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Generalized robust counterparts for constraints with bounded and unbounded uncertain parameters

Logan R. Matthews [c,a,b,*], Yannis A. Guzman [c,a,b], Christodoulos A. Floudas [a,b]

[a] Artie McFerrin Department of Chemical Engineering, Texas A&M University, College Station, TX, USA
[b] Texas A&M Energy Institute, Texas A&M University, College Station, TX, USA
[c] Department of Chemical and Biological Engineering, Princeton University, Princeton, NJ, USA

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ABSTRACT

Robust optimization has emerged as a powerful and efficient methodology for incorporating uncertain parameters into optimization models. In robust optimization, robust counterparts for uncertain constraints are created by imposing a known set of uncertain parameter realizations onto the new robust constraint. For constraints with all bounded parameters, the interval + ellipsoidal and interval + polyhedral uncertainty sets are well-established in robust optimization literature, while box, ellipsoidal, or polyhedral sets may be used for unbounded parameters. However, there has yet to be any counterparts proposed for constraints that simultaneously contain both bounded and unbounded parameters. This is crucial, as using the traditional box, ellipsoidal, or polyhedral sets with bounded parameters may impose impossible parameter realizations outside of their bounds, unnecessarily increasing the conservatism of results. In this work, robust counterparts for uncertain constraints with both bounded and unbounded uncertain parameters are derived: the generalized interval + box, generalized interval + ellipsoidal, and generalized interval + polyhedral counterparts. These counterparts reduce to the traditional box, ellipsoidal, and polyhedral counterparts if all parameters are unbounded, and reduce to the traditional interval + ellipsoidal and interval + polyhedral counterparts if all parameters are bounded. It is proven that established *a priori* probabilistic bounds remain valid for these counterparts. The importance of these developments is demonstrated with computational examples, showing the reduction of conservatism that is gained by appropriately limiting the possible realizations of the bounded parameters. The developments increase the scope and applicability of robust optimization as a tool for optimization under uncertainty.

1. Introduction

Parameter uncertainty in optimization is an important area of research that must be considered when using mathematical optimization to address many grand challenges in the world today (Floudas et al., 2016). While most mathematical models assume that parameters in a model are static, i.e. they can only realize one possible value, in most real-world applications there are many different values that a parameter can actually realize. If the parameter is crucial to determining the optimal solution of a model, an uncertain parameter realization could make an entire optimal solution infeasible, or lead to objective function values that are worse than what would be found if uncertainty was considered. For instance, if the parameter is in a safety constraint of a model of a chemical plant, its uncertain realization may lead to an optimal solution that causes unsafe operating conditions. Alternatively, uncertain price realizations could lead to much lower profits than expected from an optimal solution of many types of models. Thus, many approaches have been proposed for handling parameter uncertainty in mathematical optimization (Grossmann et al., 2016), including chance-constrained programming (Charnes and Cooper, 1959; Miller and Wagner, 1965; Jagannathan, 1974; Calfa et al., 2015), stochastic programming (Shapiro, 2008; Acevedo and Pistikopoulos, 1998), and robust optimization (Ben-Tal et al., 2009; Bertsimas et al., 2011; Gabrel et al., 2014).

Robust optimization has advantages over other methods of handling parameter uncertainty because it is deterministic in nature, provides either linear or convex conic quadratic counterparts, and requires no enumerations of scenarios. The first robust counterpart, generally referred to as the "worst-case" uncertainty set for

* Corresponding author at: Department of Chemical and Biological Engineering, Princeton University, Princeton, NJ, USA.
E-mail address: loganm@princeton.edu (L.R. Matthews).

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bounded uncertain parameters, was developed by Soyster (1973). Starting in the late 1990s (Ben-Tal and Nemirovski, 1998, 1999; El Ghaoui and Lebret, 1997; El Ghaoui et al., 1998), robust optimization theory began to expand rapidly, initially specific to linear optimization problems. For bounded uncertain parameters, the interval + ellipsoidal counterpart was developed by Ben-Tal and Nemirovski (2000), and Bertsimas and Sim (2004) derived the interval + polyhedral counterpart. These counterparts lowered the conservatism of robust optimization and introduced probabilistic bounds to characterize the probability of constraint violation for robust solutions. The theory for robust optimization was further expanded to mixed-integer linear optimization problems (Bertsimas and Sim, 2003; Lin et al., 2004; Janak et al., 2007).

