homework1

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1. Problem 3.(2) of Page 25

yes. Call the set of all real symmetric matrix "$\mathcal{S}$".

- obviously zero matrix $O \in \mathcal{S}$;
- $\forall A \in \mathcal{S}, \forall \alpha \in \mathcal{R}$, $(\alpha A)^T = \alpha A$, αA is still symmetric, so $\alpha A \in \mathcal{S}$;
- $\forall A, B \in \mathcal{S}$, $(A+B)^T = (A+B)$, $(A+B)$ is symmetric, $(A+B) \in \mathcal{S}$

in summary, the set of all real symmetric matrix is closed under addition and scalar multiplication, so it's a linear space.

2. Problem 4 of Page 25

Proof:

Let $a_1 \cdot 1 + a_2 \cdot \cos^2 t + a_3 \cdot \cos 2t = 0, a_1, a_2, a_3 \in \mathcal{R}$. Suppose $1, \cos^2 t, \cos 2t$ is linear independent,then it must has $a_1 = a_2 = a_3 = 0$. But as we all know,$\cos 2t = 2\cos^2 t - 1$, if $a_1 = 1, a_2 = -2, a_3 = 1$, it also has $a_1 \cdot 1 + a_2 \cdot \cos^2 t + a_3 \cdot \cos 2t = 0$,which is contrary to the hypothesis. So $1, \cos^2 t, \cos 2t$ is linear dependent.

3. Problem 6 of Page 25

let (η_1, η_2, η_3) be the new coordinates of vector $\mathbf{x}$, then $\mathbf{x} = \eta_1 \mathbf{x}_1 + \eta_2 \mathbf{x}_2 + \eta_3 \mathbf{x}_3$, ,which equals to

$$(\mathbf{x}_1^T, \mathbf{x}_2^T, \mathbf{x}_3^T)(\eta_1, \eta_2, \eta_3)^T = \mathbf{x}^T$$

so $(\eta_1, \eta_2, \eta_3)^T = (\mathbf{x}_1^T, \mathbf{x}_2^T, \mathbf{x}_3^T)^{-1}\mathbf{x}^T = (33, -82, 154)^T$.
new coordinates of $\mathbf{x}$: $(33, -82, 154)$

4. Problem 8 of Page 25

(1) let $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$,$\mathbf{Y} = (\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4)$, then the original equation equals to

$$\begin{cases} \mathbf{X}(1,2,0,0)^T = \mathbf{Y}(0,0,1,0)^T & (1) \\ \mathbf{X}(0,1,2,0)^T = \mathbf{Y}(0,0,0,1)^T & (2) \\ \mathbf{Y}(1,2,0,0)^T = \mathbf{X}(0,0,1,0)^T & (3) \\ \mathbf{Y}(0,1,2,0)^T = \mathbf{X}(0,0,0,1)^T & (4) \end{cases}$$

combining equation(1),(2),(3) and (4),we have

$$
\begin{cases}
\mathbf{Y}(1,0,0,0)^T = \mathbf{X}(4,8,1,-2)^T & (5)\\
\mathbf{Y}(0,1,0,0)^T = \mathbf{X}(-2,-4,0,1)^T & (6)\\
\mathbf{Y}(0,0,1,0)^T = \mathbf{X}(1,2,0,0)^T & (7)\\
\mathbf{Y}(0,0,0,1)^T = \mathbf{X}(0,1,2,0)^T & (8)
\end{cases}
$$

thus the transformation matrix ($\mathbf{Y} = \mathbf{XC}$):

$$
\mathbf{C} = \begin{pmatrix} 4 & -2 & 1 & 0 \\ 8 & -4 & 2 & 1 \\ 1 & 0 & 0 & 2 \\ -2 & 1 & 0 & 0 \end{pmatrix}
$$

(2) let **z** be the new coordinates,

$$
\mathbf{z} \begin{pmatrix} 4 & -2 & 1 & 0 \\ 8 & -4 & 2 & 1 \\ 1 & 0 & 0 & 2 \\ -2 & 1 & 0 & 0 \end{pmatrix} = (2,-1,1,1)
$$

thus,$\mathbf{z} = (2,-1,1,1)\mathbf{C}^{-1} = (-1,1,0,1)$.

