homework1

Zuyao Chen 201728008629002
zychen.uestc@gmail.com

1. Problem 3.(2) of Page 25

Yes. Call the set of all real symmetric matrix "S".

- obviously zero matrix $\boldsymbol{O} \in S$;
- $\forall \boldsymbol{A} \in S, \forall \alpha \in \mathbf{R}$, $(\alpha \boldsymbol{A})^T = \alpha \boldsymbol{A}$, $\alpha \boldsymbol{A}$ is still symmetric, so $\alpha \boldsymbol{A} \in S$;
- $\forall \boldsymbol{A}, \boldsymbol{B} \in S$, $(\boldsymbol{A} + \boldsymbol{B})^T = (\boldsymbol{A} + \boldsymbol{B})$, $(\boldsymbol{A} + \boldsymbol{B})$ is symmetric, $(\boldsymbol{A} + \boldsymbol{B}) \in S$

in summary, the set of all real symmetric matrix is closed under addition and scalar multiplication, so it's a linear space.

2. Problem 4 of Page 25

Proof:

Let $a_1 \cdot 1 + a_2 \cdot \cos^2 t + a_3 \cdot \cos 2t = 0, a_1, a_2, a_3 \in \mathbf{R}$. Suppose that $1, \cos^2 t, \cos 2t$ is linear independent,then it must has $a_1 = a_2 = a_3 = 0$. But as we all know,$\cos 2t = 2\cos^2 t - 1$, if $a_1 = 1, a_2 = -2, a_3 = 1$, it also has $a_1 \cdot 1 + a_2 \cdot \cos^2 t + a_3 \cdot \cos 2t = 0$,which is contrary to the hypothesis. So $1, \cos^2 t, \cos 2t$ is linear dependent.

3. Problem 6 of Page 25

Let $(\eta_1, \eta_2, \eta_3)^T$ be the new coordinates of vector $\boldsymbol{x}$, then $\boldsymbol{x} = \eta_1 \boldsymbol{x}_1 + \eta_2 \boldsymbol{x}_2 + \eta_3 \boldsymbol{x}_3$, which equals to

$$(\boldsymbol{x}_1^T, \boldsymbol{x}_2^T, \boldsymbol{x}_3^T)(\eta_1, \eta_2, \eta_3)^T = \boldsymbol{x}^T$$

so $(\eta_1, \eta_2, \eta_3)^T = (\boldsymbol{x}_1^T, \boldsymbol{x}_2^T, \boldsymbol{x}_3^T)^{-1} \boldsymbol{x}^T = (33, -82, 154)^T$.
new coordinates of $\boldsymbol{x}$: $(33, -82, 154)^T$

4. Problem 8 of Page 25

(1) the original equation is equivalent to

$$\begin{cases} (\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4)(1,2,0,0)^T = (\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3, \boldsymbol{y}_4)(0,0,1,0)^T & (1) \\ (\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4)(0,1,2,0)^T = (\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3, \boldsymbol{y}_4)(0,0,0,1)^T & (2) \\ (\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3, \boldsymbol{y}_4)(1,2,0,0)^T = (\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4)(0,0,1,0)^T & (3) \\ (\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3, \boldsymbol{y}_4)(0,1,2,0)^T = (\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4)(0,0,0,1)^T & (4) \end{cases}$$

combining equation(1),(2),(3) and (4),we have

$$\begin{cases} (\boldsymbol{y}_1,\boldsymbol{y}_2,\boldsymbol{y}_3,\boldsymbol{y}_4)(1,0,0,0)^T = (\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3,\boldsymbol{x}_4)(4,8,1,-2)^T & (5) \\ (\boldsymbol{y}_1,\boldsymbol{y}_2,\boldsymbol{y}_3,\boldsymbol{y}_4)(0,1,0,0)^T = (\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3,\boldsymbol{x}_4)(-2,-4,0,1)^T & (6) \\ (\boldsymbol{y}_1,\boldsymbol{y}_2,\boldsymbol{y}_3,\boldsymbol{y}_4)(0,0,1,0)^T = (\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3,\boldsymbol{x}_4)(1,2,0,0)^T & (7) \\ (\boldsymbol{y}_1,\boldsymbol{y}_2,\boldsymbol{y}_3,\boldsymbol{y}_4)(0,0,0,1)^T = (\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3,\boldsymbol{x}_4)(0,1,2,0)^T & (8) \end{cases}$$