A considerable amount of research continues in robust optimization theory in order to increase the applicability of robust optimization. Much attention has been given to problems with recourse, and adjustable robust optimization has been developed as a critical methodology for these situations (Ben-Tal et al., 2003; Chen and Zhang, 2009; Bertsimas and Goyal, 2012). Probabilistic bounds on constraint violation also have a tremendous influence on the conservatism of solutions for robust optimization, and both *a priori* and *a posteriori* probabilistic bounds have been developed for a variety of parameter distribution types (Ben-Tal and Nemirovski, 2000; Bertsimas and Sim, 2004; Kang et al., 2013; Paschalidis et al., 2008; Li et al., 2012). Recent advances by Guzman et al. (2016, 2017a,b) have greatly reduced the conservatism of probabilistic bounds, making robust optimization a practical, efficient methodology to gain strong robust results with very low computational effort. An iterative method introduced by Li and Floudas (2014) utilizing both *a priori* and *a posteriori* probabilistic bounds provides high quality solutions at desired *a posteriori* probabilities of constraint violation. Finally, uncertainty set theory itself provides an opportunity to reduce conservatism and increase applicability in robust optimization, and many approaches are being utilized to provide various uncertainty sets for different applications (Averbakh and Zhao, 2008; Bertsimas and Brown, 2009; Li et al., 2011).

A major assumption in the two most popular uncertainty sets for robust optimization, the interval + ellipsoidal and interval + polyhedral sets, is that the uncertain parameters in a constraint are subject to bounded probability distributions. The bounded parameters allow the "worst-case" box uncertainty set (i.e. interval set) to be intersected with the ellipsoidal or polyhedral uncertainty sets. However, in the case that the distributions in a given constraint are not bounded, such as the normal distribution, the box, ellipsoidal, or polyhedral sets would be required; there would not be a "worst-case" parameter realization for these distributions. Yet, to our knowledge, no robust counterparts have been developed for a scenario in which a constraint contains both bounded and unbounded parameters simultaneously. In this case, the choice of counterpart is not obvious. An interval + ellipsoidal or interval + polyhedral set would be most applicable for the bounded parameters, but would inappropriately bound the possible realizations for an unbounded parameter. Using a box, ellipsoidal, or polyhedral set would prevent this artificial limiting of parameter realizations, but would likely lead to the inclusion of impossible parameter realizations beyond the worst case values for the bounded parameters, especially at low probabilities of constraint violation. Thus, until now, the user of robust optimization is likely forced to use overly conservative methods for robust optimization in these cases.

In this work, we provide robust counterparts that are valid for constraints that contain both bounded and unbounded parameters. These "generalized robust counterparts" will be similar in form to the interval + ellipsoidal and interval + polyhedral counterparts, but will be more widely applicable. The bounded parameters will benefit from the application of the interval set for those parameters alone, while the unbounded parameters will only be influenced by the adjustable norms. In Section 2, background on the existing robust methodology and uncertainty sets will be provided. Section 3 will provide the general derivation of robust counterparts from intersecting p-norms — the foundation of the most popular uncertainty sets today. Then, Section 4 will show how this derivation can be generalized to modify which p-norms influence specific parameters; here, we also derive new generalized interval + box, generalized interval + ellipsoidal, and generalized interval + polyhedral uncertainty sets. Finally, Section 5 ensures that these new generalized robust counterparts remain applicable to the strong advancements made by Guzman et al. (2016, 2017a,b) regarding probabilistic bounds, and Section 6 demonstrates the use and impact of the new counterparts with computational examples.

