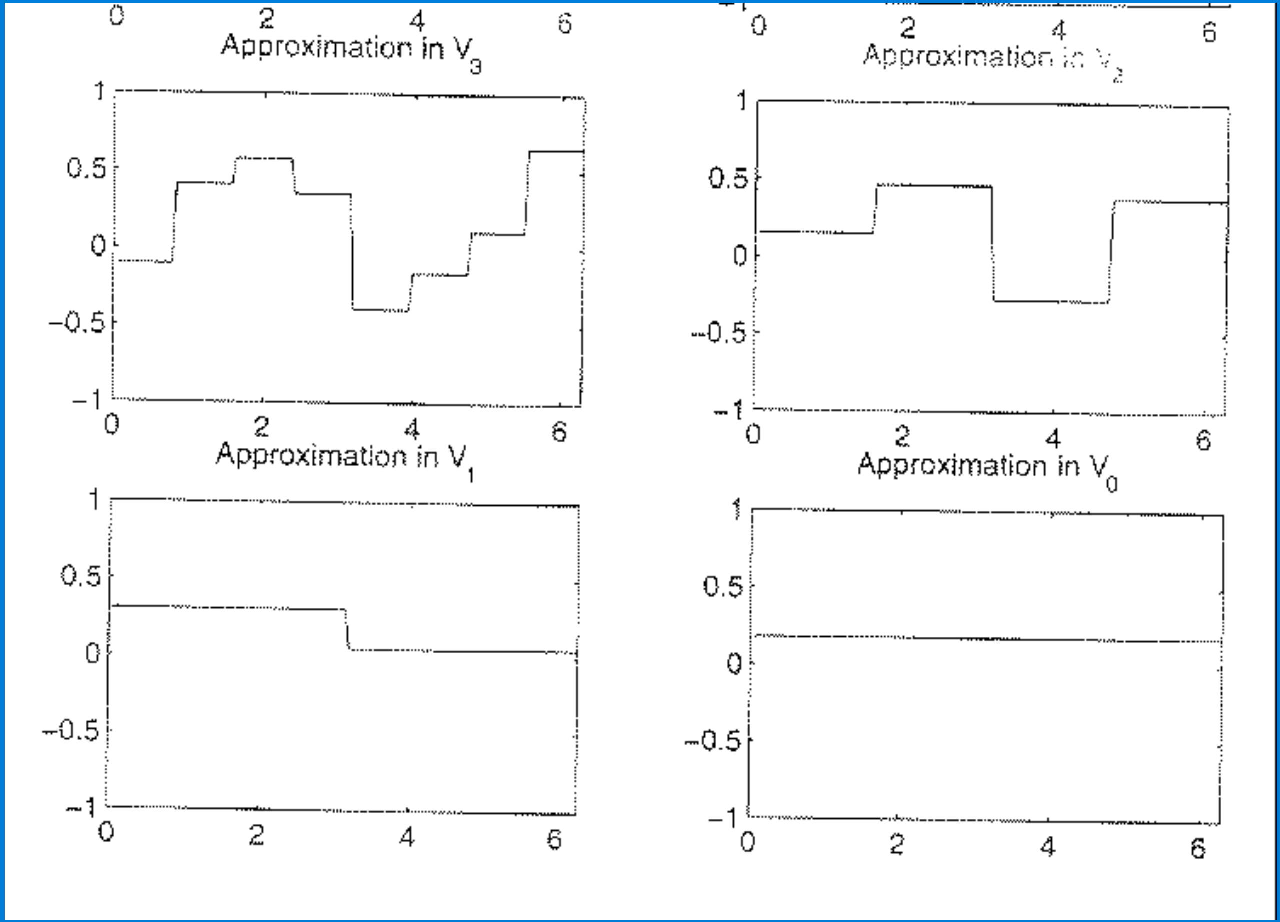
$$[6 \;\; 2 \;\; 1 \;\; -1]$$

# Example

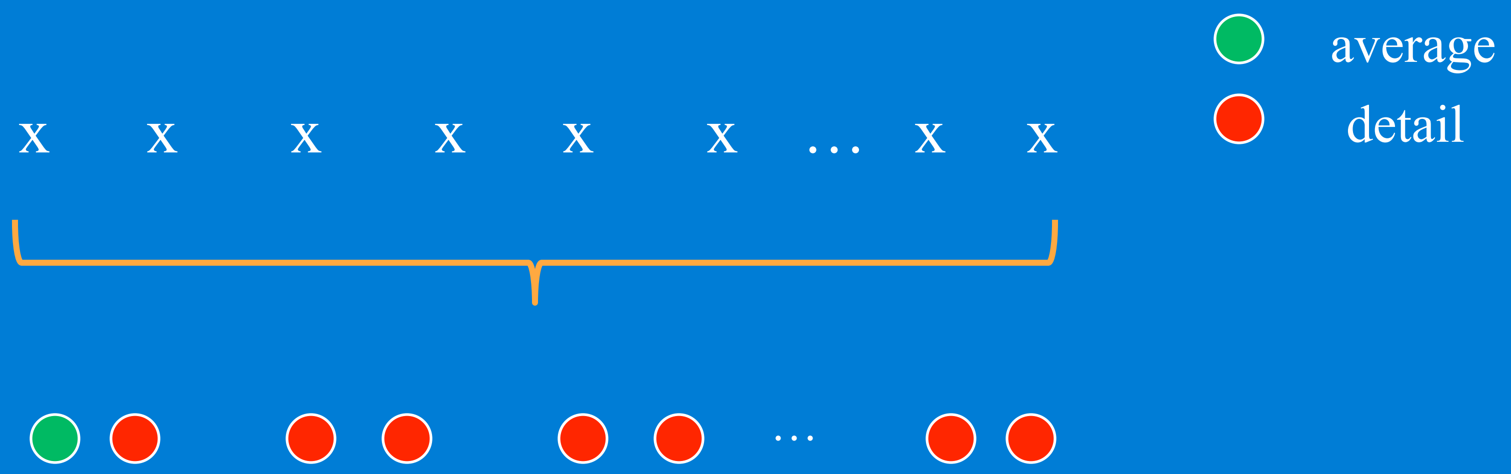


Test Function
Approximation in $V_6$
Approximation in $V_5$
Approximation in $V_4$

# Example (cont’d)



# Convention for illustrating 1D Haar wavelet decomposition (cont’d)



# 2D Haar Wavelet Transform

- The 2D Haar wavelet decomposition can be computed using 1D Haar wavelet decompositions (i.e., 2D Haar wavelet basis is separable).