thus the transformation matrix ($(\boldsymbol{y}_1,\boldsymbol{y}_2,\boldsymbol{y}_3,\boldsymbol{y}_4) = (\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3,\boldsymbol{x}_4)\boldsymbol{C}$):

$$\boldsymbol{C} = \begin{pmatrix} 4 & -2 & 1 & 0 \\ 8 & -4 & 2 & 1 \\ 1 & 0 & 0 & 2 \\ -2 & 1 & 0 & 0 \end{pmatrix}$$

(2) let $\boldsymbol{z}$ be the new coordinates,

$$(\boldsymbol{y}_1,\boldsymbol{y}_2,\boldsymbol{y}_3,\boldsymbol{y}_4)(2,-1,1,1)^T = (\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3,\boldsymbol{x}_4)\boldsymbol{C}(2,-1,1,1)^T$$

thus,$\boldsymbol{z} = \boldsymbol{C}(2,-1,1,1)^T = (11,23,4,-5)^T$.

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$\mathcal{L}(\boldsymbol{y}_1,\boldsymbol{y}_2,\boldsymbol{y}_3) = \{k_1\boldsymbol{y}_1 + k_2\boldsymbol{y}_2 + k_3\boldsymbol{y}_3\}, \boldsymbol{y}_3 = 3\boldsymbol{y}_2 - 2\boldsymbol{y}_1$, thus $\boldsymbol{y}_1, \boldsymbol{y}_2$ is one basis of $\mathcal{L}(\boldsymbol{y}_1,\boldsymbol{y}_2,\boldsymbol{y}_3)$.

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$S = V_1 \bigcap V_2 = \{(\xi_1,\xi_2,\xi_3,\xi_4)|\xi_1 = -\xi_3, \xi_2 = -\xi_4\}$,let $\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3,\boldsymbol{e}_4$ be the standard basis of $\mathbf{R}^4$. $\forall \boldsymbol{x} = (x_1,x_2,x_3,x_4) \in S$,

$$\begin{aligned} \boldsymbol{x} &= x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 + x_4\boldsymbol{e}_4 \\ &= x_1(\boldsymbol{e}_1 - \boldsymbol{e}_3) + x_2(\boldsymbol{e}_2 - \boldsymbol{e}_4) \\ &= x_1(1,0,-1,0)^T + x_2(0,1,0,-1)^T \end{aligned} \tag{9}$$

any element in S can be derived from linear combination of $\boldsymbol{e}_1 - \boldsymbol{e}_3$ and $\boldsymbol{e}_2 - \boldsymbol{e}_4$, so $\boldsymbol{e}_1 - \boldsymbol{e}_3$, $\boldsymbol{e}_2 - \boldsymbol{e}_4$ is one basis of S.

7. Problem 12 of Page 26

(1) **Proof**:

- obviously zero matrix $\boldsymbol{O} \in V$
- $\forall \boldsymbol{A}, \boldsymbol{B} \in V, \forall \alpha, \beta \in \mathbf{R}, \alpha\boldsymbol{A} + \beta\boldsymbol{B} = \begin{pmatrix} \alpha a_{11} + \beta b_{11} & * \\ * & \alpha a_{22} + \beta b_{22} \end{pmatrix} \in V$

V is closed under addition and scalar multiplication , thus it is a subspace of $\mathbf{R}^{2\text{X}2}$.

(2) let $\boldsymbol{e}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \boldsymbol{e}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \boldsymbol{e}_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \forall \boldsymbol{A} \in V$,

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{pmatrix} = a_{11}\boldsymbol{e}_1 + a_{12}\boldsymbol{e}_2 + a_{21}\boldsymbol{e}_3$$

$\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3$ is linear independent, so the dimension of subspace V is 3 and $\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3$ is one basis of V.

