In[173]:=

```
(*Description: Derivation of fixed point equation*)
```

In[175]:=

(*1.0) Use equation derived in other Mathematica script using the substitution $A=\frac{c_0}{1-z^N}$ and $B=\frac{c_{-1}}{1-zc^N}$ *)

In[176]:=

$$c_{DFT,n}[n\_, A\_, B\_] := A * z^n + B * zc^{M-n-1}$$

In[177]:=

$$c_{DFT,-n}[n\_, A\_, B\_] := A * z^{M-n} + B * zc^{n-1}$$

In[178]:=

```
(*1.1) Show that those are the same equations as in Eq.[12,13]*)
```

In[179]:=

$$\text{Solve}[\{c_{DFT,n}[0, A, B] == c0,\ c_{DFT,-n}[1, A, B] == cm1\}, \{A, B\}] \ // \ \text{FullSimplify}$$

Out[179]=

$$\left\{\left\{A \to \frac{z\left(c0\, zc - cm1\, zc^M\right)}{z\, zc - z^M\, zc^M},\ B \to \frac{\left(cm1\, z - c0\, z^M\right) zc}{z\, zc - z^M\, zc^M}\right\}\right\}$$

In[181]:=

(*1.2) Check if those expressions agree with Eq.[A16] using r^2=z*zc*)

In[182]:=

```
(*1.2.1.For A*)
```

In[183]:=

$$\frac{z\left(c0\, zc - cm1\, zc^M\right)}{z * zc - z^M * zc^M} - \left(\frac{c0 - cm1 * zc^{M-1}}{1 - (zc * z)^{M-1}}\right) \ // \ \text{FullSimplify}$$

Out[183]=

$$z\left(c0\, zc - cm1\, zc^M\right)\left(\frac{1}{z\, zc - z^M\, zc^M} + \frac{1}{-z\, zc + (z\, zc)^M}\right)$$

In[184]:=

(*1.2.2) Replace z*zc with z*zc=r^2*)

In[146]:=

$$\left(\frac{1}{r^2 - \left(r^2\right)^M} + \frac{1}{-r^2 + \left(r^2\right)^M}\right) \ // \ \text{FullSimplify}$$

Out[146]=

0

In[185]:=

```
(*1.2.3) For B*)
```

In[148]:=

$$\frac{\left(cm1\, z - c0\, z^M\right) zc}{z\, zc - z^M\, zc^M} - \left(\frac{cm1 - c0 * z^{M-1}}{1 - r^{2*M-2}}\right) \ // \ \text{FullSimplify}$$

Out[148]=

$$-\frac{cm1 - c0\, z^{-1+M}}{1 - r^{-2+2M}} + \frac{\left(cm1\, z - c0\, z^M\right) zc}{z\, zc - z^M\, zc^M}$$

In[186]:=

```
(*1.2.4) Replace z*zc with z*zc=r^2*)
```

In[187]:=

$$(*1.2.5)\ \text{with}\ \frac{(\text{cm1 z}-\text{c0 z}^{M})\ \text{zc}}{\text{z zc}-\text{z}^{M}\ \text{zc}^{M}}=\frac{(\text{cm1}-\text{c0 z}^{M-1})\ \text{zc*z}}{\text{z zc}-\text{z}^{M}\ \text{zc}^{M}}*)$$

In[188]:=

$$\frac{(\text{cm1}-\text{c0 z}^{M-1})\ r^2}{r^2-r^{2*M}}-\left(\frac{\text{cm1}-\text{c0}*\text{z}^{M-1}}{1-r^{2*M-2}}\right)\ //\ \text{FullSimplify}$$