- Two decompositions
  - Standard decomposition
  - Non-standard decomposition
- Each decomposition corresponds to a different set of 2D basis functions.

# Standard Haar wavelet decomposition

- Steps

  (1) Compute 1D Haar wavelet decomposition of each row of the original pixel values.

  (2) Compute 1D Haar wavelet decomposition of each column of the row-transformed pixels.

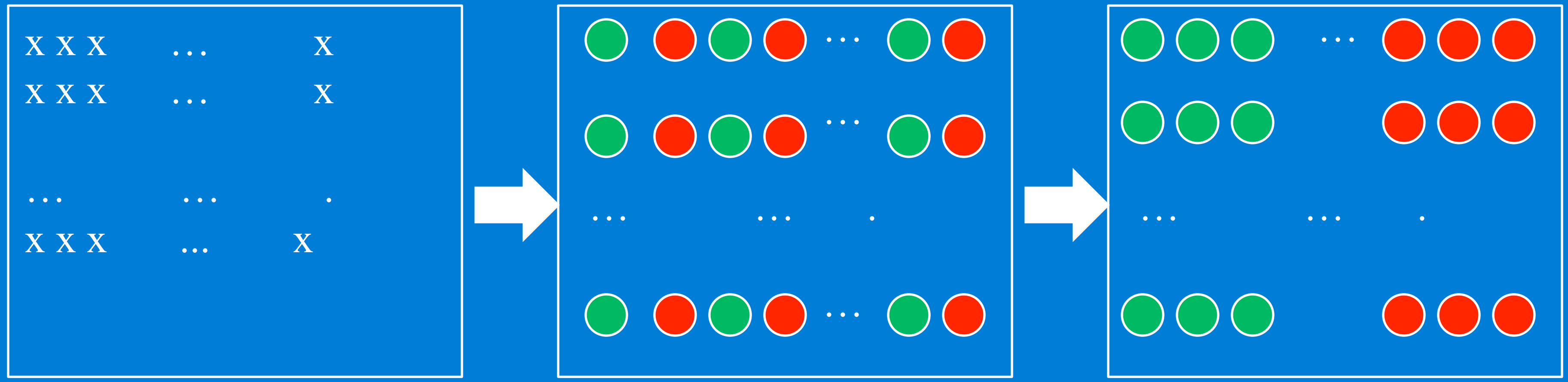
# Standard Haar wavelet decomposition (cont’d)