8. Additional Problem

Question:

if W_1, W_2, W_3 are subspaces of W, then $(W_1 \bigcap W_3) + (W_2 \bigcap W_3) \subset (W_1 + W_2) \bigcap W_3$. Can the left be equivalent to the right? under which condition?

Solution:

let $W = \mathbf{R}^2$,$W_1 = \mathcal{L}\{(1,0)\}$, $W_2 = \mathcal{L}\{(0,1)\}$, $W_3 = \mathcal{L}\{(1,1)\}$.Then

$$W_1 \bigcap W_3 = \{\mathbf{0}\}\,; W_2 \bigcap W_3 = \{\mathbf{0}\}\,; W_1 + W_2 = \mathbf{R}^2; (W_1 + W_2) \bigcap W_3 = W_3$$

Thus $(W_1 + W_2) \bigcap W_3 \not\subset (W_1 \bigcap W_3) + (W_2 \bigcap W_3)$.

- if $W_3 = \{\mathbf{0}\}$, then the left equals to the right;
- if $(W_1 + W_2) \subset W_3$, then $(W_1 \bigcap W_3) + (W_2 \bigcap W_3) = W_1 + W_2 = (W_1 + W_2) \bigcap W_3$.

9. Problem 1.(2) of Page 77

Yes. $\forall \boldsymbol{X}_1, \boldsymbol{X}_2 \in \mathbf{R}^{nxn}$,$\forall \alpha, \beta \in \mathbf{R}$

$$T(\alpha \boldsymbol{X}_1 + \beta \boldsymbol{X}_2) = \boldsymbol{B}(\alpha \boldsymbol{X}_1 + \beta \boldsymbol{X}_2)\boldsymbol{C} = \alpha \boldsymbol{B}\boldsymbol{X}_1\boldsymbol{C} + \beta \boldsymbol{B}\boldsymbol{X}_2\boldsymbol{C} = \alpha T(\boldsymbol{X}_1) + \beta T(\boldsymbol{X}_2)$$

thus T is linear transformation.

10. Problem 6 of Page 78

Let $\boldsymbol{g} = (\boldsymbol{x}_1, \boldsymbol{x}_2, .., \boldsymbol{x}_6)$,

$$\begin{cases} \dfrac{\partial \boldsymbol{x}_1}{\partial t} = a\boldsymbol{x}_1 - b\boldsymbol{x}_2 \\ \dfrac{\partial \boldsymbol{x}_2}{\partial t} = b\boldsymbol{x}_1 + a\boldsymbol{x}_2 \\ \dfrac{\partial \boldsymbol{x}_3}{\partial t} = \boldsymbol{x}_1 + a\boldsymbol{x}_3 - b\boldsymbol{x}_4 \\ \dfrac{\partial \boldsymbol{x}_4}{\partial t} = \boldsymbol{x}_2 + b\boldsymbol{x}_3 + a\boldsymbol{x}_4 \\ \dfrac{\partial \boldsymbol{x}_5}{\partial t} = \boldsymbol{x}_3 + a\boldsymbol{x}_5 - b\boldsymbol{x}_6 \\ \dfrac{\partial \boldsymbol{x}_6}{\partial t} = \boldsymbol{x}_4 + b\boldsymbol{x}_5 + a\boldsymbol{x}_6 \end{cases}$$

thus

$$\begin{aligned} \nabla_t(\boldsymbol{g}) &= (\frac{\partial \boldsymbol{x}_1}{\partial t}, \frac{\partial \boldsymbol{x}_2}{\partial t}, ..., \frac{\partial \boldsymbol{x}_6}{\partial t}) \\ &= \boldsymbol{g} \begin{pmatrix} a & b & 1 & 0 & 0 & 0 \\ -b & a & 0 & 1 & 0 & 0 \\ 0 & 0 & a & b & 1 & 0 \\ 0 & 0 & -b & a & 0 & 1 \\ 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & -b & a \end{pmatrix} \\ &= \boldsymbol{g}\boldsymbol{D} \end{aligned}$$

$\boldsymbol{D}$ is what we wanted.