Out[188]=

0

In[189]:=

```
(*1.3) SUmmary: Eq.[A16] is correct*)
```

In[196]:=

```
(*2) Derive equation for z*)
```

In[194]:=

```
(*2.1) Solve c1 for z*)
```

In[191]:=

$c_{DFT,n}[1, A, B]$

Out[191]=

$A\,z + B\,zc^{-2+M}$

In[192]:=

Solve$[c_{DFT,n}[1, A, B] == c1, z]$ // FullSimplify

Out[192]=

$$\left\{\left\{z \to \frac{c1 - B\,zc^{-2+M}}{A}\right\}\right\}$$

In[197]:=

```
(*2.2) Insert B and A definition in z*)
```

In[198]:=

Solve$[c_{DFT,n}[0, A, B] == c0, A]$ // FullSimplify

Out[198]=

$\{\{A \to c0 - B\,zc^{-1+M}\}\}$

In[199]:=

```
(*2.3) Use expression from 2.1) and insert term from 2.2)*)
```

In[200]:=

$$\text{zt}[A\_, B\_] := \frac{c1 - B\,zc^{-2+M}}{A}$$

In[201]:=

$\text{zt}[c0 - B\,zc^{-1+M}, B]$

Out[201]=

$$\frac{c1 - B\,zc^{-2+M}}{c0 - B\,zc^{-1+M}}$$

In[204]:=

```
(*2.4) Reproducing Eq.[A17] by using expression from 2.3)
  and insert expression for B from 1.2.3)*)
```

In[205]:=

$$zt2[B\_] := \frac{c1 - B\, zc^{-2+M}}{c0 - B\, zc^{-1+M}}$$

In[206]:=

$$zt2\left[\frac{cm1 - c0 * z^{M-1}}{1 - r^{2*M-2}}\right]$$

Out[206]=

$$\frac{c1 - \frac{\left(cm1 - c0\, z^{-1+M}\right) zc^{-2+M}}{1 - r^{-2+2M}}}{c0 - \frac{\left(cm1 - c0\, z^{-1+M}\right) zc^{-1+M}}{1 - r^{-2+2M}}}$$

In[207]:=

```
(*2.5) Summary: This expression agrees with Eq. [A18]*)
```

In[208]:=

```
(*3) Derive a fixed point equation*)
```

In[209]:=

(*3.1) Multiply $z = \frac{c1 - \frac{\left(cm1 - c0\, z^{-1+M}\right) zc^{-2+M}}{1 - r^{-2+2M}}}{c0 - \frac{\left(cm1 - c0\, z^{-1+M}\right) zc^{-1+M}}{1 - r^{-2+2M}}}$ with the dominator to get rid of it*)

In[213]:=

```
(*3.2.1) The left hand side of the equation given in 3.1) yields then:*)
```

In[211]:=

$$z * \left(c0 - \frac{\left(cm1 - c0\, z^{-1+M}\right) zc^{-1+M}}{1 - r^{-2+2M}}\right) \text{ // Expand}$$

Out[211]=

$$c0\, z - \frac{cm1\, z\, zc^{-1+M}}{1 - r^{-2+2M}} + \frac{c0\, z^{M}\, zc^{-1+M}}{1 - r^{-2+2M}}$$

In[225]:=

(*Use r^2=z*zc*)

In[226]:=

$$c0\, z - \frac{cm1\, r^2\, zc^{-2+M}}{1 - r^{-2+2M}} + \frac{c0\, z\, r^{2*M-2}}{1 - r^{-2+2M}} \text{ // Expand}$$

Out[226]=

$$c0\, z + \frac{c0\, r^{-2+2M}\, z}{1 - r^{-2+2M}} - \frac{cm1\, r^2\, zc^{-2+M}}{1 - r^{-2+2M}}$$

In[212]:=

```
(*3.2.2) The right hand side of the equation (see 3.1))*)
```

In[214]:=

$$c1 - \frac{\left(cm1 - c0\, z^{-1+M}\right) zc^{-2+M}}{1 - r^{-2+2M}} \text{ // Expand}$$

Out[214]=

$$c1 - \frac{cm1\, zc^{-2+M}}{1 - r^{-2+2M}} + \frac{c0\, z^{-1+M}\, zc^{-2+M}}{1 - r^{-2+2M}}$$

In[215]:=

$$\text{RHS}[z\_] := c1 - \frac{cm1\, zc^{-2+M}}{1 - r^{-2+2M}} + \frac{c0\, z^{-1+M}\, zc^{-2+M}}{1 - r^{-2+2M}}$$