average

detail

(1) row-wise Haar decomposition:

re-arrange terms



# Standard Haar wavelet decomposition (cont'd)

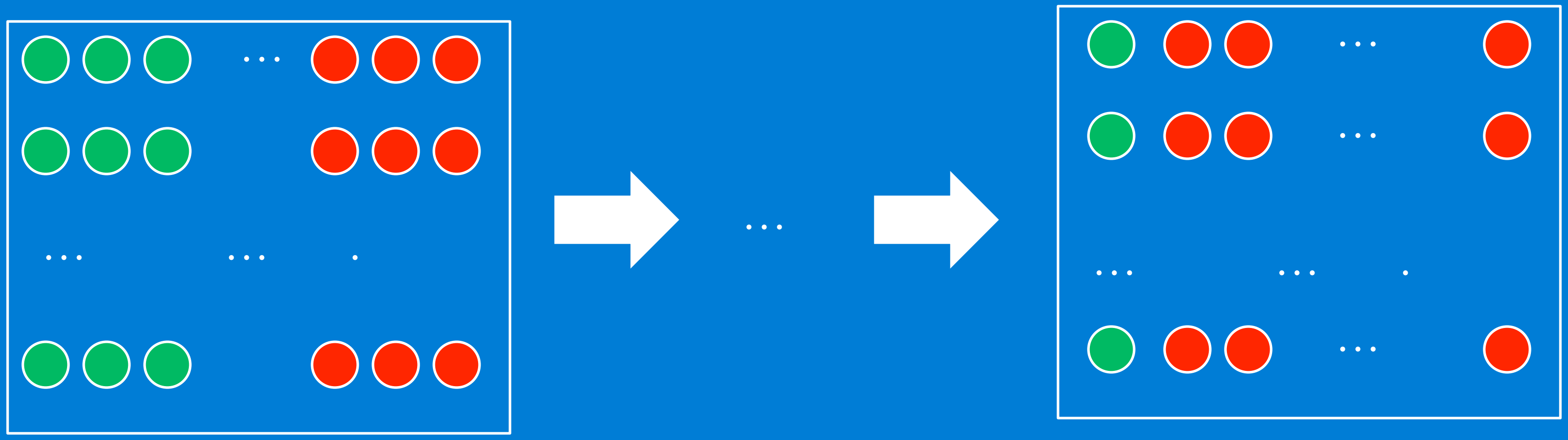
average

detail

(1) row-wise Haar decomposition:

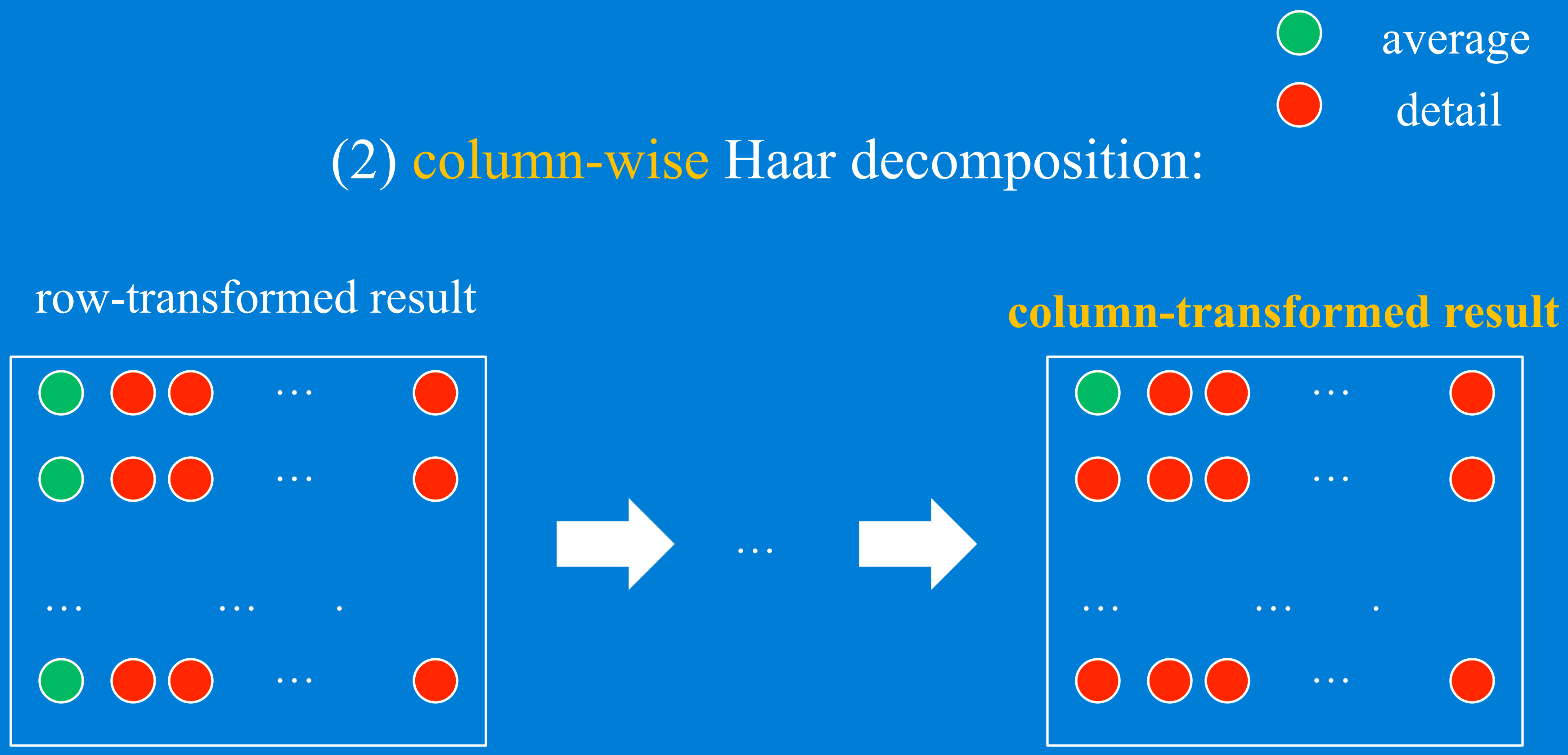
from previous slide:

**row-transformed result**

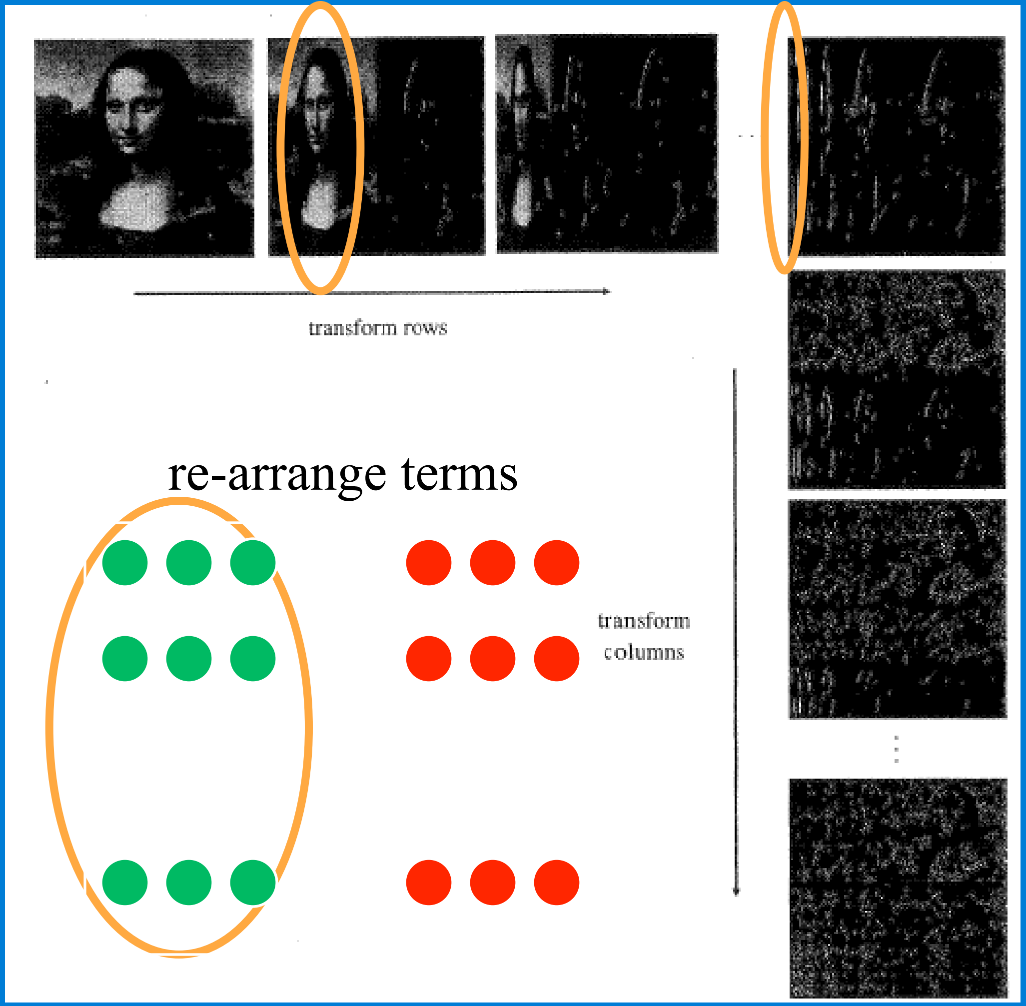


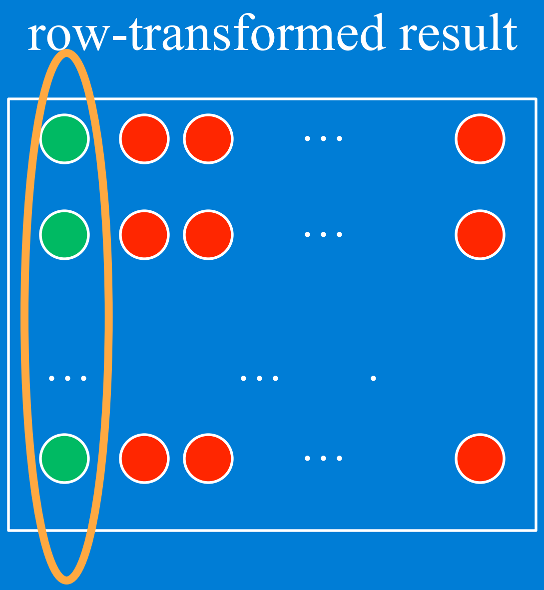
# Standard Haar wavelet decomposition (cont’d)

(2) column-wise Haar decomposition:

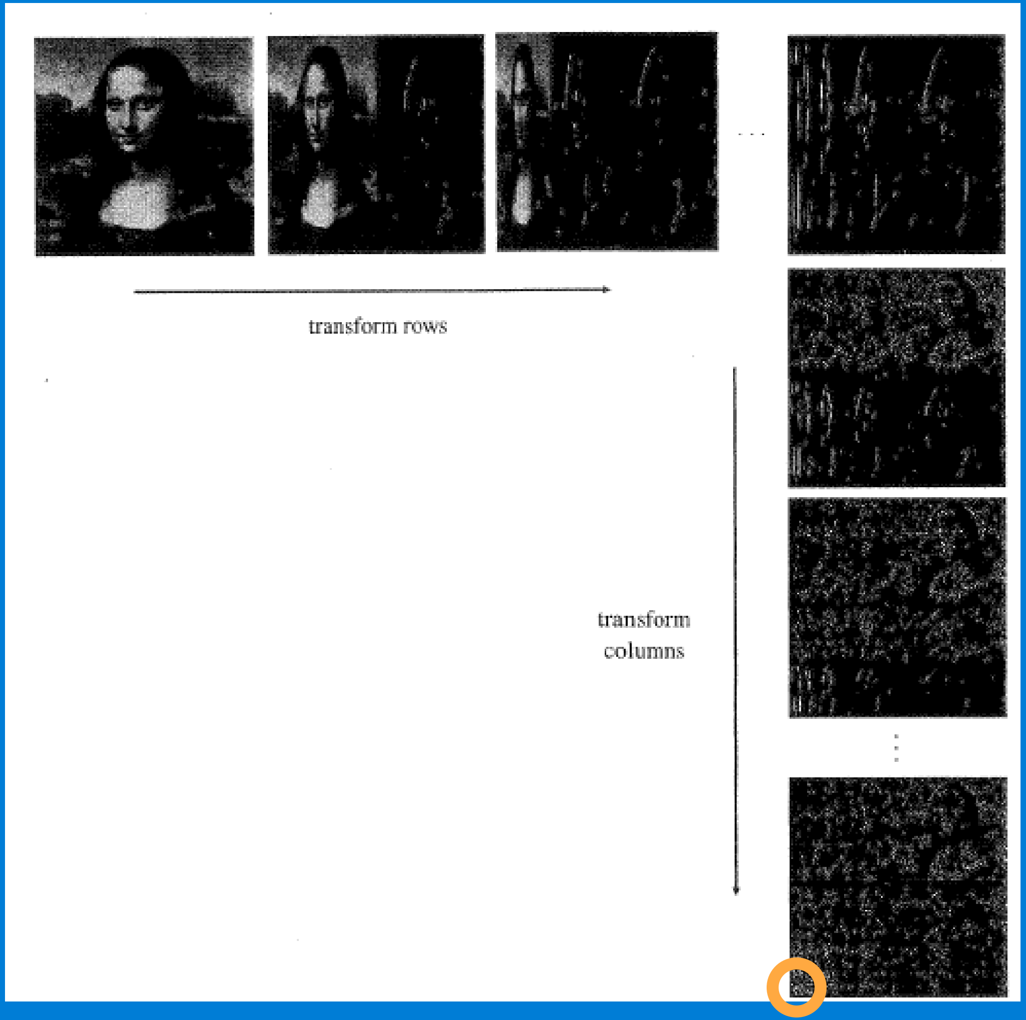


# Example

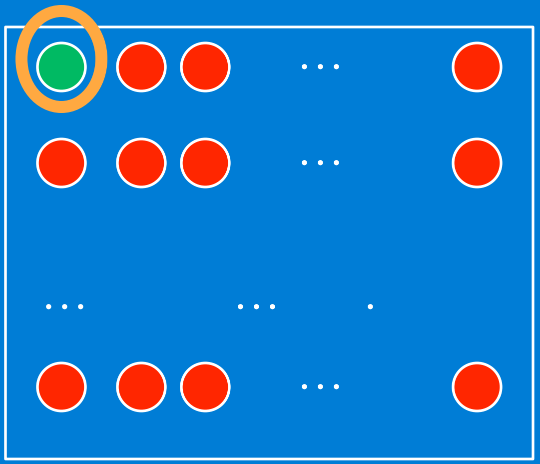

transform rows
re-arrange terms
transform columns


row-transformed result

# Example (cont’d)




column-transformed result



# Non-standard Haar wavelet decomposition

- Alternates between operations on rows and columns.

  (1) Perform one level decomposition in each row (i.e., one step of horizontal pairwise averaging and differencing).