2. Background

In general, robust optimization literature has focused on linear (LP) or mixed-integer linear (MILP) optimization problems with uncertain parameters in the form of Problem (1):

$$\begin{aligned} \max_{x,y} \quad & \sum_k \tilde{c}_k x_k + \sum_\ell \tilde{d}_\ell y_\ell \\ \text{s.t.} \quad & \sum_k \tilde{a}_{ik} x_k + \sum_\ell \tilde{b}_{i\ell} y_\ell \le \tilde{p}_i \quad \forall i \\ & y_\ell \in \{0,1\} \quad \forall \ell \end{aligned} \tag{1}$$

where parameters denoted with a tilde may be uncertain. As noted in Guzman et al. (2016), however, traditional robust optimization is applicable when uncertain parameters participate linearly in a constraint, or are coefficients of variables that participate linearly in a constraint. This may be achieved via reformulations of nonlinear constraints using new auxiliary variables (Guzman et al., 2016). Similarly, the introduction of an auxiliary variable representing the objective function value can allow objective function uncertainty to be considered as well (Li et al., 2012). With these points considered, we will follow the traditional convention of robust optimization literature by considering problems of the form in (1) moving forward.

In general, an uncertain parameter $\tilde{a}$ can be defined using a nominal value a, normally set to $\mathbf{E}\tilde{a}$, and a perturbation whose contribution to the actual parameter realization is determined by the product of random variable ξ with positive constant $\hat{a}$:

$$\tilde{a} = a + \xi\hat{a}. \tag{2}$$

With this definition of $\tilde{a}$, models with uncertain parameters participating linearly in their constraints or objective function can be represented without loss of generality as:

$$\begin{aligned} \max_{x} \quad & \sum_j c_j x_j \\ \text{s.t.} \quad & \sum_j a_{ij} x_j + \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij} x_j \le b_i \quad \forall i \end{aligned} \tag{3}$$

where $J_i := \{j : \hat{a}_{ij} > 0\}$ is the set of indices in constraint i where parameters a_{ij} are subject to uncertainty and variables x_j may be integer or continuous. If a parameter in constraint i is not uncertain, then only its nominal value a_{ij} must be included in the constraint.

At this point, random variables ξ have been introduced to the constraints, incorporating uncertainty, but creating a non-deterministic formulation. In order to create a deterministic formulation for which uncertain realizations of ξ are imposed on the model, uncertainty sets are utilized. Imposing entire uncertainty sets for ξ onto the model ensures that the optimal solution will remain feasible for all of the uncertain realizations in the uncertainty set, unlike a nominal model in which feasibility is only guaranteed for one specific vector of parameter values. The uncertainty set imposed on a model is defined as U_i, and allows the robust counterpart of (3) to be written as:

$$\begin{aligned} \max_x \quad & \sum_j c_j x_j \\ \text{s.t.} \quad & \sum_j a_{ij} x_j + \max_{\xi_i \in U_i} \left\{ \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij} x_j \right\} \le b_i \quad \forall i. \end{aligned} \tag{4}$$

The counterpart becomes deterministic when the inner maximization, representing the imposition of the uncertainty set on the constraint, is replaced by its dual. This will be discussed in detail in the following sections.

As uncertainty sets contain possible realizations of ξ_i, and thus $\tilde{a}$, they are generally constructed geometrically by including parameter points within some norm distance around the nominal values, $\xi_i = \mathbf{0}$. Here we highlight the five most common uncertainty sets in robust optimization, and the norms used to create them. The worst-case interval set proposed by Soyster (1973) is a specific instance of the box uncertainty set (Li et al., 2011) when parameters are bounded. The box uncertainty set can be defined using an ∞-norm distance about the nominal point:

$$U_i^B = \{\xi_i : \|\xi_i\|_\infty = \max_{j \in J_i} |\xi_{ij}| \le \Psi_i\}. \tag{B}$$