In[217]:=

$$\text{RHS}\left[\frac{r^2}{zc}\right] \text{ // FullSimplify // Expand}$$

Out[217]=

$$c1 + \frac{cm1\, r^2\, zc^{-2+M}}{-r^2 + r^{2M}} - \frac{c0\left(\frac{r^2}{zc}\right)^M zc^{-1+M}}{-r^2 + r^{2M}}$$

In[219]:=

```
(*3.2.2.1) Need to be simplified by hand: use r^2=zc*z*)
```

In[221]:=

$$c1 - \frac{cm1\, zc^{-2+M}}{1 - r^{-2+2M}} + \frac{c0\, z\, r^{2*M-4}}{1 - r^{-2+2M}} \text{ // Expand}$$

Out[221]=

$$c1 + \frac{c0\, r^{-4+2M}\, z}{1 - r^{-2+2M}} - \frac{cm1\, zc^{-2+M}}{1 - r^{-2+2M}}$$

In[227]:=

(*3.3.1) We have the equation LHS=RHS: $c0\ z + \frac{c0\ r^{-2+2\,M}\ z}{1-r^{-2+2\,M}} - \frac{cm1\ r^2\ zc^{-2+M}}{1-r^{-2+2\,M}} = c1 + \frac{c0\ r^{-4+2\,M}\ z}{1-r^{-2+2\,M}} - \frac{cm1\ zc^{-2+M}}{1-r^{-2+2\,M}}$ *)

In[228]:=

(*3.3.2) We want to solve for c0*z so we need to subtract the equation by $-\left(\frac{c0\ r^{-2+2\,M}\ z}{1-r^{-2+2\,M}} - \frac{cm1\ r^2\ zc^{-2+M}}{1-r^{-2+2\,M}}\right)$*)

(*3.3.3) c0*z=...*)

In[229]:=

$$c1 + \frac{c0\, r^{-4+2M}\, z}{1 - r^{-2+2M}} - \frac{cm1\, zc^{-2+M}}{1 - r^{-2+2M}} - \left(\frac{c0\, r^{-2+2M}\, z}{1 - r^{-2+2M}} - \frac{cm1\, r^2\, zc^{-2+M}}{1 - r^{-2+2M}}\right) \text{ // FullSimplify}$$

Out[229]=

$$c1 + \frac{\left(-1 + r^2\right)\left(-c0\, r^{2M}\, z\, zc^2 + cm1\, r^4\, zc^M\right)}{r^2\left(r^2 - r^{2M}\right) zc^2}$$

In[231]:=

```
(*3.3.4) z=...*)
```

In[232]:=

$$\left(c1 + \frac{\left(-1 + r^2\right)\left(-c0\, r^{2M}\, z\, zc^2 + cm1\, r^4\, zc^M\right)}{r^2\left(r^2 - r^{2M}\right) zc^2}\right) \Big/ c0 \text{ // FullSimplify}$$

Out[232]=

$$\frac{c1 + \frac{\left(-1+r^2\right)\left(-c0\, r^{2M}\, z\, zc^2 + cm1\, r^4\, zc^M\right)}{r^2\left(r^2 - r^{2M}\right) zc^2}}{c0}$$

In[238]:=

```
(*3.4) This is final fixed point equation for z. However mathematica struggles to
  further simplify the equation- so we do it pseudo manually in the following*)
```

In[237]:=

```
(*4.0) Simplify fixed point equation pseudo manually *)
```

In[236]:=

$$\frac{c1+\frac{\left(-1+r^2\right)\left(-c0\,r^{2M}\,z\,zc^2+cm1\,r^4\,zc^M\right)}{r^2\left(r^2-r^{2M}\right)zc^2}}{c0}\ \text{// Expand // FullSimplify}$$

Out[236]=

$$\frac{c1+\frac{\left(-1+r^2\right)\left(-c0\,r^{2M}\,z\,zc^2+cm1\,r^4\,zc^M\right)}{r^2\left(r^2-r^{2M}\right)zc^2}}{c0}$$

In[239]:=

```
(*4.1) z-c1/c0=...*)
```

In[240]:=

$$\frac{\frac{\left(-1+r^2\right)\left(-c0\,r^{2M}\,z\,zc^2+cm1\,r^4\,zc^M\right)}{r^2\left(r^2-r^{2M}\right)zc^2}}{c0}\ \text{// FullSimplify}$$