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$$T_1\boldsymbol{E}_{11} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} = a\boldsymbol{E}_{11} + c\boldsymbol{E}_{21}, T_1\boldsymbol{E}_{12} = \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} = a\boldsymbol{E}_{12} + c\boldsymbol{E}_{22},$$

$$T_1\boldsymbol{E}_{21} = \begin{pmatrix} b & 0 \\ d & 0 \end{pmatrix} = b\boldsymbol{E}_{11} + d\boldsymbol{E}_{21}, T_1\boldsymbol{E}_{22} = \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} = b\boldsymbol{E}_{12} + d\boldsymbol{E}_{22}$$

$$\begin{aligned} T_1(\boldsymbol{E}_{11}, \boldsymbol{E}_{12}, \boldsymbol{E}_{21}, \boldsymbol{E}_{22}) &= (T_1\boldsymbol{E}_{11}, T_1\boldsymbol{E}_{12}, T_1\boldsymbol{E}_{21}, T_1\boldsymbol{E}_{22}) \\ &= (\boldsymbol{E}_{11}, \boldsymbol{E}_{12}, \boldsymbol{E}_{21}, \boldsymbol{E}_{22}) \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} \end{aligned}$$

$$\boldsymbol{A}_1 = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}$$

similarly,we can get

$$\boldsymbol{A}_2 = \begin{pmatrix} a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & b & d \end{pmatrix}, \boldsymbol{A}_3 = \boldsymbol{A}_1\boldsymbol{A}_2$$

12. Problem 9 of Page 78

Proof:

considering the linear combination of $\boldsymbol{x}, T\boldsymbol{x}, ..., T^{k-1}\boldsymbol{x}$

$$\alpha_1\boldsymbol{x} + \alpha_2 T\boldsymbol{x} + ... + \alpha_k T^{k-1}\boldsymbol{x} = \boldsymbol{0},$$

apply transformation T for both sides,we have

$$\alpha_1 T\boldsymbol{x} + \alpha_2 T^2\boldsymbol{x} + ... + \alpha_{k-1}T^{k-1}\boldsymbol{x} + \alpha_k T^k\boldsymbol{x} = \boldsymbol{0}$$

as $T^k\boldsymbol{x} = \boldsymbol{0}, T^{k-1}\boldsymbol{x} \neq \boldsymbol{0}$,

$$\alpha_1 T\boldsymbol{x} + \alpha_2 T^2\boldsymbol{x} + ... + \alpha_{k-1}T^{k-1}\boldsymbol{x} = \boldsymbol{0}$$

repeat the transformation above until

$$\alpha_1 T^{k-1}\boldsymbol{x} = \boldsymbol{0}$$

so we get $\alpha_1 = 0$,then go back

$$\alpha_1 T^{k-2}\boldsymbol{x} + \alpha_2 T^{k-1}\boldsymbol{x} = \boldsymbol{0}$$

we get $\alpha_2 = 0$,then from

$$\alpha_1 T^{k-3}\boldsymbol{x} + \alpha_2 T^{k-2}\boldsymbol{x} + \alpha_3 T^{k-1}\boldsymbol{x} = \boldsymbol{0}$$

we get $\alpha_3 = 0$, similarly we can get $\alpha_1 = \alpha_2 = ... = \alpha_k = 0$.
Thus $\boldsymbol{x}, T\boldsymbol{x}, ..., T^{k-1}\boldsymbol{x}$ is linear independent.

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$$T\boldsymbol{x} = \boldsymbol{x}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

thus,

$$T^2\boldsymbol{x} = \boldsymbol{x}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$R(T^2) = \left\{\boldsymbol{y}|\boldsymbol{y} = T^2\boldsymbol{x}, \boldsymbol{x} \in \mathbf{R}^3\right\} = \{(0,0,\xi_1)|\xi_1 \in \mathbf{R}\}$$
$$N(T^2) = \left\{\boldsymbol{x}|T^2\boldsymbol{x} = \mathbf{0}, \boldsymbol{x} \in \mathbf{R}^3\right\} = \{(0,\xi_2,\xi_3)|\xi_2,\xi_3 \in \mathbf{R}\}$$

for $R(T^2)$,its dimension is 1 and $(0,0,1)$ is one basis ;
for $N(T^2)$,its dimension is 2 and $(0,1,0),(0,0,1)$ is one basis.

