homework3

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$\Delta_1 = 5, \Delta_2 = 1, \Delta_3 = 1, \Delta_4 = -7,$

$$L_1 = \begin{pmatrix} 1 & & & \\ \frac{2}{5} & 1 & & \\ \frac{-4}{5} & & 1 & \\ 0 & & & 1 \end{pmatrix}, L_1^{-1} = \begin{pmatrix} 1 & & & \\ \frac{-2}{5} & 1 & & \\ \frac{4}{5} & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

then

$$A^{(1)} = L_1^{-1} A^{(0)} = \begin{pmatrix} 5 & 2 & -4 & 0 \\ 0 & 1/5 & -2/5 & 1 \\ 0 & -2/5 & 9/5 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$

$$L_2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & -2 & 1 & \\ & 5 & & 1 \end{pmatrix}, L_2^{-1} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & 2 & 1 & \\ & -5 & & 1 \end{pmatrix}$$

$$A^{(2)} = L_2^{-1} A^{(1)} = \begin{pmatrix} 5 & 2 & -4 & 0 \\ 0 & 1/5 & -2/5 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & -3 \end{pmatrix}$$

$$L_3 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & 2 & 1 \end{pmatrix}, L_3^{-1} = \begin{pmatrix} 1 & & & \\ & & & \\ & & 1 & \\ & & -2 & 1 \end{pmatrix}$$

$$A^{(3)} = L_3^{-1} A^{(2)} = \begin{pmatrix} 5 & 2 & -4 & 0 \\ 0 & 1/5 & -2/5 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -7 \end{pmatrix}$$

$$= \text{diag}(5, 1/5, 1, -7) \begin{pmatrix} 1 & 2/5 & -4/5 & 0 \\ 0 & 1/5 & -2 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} = DU$$

thus

$$L = L_1 L_2 L_3 = \begin{pmatrix} 1 & & & \\ 2/5 & 1 & & \\ -4/5 & -2 & 1 & \\ 0 & 5 & 2 & 1 \end{pmatrix}$$

Doolittle decomposition:

$$A = L(DU) = L\hat{U}$$

here

$$\hat{U} = DU = \begin{pmatrix} 5 & 2 & -4 & 0 \\ 0 & 1/5 & -2/5 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -7 \end{pmatrix}$$

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Proof:
since

$$L_1 = \begin{pmatrix} 1 & & & \\ c_{21} & 1 & & \\ \vdots & & \ddots & \\ c_{n1} & & & 1 \end{pmatrix}, L_1^{-1} = \begin{pmatrix} 1 & & & \\ -c_{21} & 1 & & \\ \vdots & & \ddots & \\ -c_{n1} & & & 1 \end{pmatrix}$$

here $c_{i1} = a_{i1}/a_{11}$ $(i = 2, 3, ..., n)$,

$$A^{(1)} = L_1^{-1} A^{(0)} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & \vdots & & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

thus

$$B = \begin{pmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & & \vdots & \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}$$

since A is a real symmetric positive-definite matrix, B is a real symmetric positive-definite matrix and its diagonal elements are unchanging.

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$\Delta_1 = 5, \Delta_2 = 1, \Delta_3 = 1,$

$$L_1 = \begin{pmatrix} 1 & & \\ 2/5 & 1 & \\ -4/5 & & 1 \end{pmatrix}, A^{(1)} = L_1^{-1} A^{(0)} = \begin{pmatrix} 5 & 2 & -4 \\ 0 & 1/5 & -2/5 \\ 0 & -2/5 & 9/5 \end{pmatrix}$$

$$L_2 = \begin{pmatrix} 1 & & \\ & 1 & \\ & -2 & 1 \end{pmatrix}, A^{(2)} = L_2^{-1} A^{(1)} = \begin{pmatrix} 5 & 2 & -4 \\ 0 & 1/5 & -2/5 \\ 0 & 0 & 1 \end{pmatrix} = \text{diag}(5, 1/5, 1) \begin{pmatrix} 1 & 2/5 & -4/5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

thus

$$
\begin{aligned}
G = L\tilde{D} &= L_1 L_2 \mathbf{diag}(\sqrt{5}, \sqrt{1/5}, 1) \\
&= \begin{pmatrix} \sqrt{5} & & \\ 2/\sqrt{5} & 1/\sqrt{5} & \\ -4/\sqrt{5} & -2/\sqrt{5} & 1 \end{pmatrix}
\end{aligned}
$$

$$A = GG^T$$

4. Problem 2 of Page 219

$T_{12}(c, s) : c = 2/\sqrt(13), s = 3/\sqrt(13),$

$$
T_{12} = \begin{pmatrix} 2/\sqrt{13} & 3/\sqrt{13} & & \\ -3/\sqrt{13} & 2/\sqrt{13} & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, T_{12}\boldsymbol{x} = (\sqrt{13}, 0, 0, 5)^T
$$