Similarly, a polyhedral or ellipsoidal shape may be used, based on the 1-norm or 2-norm respectively (Ben-Tal and Nemirovski, 1998, 1999, 2000; El Ghaoui et al., 1998; El Ghaoui and Lebret, 1997; Li et al., 2011; Bertsimas and Sim, 2004):

$$U_i^P = \{\xi_i : \|\xi_i\|_1 = \sum_{j \in J_i} |\xi_{ij}| \le \Gamma_i\} \tag{P}$$

$$U_i^E = \left\{ \xi_i : \|\xi_i\|_2 = \sqrt{\sum_{j \in J_i} \xi_{ij}^2} \le \Omega_i \right\}. \tag{E}$$

The set (P) is referred to as the polyhedral uncertainty set and (E) as the ellipsoidal uncertainty set. In general, a polyhedral uncertainty set does not have to involve the 1-norm, as various polyhedral geometries may be created from constraints on ξ_i. For this manuscript, however, the convention of Li et al. (2011, 2012) and Guzman et al. (2016, 2017a,b) will be followed in distinguishing the three basic, traditional uncertainty set geometries created via the ∞-norm, 1-norm, and 2-norm as the box, polyhedral, and ellipsoidal uncertainty sets.

These three uncertainty sets have size parameters, Ψ_i, Γ_i, and Ω_i, which can be denoted for a general uncertainty set as Δ_i. Increasing the size of Δ_i increases the size of the uncertainty set, and thus these uncertainty sets are ideal for unbounded parameters as the size parameter Δ_i may always be increased as necessary to include more parameter realizations. If, however, the parameters are bounded, meaning they can be defined such that $\xi_{ij} \in [-1, 1]$, then when $\Psi_i = 1$, all possible parameter realizations are included in the box uncertainty set ($\Psi_i = 1$ defines the interval set). Based on the structure of the uncertainty sets, all possible realizations of bounded uncertain parameters are included in the polyhedral set when $\Gamma_i = |J_i|$, and in the ellipsoidal set when $\Omega_i = \sqrt{|J_i|}$ (Li et al., 2011). However, with these values of Γ_i and Ω_i, spurious, impossible parameter points beyond the worst-case set are inappropriately included for bounded parameters due to the geometry of the underlying norm. This can be handled by intersecting the interval uncertainty set with other geometries; Ben-Tal and Nemirovski (2000) proposed the interval + ellipsoidal uncertainty set:

$$U_i^{IE} = \{\xi_i : \|\xi_i\|_2 \le \Omega_i, \|\xi_i\|_\infty \le 1\}. \tag{IE}$$

The interval + polyhedral uncertainty set, also often referred to as a budget uncertainty set, was proposed by Bertsimas and Sim (2004):

$$U_i^{IP} = \{\xi_i : \|\xi_i\|_1 \le \Gamma_i, \|\xi_i\|_\infty \le 1\}. \tag{IP}$$

These latter two uncertainty sets provide the benefits of the polyhedral or ellipsoidal geometries, while also ensuring that no parameters realize values beyond their bounds. Of course, if the parameters are unbounded, then there are no Δ_i values for which all possible parameter realizations are included, and the interval sets should not be utilized.

These five uncertainty sets can be utilized to create robust counterparts to general uncertain constraint i of the form:

$$\sum_j a_{ij} x_j + \max_{\xi_i \in U_i} \left\{ \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij} x_j \right\} \le b_i. \tag{5}$$

Yet, none of these uncertainty set definitions are appropriate for a constraint with both bounded and unbounded uncertain parameters. The interval + polyhedral or interval + ellipsoidal uncertainty sets simply do not apply in this case, as not all elements of ξ_i are bounded between -1 and 1. The box, ellipsoidal, and polyhedral sets, on the other hand, would allow parameter realizations for bounded parameters that are outside of their bounds when $\Psi_i > 1$, $\Omega_i > \sqrt{|J_i|}$, and $\Gamma_i > |J_i|$. Thus, Section 4 will provide new uncertainty sets and robust counterparts for constraints with both bounded and unbounded uncertain parameters.