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(1) $(\boldsymbol{y}_1,\boldsymbol{y}_2,\boldsymbol{y}_3) = (\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3)\boldsymbol{C}$, thus

$$\begin{aligned}\boldsymbol{C} &= \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & -1 \\ -1 & -1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -2 & -1.5 & 1.5 \\ 1 & 1.5 & 1.5 \\ 1 & 0.5 & -2.5 \end{pmatrix}\end{aligned}$$

(2) $T(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3) = (\boldsymbol{y}_1,\boldsymbol{y}_2,\boldsymbol{y}_3) = (\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3)\boldsymbol{C}$, thus the matrix is $\boldsymbol{C}$

(3) $T(\boldsymbol{y}_1,\boldsymbol{y}_2,\boldsymbol{y}_3) = (\boldsymbol{y}_1,\boldsymbol{y}_2,\boldsymbol{y}_3)\boldsymbol{C}$, thus the matrix is $\boldsymbol{C}$

15. Problem 12 of Page 79

$$\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0 \tag{10}$$

we can get $\lambda_1 = -2, \lambda_2 = \lambda_3 = 1$ from equation(10).
for $\lambda_1 = -2$,solving the equation $(\boldsymbol{A} - \lambda_1\boldsymbol{I})\boldsymbol{x} = \mathbf{0}$, we get a basic solution:$(0,0,1)^T$.all eigenvectors of T are

$$k\boldsymbol{x}_3, k \in \mathbf{R}, k \neq 0$$

for $\lambda_2 = \lambda_3 = 1$, we get a basic solution:$(3,-6,20)^T$,all eigenvectors of T are

$$k(3\boldsymbol{x}_1 - 6\boldsymbol{x}_2 + 20\boldsymbol{x}_3), k \in \mathbf{R}, k \neq 0$$

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From $\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = 0$ we can get $\lambda_1 = 2, \lambda_2 = \lambda_3 = 1$. for $\lambda_1 = 2$, we get a basic solution $(0,0,1)^T$; for $\lambda_2 = \lambda_3 = 1$, we get a basic solution $(-1,-2,1)^T$. then we obtain a non-singular matrix:

$$\boldsymbol{P}_1 = \begin{pmatrix} 0 & -1 & 1 \\ 0 & -2 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

further,

$$\boldsymbol{P}_1^{-1}\boldsymbol{A}\boldsymbol{P}_1 = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

17. Problem 15 of Page 79

Let $\psi(\lambda) = (2\lambda^4 - 12\lambda^3 + 19\lambda^2 - 29\lambda + 37)$, the characteristic polynomial of A

$$\varphi(\lambda) = \det(\boldsymbol{A} - \lambda\boldsymbol{I}) = (\lambda - 1)(\lambda - 5) + 2 = \lambda^2 - 6\lambda + 7$$

so

$$\begin{aligned} \psi(\lambda) &= \varphi(\lambda)(2\lambda^2 + 5) + \lambda + 2 \\ \psi(\boldsymbol{A}) &= \varphi(\boldsymbol{A})(2\boldsymbol{A}^2 + 5\boldsymbol{I}) + \boldsymbol{A} + 2\boldsymbol{I} \\ &= \boldsymbol{A} + 2\boldsymbol{I} \end{aligned}$$

the original problem is equivalent to solving $\psi(\boldsymbol{A})^{-1}$,

$$\psi(\boldsymbol{A})^{-1} = (\boldsymbol{A} + 2\boldsymbol{I})^{-1} = \frac{1}{23}\begin{pmatrix} 7 & 1 \\ -2 & 3 \end{pmatrix}$$