$T_14(c, s) : c = \sqrt{13}/\sqrt{38}, s = 5/\sqrt{38},$

$$
T_{14} = \begin{pmatrix} \sqrt{13}/\sqrt{38} & & & 5/\sqrt{38} \\ & 1 & & \\ & & 1 & \\ -5/\sqrt{38} & & & \sqrt{13}/\sqrt{38} \end{pmatrix}, T_{12}\boldsymbol{x} = (\sqrt{38}, 0, 0, 0)^T
$$

thus

$$
T = T_{14}T_{12} = \begin{pmatrix} 2/\sqrt{38} & 3/\sqrt{38} & 0 & 5/\sqrt{38} \\ -3/\sqrt{13} & 2/\sqrt{13} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -10/\sqrt{13*38} & -15/\sqrt{13*38} & 0 & \sqrt{13/38} \end{pmatrix}, T\boldsymbol{x} = \sqrt{38}\boldsymbol{e}_1
$$

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$$
\begin{aligned}
(\boldsymbol{Hx})^T &= \boldsymbol{x}^T - a(\boldsymbol{x}, \boldsymbol{w})\boldsymbol{w}^T \\
\boldsymbol{Hx}(\boldsymbol{Hx})^T &= (\boldsymbol{x} - a(\boldsymbol{x}, \boldsymbol{w})\boldsymbol{w})(\boldsymbol{x}^T - a(\boldsymbol{x}, \boldsymbol{w})\boldsymbol{w}^T) \\
&= \boldsymbol{xx}^T - 2a(\boldsymbol{x}, \boldsymbol{w})\boldsymbol{wx}^T + a^2(\boldsymbol{x}, \boldsymbol{w})^2\boldsymbol{ww}^T \\
&= \boldsymbol{xx}^T - 2a(\boldsymbol{wx}^T)^2 + a^2(\boldsymbol{wx}^T)^2
\end{aligned}
$$

let $\boldsymbol{HH}^T = \boldsymbol{H}^T\boldsymbol{H} = I$,

$$-2a(\boldsymbol{wx}^T)^2 + a^2(\boldsymbol{wx}^T)^2 = 0$$

hence $a = 0, 2$.

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$$b_1 = (2,0,2)^T, T_{13} = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

then

$$T_{13}A^{(0)} = \begin{pmatrix} 4/\sqrt{2} & 3/\sqrt{2} & 3/\sqrt{2} \\ 0 & 2 & 2 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$b_2 = (2, -1/\sqrt{2})^T, T_{12} = \begin{pmatrix} 2\sqrt{2}/3 & -1/3 \\ 1/3 & 2\sqrt{2}/3 \end{pmatrix}$$

$$T_{12}A^{(1)} = \begin{pmatrix} 3\sqrt{2}/2 & 7\sqrt{2}/6 \\ 0 & 4/3 \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & \\ & T_{12} \end{pmatrix} T_{13} = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/(3\sqrt{2}) & 2\sqrt{2}/3 & -1/(3\sqrt{2}) \\ -2/3 & 1/3 & 2/3 \end{pmatrix}$$

$$Q = T^T, R = \begin{pmatrix} 4/\sqrt{2} & 3/\sqrt{2} & 3/\sqrt{2} \\ & 3\sqrt{2}/2 & 7\sqrt{2}/6 \\ & 0 & 4/3 \end{pmatrix}$$

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$$\boldsymbol{b}_1 = (0,1,0)^T, \boldsymbol{b}_1 - |\boldsymbol{b}_1|\boldsymbol{e}_1 = (-1,1,0)^T, \boldsymbol{u} = \frac{1}{\sqrt{2}}(-1,1,0)^T$$

then

$$H_1 = \boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, H_1 A^{(0)} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 1 \\ 0 & 3 & 2 \end{pmatrix}$$

$$\boldsymbol{b}_2 = (4,3)^T, \boldsymbol{b}_2 - |\boldsymbol{b}_2|\boldsymbol{e}_1 = (-1,3)^T, \boldsymbol{u} = \frac{1}{\sqrt{10}}(-1,3)^T$$

$$\boldsymbol{H}_2 = \boldsymbol{I} - 2\boldsymbol{u}\boldsymbol{u}^T = \begin{pmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{pmatrix}, H_2 A^{(1)} = \begin{pmatrix} 5 & 2 \\ 0 & -1 \end{pmatrix}$$