Out[240]=

$$\frac{\left(-1+r^2\right)\left(-c0\,r^{2M}\,z\,zc^2+cm1\,r^4\,zc^M\right)}{c0\,r^2\left(r^2-r^{2M}\right)zc^2}$$

In[241]:=

(*4.2) divide nominator and denominator by zc^2 and by r^4*)

In[242]:=

```
(*4.2.1) Nominator:*)
```

In[243]:=

$$\left(-1+r^2\right)\left(-c0\,r^{2M}\,z\,zc^2+cm1\,r^4\,zc^M\right)*\left(zc^{-2}*r^{-4}\right)\ \text{// FullSimplify}$$

Out[243]=

$$(-1+r)\,(1+r)\left(-c0\,r^{-4+2M}\,z+cm1\,zc^{-2+M}\right)$$

In[244]:=

$$(-1+r)\,(1+r)\ \text{// FullSimplify}$$

Out[244]=

$$-1+r^2$$

In[245]:=

```
(*4.2.2) Nominator final shape is:*)
```

In[246]:=

$$\left(-1+r^2\right)\left(-c0\,r^{-4+2M}\,z+cm1\,zc^{-2+M}\right)\ \text{// FullSimplify}$$

Out[246]=

$$\left(-1+r^2\right)\left(-c0\,r^{-4+2M}\,z+cm1\,zc^{-2+M}\right)$$

In[247]:=

```
(*4.2.3) Denominator:*)
```

In[248]:=

$$c0\,r^2\left(r^2-r^{2M}\right)zc^2*\left(zc^{-2}*r^{-4}\right)\ \text{// FullSimplify}$$

Out[248]=

$$c0-c0\,r^{-2+2M}$$

In[249]:=

```
(*4.3) Putting the fraction back together*)
```

In[250]:=

$$\frac{\left(-1+r^2\right)\left(-c0\, r^{-4+2M} z + cm1\, zc^{-2+M}\right)}{c0 - c0\, r^{-2+2M}} \text{ // FullSimplify}$$

Out[250]=

$$\frac{\left(-1+r^2\right)\left(-c0\, r^{-4+2M} z + cm1\, zc^{-2+M}\right)}{c0 - c0\, r^{-2+2M}}$$

In[251]:=

```
(*4.3.1) cancel out the c0 factor*)
```

In[252]:=

$$\frac{\left(-1+r^2\right)}{1-r^{-2+2M}} * \left(-r^{-4+2M} z + \frac{cm1}{c0} zc^{-2+M}\right) \text{ // FullSimplify}$$

Out[252]=

$$\frac{\left(-1+r^2\right)\left(-r^{-4+2M} z + \frac{cm1\, zc^{-2+M}}{c0}\right)}{1-r^{-2+2M}}$$

In[253]:=

(*4.3.2) define $\frac{cm1}{c0}:=\chi$*)

In[256]:=

$$\frac{\left(-1+r^2\right)}{1-r^{-2+2M}} \left(-r^{-4+2M} z + \chi * zc^{-2+M}\right)$$

Out[256]=

$$\frac{\left(-1+r^2\right)\left(-r^{-4+2M} z + zc^{-2+M} \chi\right)}{1-r^{-2+2M}}$$

In[257]:=

```
(*4.4) add back the term  c1/c0 (see 4.1) to obtain an equation for z*)
```

In[258]:=

(*define $\rho=\frac{c1}{c0}$*)

In[259]:=

$$\rho + \frac{\left(-1+r^2\right)}{1-r^{-2+2M}} \left(-r^{-4+2M} z + \chi * zc^{-2+M}\right)$$

Out[259]=

$$\rho + \frac{\left(-1+r^2\right)\left(-r^{-4+2M} z + zc^{-2+M} \chi\right)}{1-r^{-2+2M}}$$

In[262]:=

(*4.5) the final equation for z is $z=\rho-\frac{1-r^2}{1-r^{2M-2}}\left(\chi * zc^{M-2}-r^{2M-4} z\right)$*)