5. Problem 10 of Page 26

call the span space "$\mathcal{S}$",the linear combination of $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3$ can be written as

$$
\begin{aligned}
&k_1(\mathbf{x}_1 - 2\mathbf{x}_2 + 3\mathbf{x}_3) + k_2(2\mathbf{x}_1 + 3\mathbf{x}_2 + 2\mathbf{x}_3) + k_3(4\mathbf{x}_1 + 13\mathbf{x}_2) \\
=&(k_1 + 2K_2 + 4k_3)\mathbf{x}_1 + (-2k_1 + 3k_2 + 13k_3)\mathbf{x}_2 + (3k_1 + 2k_2)\mathbf{x}_3
\end{aligned} \tag{9}
$$

it's a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, thus $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ is one basis of space $\mathcal{S}$

6. Problem 11 of Page 26

$\mathbf{S} = \mathbf{V}_1 \bigcap \mathbf{V}_2 = \{(\zeta_1, \zeta_2, \zeta_3, \zeta_4) | \zeta_1 = -\zeta_3, \zeta_2 = \zeta_4\}$,let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4$ be the standard basis of $\mathbf{R}^4$. $\forall \mathbf{x} = (x_1, x_2, x_3, x_4) \in \mathbf{S}$,

$$
\begin{aligned}
\mathbf{x} &= x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 + x_4\mathbf{e}_4 \\
&= x_1(\mathbf{e}_1 - \mathbf{e}_3) + x_2(\mathbf{e}_2 + \mathbf{e}_4)
\end{aligned} \tag{10}
$$

any element in **S** can be derived from linear combination of $\mathbf{e}_1 - \mathbf{e}_3$ and $\mathbf{e}_2 + \mathbf{e}_4$, so $\mathbf{e}_1 - \mathbf{e}_3$, $\mathbf{e}_2 + \mathbf{e}_4$ is one basis of **S**.

7. Problem 12 of Page 26

(1) **Proof**:

- obviously zero matrix $O \in \mathbf{V}$
- $\forall A, B \in \mathbf{V}, \forall \alpha, \beta \in \mathbf{R}, \alpha A + \beta B = \begin{pmatrix} \alpha a_{11} + \beta b_{11} & * \\ * & \alpha a_{22} + \beta b_{22} \end{pmatrix} \in \mathbf{V}$

V is closed under addition and scalar multiplication , thus it is a subspace of $\mathbf{R}^{2x2}$.

(2) let $\mathbf{e}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{e}_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \forall A \in \mathbf{V},$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & -a_{11} \end{pmatrix} = a_{11}\mathbf{e}_1 + a_{12}\mathbf{e}_2 + a_{21}\mathbf{e}_3$$

$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is linear independent, so the dimension of subspace **V** is 3 and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is one basis of **V**.

8. Additional Problem

Question:
if $\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3$ are subspaces of **W**, then $(\mathbf{W}_1 \bigcap \mathbf{W}_3) + (\mathbf{W}_2 \bigcap \mathbf{W}_3) \subset (\mathbf{W}_1 + \mathbf{W}_2) \bigcap \mathbf{W}_3$. Can the left equal to the right under some circumstances?

9. Problem 1.(2) of Page 77

Yes. $\forall X_1, X_2 \in \mathbf{R}^{nxn}$,$\forall \alpha, \beta \in \mathbf{R}$

$$T(\alpha X_1 + \beta X_2) = B(\alpha X_1 + \beta X_2)C = \alpha B X_1 C + \beta B X_2 C = \alpha T(X_1) + \beta T(X_2)$$

thus T is linear transformation.

10. Problem 6 of Page 78

let $A = (x_1, x_2, .., x_6)^T$,

$$\begin{cases} \dfrac{\partial x_1}{\partial t} = ax_1 - bx_2 \\ \dfrac{\partial x_2}{\partial t} = bx_1 + ax_2 \\ \dfrac{\partial x_3}{\partial t} = (1+at)x_1 - btx_2, \\ \dfrac{\partial x_4}{\partial t} = btx_1 + (1+at)x_2, \\ \dfrac{\partial x_5}{\partial t} = (t + \dfrac{1}{2}at^2)x_1 - \dfrac{1}{2}bt^2 x_2, \\ \dfrac{\partial x_6}{\partial t} = \dfrac{1}{2}bt^2 x_1 + (t + \dfrac{1}{2}at^2)x_2 \end{cases}$$

thus

$$\begin{aligned} \nabla_t(A) &= (\frac{\partial x_1}{\partial t}, \frac{\partial x_2}{\partial t}, ..., \frac{\partial x_6}{\partial t})^T \\ &= \begin{pmatrix} a & -b & 0 & 0 & 0 & 0 \\ b & a & 0 & 0 & 0 & 0 \\ 1+at & -bt & 0 & 0 & 0 & 0 \\ bt & 1+at & 0 & 0 & 0 & 0 \\ t+\frac{1}{2}at^2 & -\frac{1}{2}bt^2 & 0 & 0 & 0 & 0 \\ \frac{1}{2}bt^2 & t+\frac{1}{2}at^2 & 0 & 0 & 0 & 0 \end{pmatrix} A \\ &= DA \end{aligned}$$

$\forall B = (b_1, b_2, ..., b_6)^T \in \mathbf{V}$,

$$\nabla_t(B) = DB$$