Hence

$$H = \begin{pmatrix} 1 & \\ & H_2 \end{pmatrix} H_1 = \begin{pmatrix} 0 & 1 & 0 \\ 4/5 & 0 & 3/5 \\ 3/5 & 0 & -4/5 \end{pmatrix}, Q = H^T, R = \begin{pmatrix} 1 & 1 & 1 \\ & 5 & 2 \\ & & -1 \end{pmatrix}$$

8. Problem 1 of Page 225

(1)

$$A \to B = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1/2 & -1/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

take a_1, a_2,

$$F = (a_1, a_2) = \begin{pmatrix} 1 & 2 \\ 0 & 2 \\ 1 & 0 \end{pmatrix}, G = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1/2 & -1/2 \end{pmatrix}$$

(2)

$$A \to B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

take a_1, a_2,

$$F = (a_1, a_2) = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ -1 & -1 \\ 1 & 1 \end{pmatrix}, G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 \end{pmatrix}$$

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Proof:
since rank$(\boldsymbol{B}) = r$,we have

$$\boldsymbol{B} = \boldsymbol{QR}$$

here $\boldsymbol{Q} \in \mathbb{R}_r^{m\times r}, \boldsymbol{R} \in \mathbb{R}_r^{r\times r}$ and $\boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I}$, $\boldsymbol{R}$ is a non-singular triangular matrix.Hence

$$\boldsymbol{B}^T\boldsymbol{B} = (\boldsymbol{QR})^T\boldsymbol{QR} = \boldsymbol{R}^T\boldsymbol{Q}^T\boldsymbol{QR} = \boldsymbol{R}^T\boldsymbol{R}$$

$\boldsymbol{B}^T\boldsymbol{B}$ is non-singular.

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Proof:

- if rank$(\boldsymbol{A}) = m$, then we have $\boldsymbol{A} = \boldsymbol{QR}, \boldsymbol{Q} \in \mathbf{C}_m^{n\times m}, \boldsymbol{Q}^H\boldsymbol{Q} = \boldsymbol{I}, \boldsymbol{R} \in \mathbf{C}_m^{m\times m}$, let $\boldsymbol{B} = \boldsymbol{R}^{-1}\boldsymbol{Q}^H \in \mathbf{C}^{m\times n}$, then $\boldsymbol{BA} = \boldsymbol{R}^{-1}\boldsymbol{Q}^H\boldsymbol{QR} = \boldsymbol{I}$;

- if $\boldsymbol{BA} = \boldsymbol{I}$, let rank$(\boldsymbol{B}) = r \le n$, then $\boldsymbol{B} = \boldsymbol{QR}, \boldsymbol{Q} \in \mathbf{C}_r^{m\times r}, \boldsymbol{R} \in \mathbf{C}_r^{r\times n}$,

$$\boldsymbol{BA} = \boldsymbol{QRA} = \boldsymbol{I}$$
$$\boldsymbol{RA} = \boldsymbol{Q}^H$$

  rank$(\boldsymbol{RA}) = r$ and rank$(\boldsymbol{R}) = r$, thus rank$(\boldsymbol{A}) = m$.

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Proof:
$\boldsymbol{F} = \boldsymbol{Q}_1\boldsymbol{R}_1, \boldsymbol{Q}_1 \in \mathbf{C}_r^{m\times r}, \boldsymbol{R}_1 \in \mathbf{C}_r^{r\times r}$,

$$\boldsymbol{FG} = \boldsymbol{Q}_1\boldsymbol{R}_1\boldsymbol{G} = \boldsymbol{Q}_1\boldsymbol{R}$$

$\boldsymbol{R} = \boldsymbol{R}_1\boldsymbol{G} \in \mathbf{C}_r^{r\times n}$,thus

$$\text{rank}(\boldsymbol{FG}) = r$$

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13. Problem 3 of Page 233

14. Problem 4 of Page 233

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