Design and Analysis of Algorithms

Assignment-1

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CS6A

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Q. Find the asymptotic order of the solution for the below recurrence equation using suitable methods. You may assume $T(1)=1$, the recurrence is for $n>1$

$$T(n) = 2T(n/2) + \frac{n}{\log n}$$

Ans. Given,

$$T(n) = 2T(n/2) + \frac{n}{\log n} \quad (i)$$

Put $n = \frac{n}{2}$ in eq (i)

We get,

$$T(\tfrac{n}{2}) = 2T(n/4) + \frac{n}{2\log \frac{n}{2}} \quad (ii)$$

Substitute the value of $T(\frac{n}{2})$ from (ii) to (i)

$$T(n) = 2\left(2T(n/4) + \frac{n}{2\log\frac{n}{2}}\right) + \frac{n}{\log n}$$

$$\Rightarrow T(n) = 4T(n/4) + \frac{n}{\log\frac{n}{2}} + \frac{n}{\log n} \quad (iii)$$

Now Put $n = \frac{n}{4}$ in eq (i)

We get,

$$T(\tfrac{n}{4}) = 2T(n/8) + \frac{n}{4\log\frac{n}{4}} \quad (iv)$$

Substitute the value of $T(\frac{n}{4})$ from eq (iv) to eq (iii)

$$T(n) = 4\left(2T(n/8) + \frac{n}{4\log\frac{n}{4}}\right) + \frac{n}{\log\frac{n}{2}} + \frac{n}{\log n}$$

$$\Rightarrow T(n) = 8T(n/8) + \frac{n}{\log\frac{n}{4}} + \frac{n}{\log\frac{n}{2}} + \frac{n}{\log n} \quad (v)$$

From eq (v) the general equation for $T(n)$ is

$$T(n) = 2^k T(n/2^k) + \sum_{i=0}^{k-1} \frac{n}{\log\frac{n}{2^i}}$$

$$= T(n) = 2^k T(n/2^k) + n\sum_{i=0}^{k-1} \frac{1}{\log\frac{n}{2^i}}$$

$$= 2^k T\left(\frac{n}{2^k}\right) + n \cdot \sum_{i=0}^{k-1} \frac{1}{\log n - \log 2^i}$$

$$= 2^k T\left(\frac{n}{2^k}\right) + n \cdot \sum_{i=0}^{k-1} \frac{1}{\log n - i}$$

Now at base case,

$$\frac{n}{2^k} = 1 \quad \Rightarrow \quad n = 2^k$$

$$\therefore \log n = k$$

So we get,

$$T(n) = n\, T(1) + n \cdot \sum_{i=0}^{k-1} \frac{1}{k-i}$$

$$\Rightarrow T(n) = n \cdot \Theta(1) + n \cdot \sum_{i=0}^{k-1} \frac{1}{k-i}$$

$$(\because T(1) = \Theta(1))$$

$$\sum_{i=0}^{k-1} \frac{1}{k-i} = \int_0^{k-1} \frac{1}{k-i}\, di$$

$$= \left[-\log(k-i)\right]_0^{k-1}$$

$$= [-\log(k-k+1)] - [-\log(k)]$$

$$= 0 + \log k = \log k$$

$$\Rightarrow T(n) = n \cdot \Theta(1) + n \cdot \log k$$

But we know, $k = \log n$

$$\Rightarrow T(n) = n \cdot \Theta(1) + n \cdot \log \log n$$

$$\therefore T(n) \in O(n \cdot \log \log n)$$