18. Problem 16 of Page 79

(1) the characteristic polynomial

$$\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = (\lambda - 9)(\lambda + 9)^2$$

the minimum polynomial

$$m(\lambda) = (\lambda - 9)(\lambda + 9)$$

(2) the characteristic polynomial

$$\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = (\lambda^2 - 2a_0\lambda + a_0^2 + a_1^2 + a_2^2 + a_3^2)^2$$

the minimum polynomial

$$m(\lambda) = \lambda^2 - 2a_0\lambda + a_0^2 + a_1^2 + a_2^2 + a_3^2$$

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Proof:

$\det(\boldsymbol{A} - \lambda\boldsymbol{I}) = \det(\boldsymbol{A}^T - \lambda\boldsymbol{I})$, thus A and A^T have the same characteristic polynomial and same minimum polynomial.

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Proof:

$V_{\lambda_0} = \{\boldsymbol{x}|T_1\boldsymbol{x} = \lambda_0\boldsymbol{x}, \boldsymbol{x} \in V^n\}, \forall\boldsymbol{x} \in V_{\lambda_0},$

$$T_1(T_2\boldsymbol{x}) = T_2(T_1\boldsymbol{x}) = T_2(\lambda_0\boldsymbol{x}) = \lambda_0(T_2\boldsymbol{x}), (T_2\boldsymbol{x}) \in V^n$$

$T_2\boldsymbol{x}$ still belongs to V_{λ_0},thus V_{λ_0} is the invariant subspace of T_2.

21. Additional Problem

(1) **Proof:**

$$N(f(T)) = \{\boldsymbol{x}|f(T)\boldsymbol{x} = \mathbf{0}, \boldsymbol{x} \in V^n\}, N(g(T)) = \{\boldsymbol{x}|g(T)\boldsymbol{x} = \mathbf{0}, \boldsymbol{x} \in V^n\}$$

two polynomials $f(\lambda)$ and $g(\lambda)$ are relatively prime,thus $f(\lambda) = 0$ and $g(\lambda) = 0$ have different roots.$\forall \boldsymbol{x} \in N(f(T)) \bigcap N(g(T))$,

$$\begin{cases} f(T)\boldsymbol{x} = \mathbf{0} \\ g(T)\boldsymbol{x} = \mathbf{0} \end{cases} \tag{11}$$

the equation(11) has a unique solution of $\boldsymbol{x} = \mathbf{0}$. thus $N(f(T)) \bigcap N(g(T)) = \{\mathbf{0}\}$

(2)

$$N(f(T)) \bigoplus N(g(T)) = \{\boldsymbol{z}|\boldsymbol{z} = \boldsymbol{x}_1 + \boldsymbol{x}_2, \boldsymbol{x}_1 \in N(f(T)), \boldsymbol{x}_2 \in N(g(T))\},$$
$$N(f(T)g(T)) = \{\boldsymbol{x}|f(T)g(T)\boldsymbol{x} = \mathbf{0}, \boldsymbol{x} \in V^n\}$$

$\forall \boldsymbol{z} \in N(f(T)) \bigoplus N(g(T))$,

$$\begin{aligned} f(T)g(T)\boldsymbol{z} &= f(T)g(T)(\boldsymbol{x}_1 + \boldsymbol{x}_2) \\ &= f(T)(g(T)\boldsymbol{x}_1 + g(T)\boldsymbol{x}_2) \\ &= f(T)(g(T)\boldsymbol{x}_1 + \mathbf{0}) \\ &= f(T)g(T)\boldsymbol{x}_1 \end{aligned}$$

$f(\lambda)g(\lambda) = g(\lambda)f(\lambda)$, thus $f(T)g(T) = g(T)f(T)$,then

$$f(T)g(T)\boldsymbol{z} = f(T)g(T)\boldsymbol{x}_1 = g(T)f(T)\boldsymbol{x}_1 = \mathbf{0}$$

$\boldsymbol{z} \in V^n$,so $N(f(T)) \bigoplus N(g(T)) = N(f(T)g(T))$