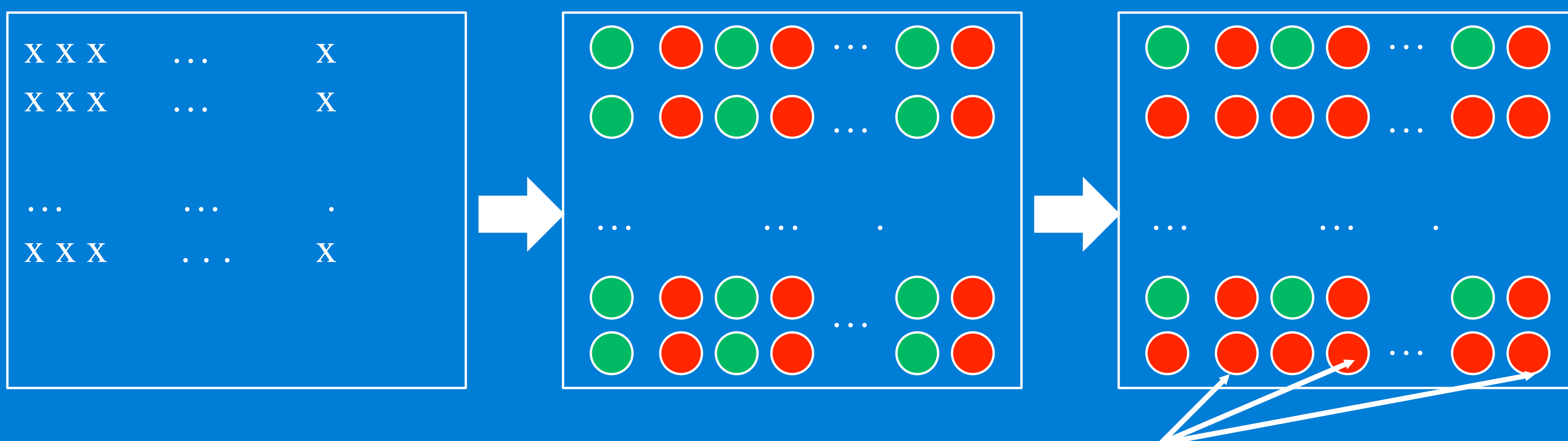
  (2) Perform one level decomposition in each column from step 1 (i.e., one step of vertical pairwise averaging and differencing).

  (3) Repeat the process on the quadrant containing averages only (i.e., in both directions).

# Non-standard Haar wavelet decomposition (cont’d)

one level, horizontal
Haar decomposition:

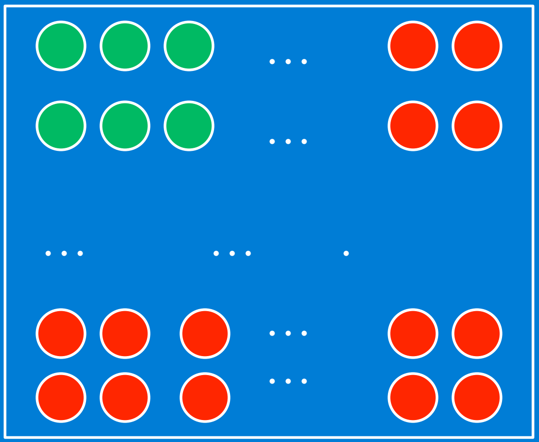
one level, vertical
Haar decomposition:




Note: averaging/differencing
of detail coefficients shown

# Non-standard Haar wavelet decomposition (cont’d)

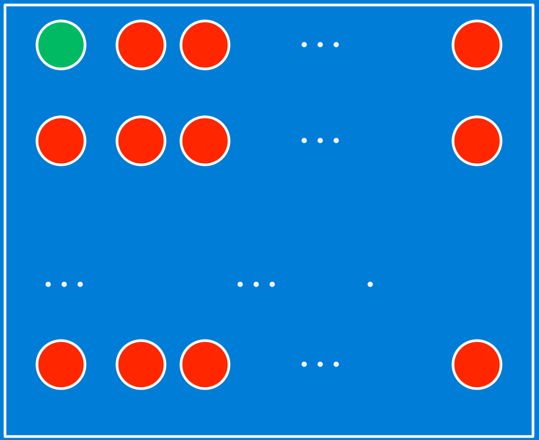
re-arrange terms



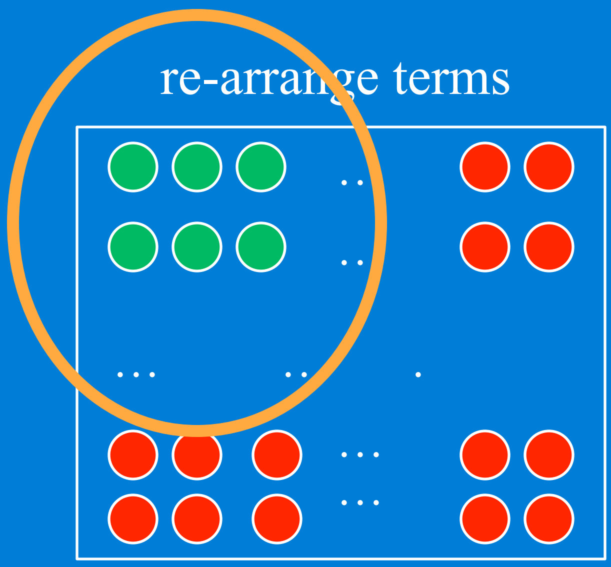
one level, horizontal
Haar decomposition
on “green” quadrant

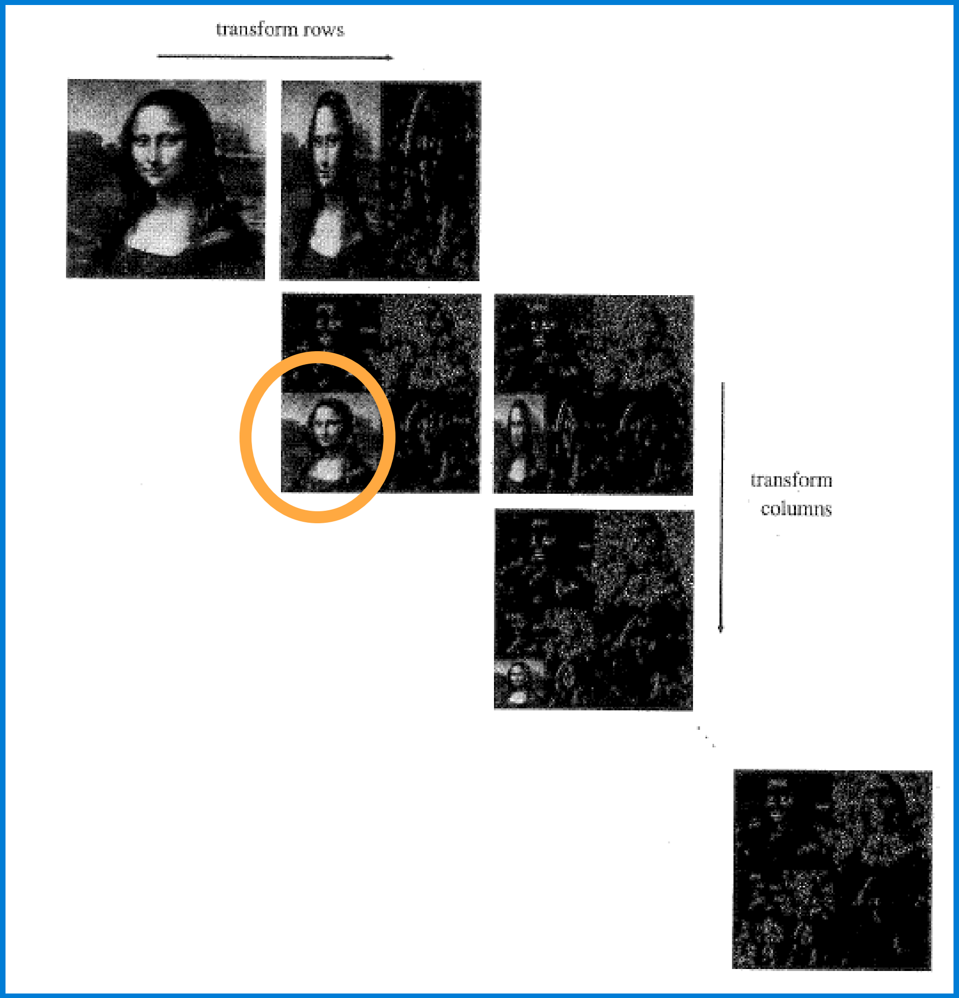
one level, vertical
Haar decomposition
on “green” quadrant

…



# Example



# Example (cont’d)