3. General derivation of robust counterparts for uncertainty sets based on the intersection of multiple norm constraints

The use of uncertainty sets that combine multiple norm constraints for the random variables is advantageous, especially in the case when parameters in the uncertain constraint are bounded. As noted in Section 2, the interval + ellipsoidal and interval + polyhedral uncertainty sets are of this type and are widely popular in robust optimization. Before considering the case of uncertain constraints with both bounded and unbounded parameters, it is important to understand the process for deriving these uncertainty sets with multiple norm constraints. Derivations of this type have been done before in the context of the interval + ellipsoidal set (Ben-Tal and Nemirovski, 2000) and interval + polyhedral set (Bertsimas and Sim, 2004), and Ben-Tal et al. (2015) discuss the combining of multiple uncertainty regions based on p-norm constraints in their comprehensive work on robust counterparts for nonlinear constraints. This section is thus presented to clearly reiterate the details of this derivation process in order to lay the foundation for the novel development of robust counterparts in Section 4 which distinguish between bounded and unbounded parameters in the same constraint.

The uncertainty set is imposed through the inner maximization of constraint (5), specifically

$$\max_{\xi_i \in U_i} \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij} x_j. \tag{6}$$

Assuming that the uncertainty set U_i is defined by an intersection of p-norms, this creates the maximization problem in (7).

$$\begin{aligned} \max_{\xi_i} \quad & \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij} x_j \\ \text{s.t.} \quad & \|\xi_i\|_{p_1} \le \Delta_i^{(1)} \\ & \|\xi_i\|_{p_2} \le \Delta_i^{(2)} \end{aligned} \tag{7}$$

Here, the p-norms are general, convex p-norms with $p \ge 1$ and are distinguished through indices p_1 and p_2, with corresponding parameters $\Delta_i^{(1)}$ and $\Delta_i^{(2)}$ bounding the norms.

This problem is a conic optimization problem, as can be seen through the following transformation. For simplicity in the following definitions, the value $n := |J_i|$. Let the convex cone K_k represent the norm cone created by an individual p-norm constraint, defined as such (Boyd and Vandenberghe, 2004):

$$K_k := \left\{ \left(\xi_i; \Delta_i^{(k)} \right) \in \mathbb{R}^{n+1} : \|\xi_i\|_{p_k} \le \Delta_i^{(k)} \right\}.$$

Then, the Cartesian product of the closed convex cones K_1 and K_2 is also a closed convex cone K (Dattorro, 2005), defined by:

$$\begin{aligned} K &= K_1 \times K_2 \\ &= \left\{ \left(\xi_i; \Delta_i^{(1)}; \xi_i; \Delta_i^{(2)} \right) \in \mathbb{R}^{2n+2} : \|\xi_i\|_{p_1} \le \Delta_i^{(1)}, \|\xi_i\|_{p_2} \le \Delta_i^{(2)} \right\}. \end{aligned} \tag{8}$$

Defining vector

$$\gamma := \left(\mathbf{0}_{n\times 1}; \Delta_i^{(1)}; \mathbf{0}_{n\times 1}; \Delta_i^{(2)} \right) \in \mathbb{R}^{2n+2}$$

and matrix

$$A := (-I_{n\times n}, \mathbf{0}_{1\times n}, -I_{n\times n}, \mathbf{0}_{1\times n}) \in \mathbb{R}^{n\times(2n+2)},$$

the original inner maximization problem (7) can be equivalently rewritten in (9).

$$\begin{aligned} \max_{\xi_i} \quad & \beta^T \xi_i \\ \text{s.t.} \quad & \gamma - A^T \xi_i \in K \end{aligned} \tag{9}$$

Here, the constants in the objective function are grouped into parameter $\beta \in \mathbb{R}^n$, such that $\beta_j := \hat{a}_{ij} x_j, \forall j \in J_i$.

The problem (9) is a conic optimization problem for which strong duality holds and whose dual will be used formulate the robust counterpart. To find the dual, it is first noted that for norm cone K_k, its dual K_k^* is defined as,

$$K_k^* := \left\{ \left(u_i^{(k)}; v_i^{(k)} \right) : \|u_i^{(k)}\|_{q_k} \le v_i^{(k)} \right\}, \tag{10}$$

where $u_i^{(k)} \in \mathbb{R}^n$, $v_i^{(k)} \in \mathbb{R}$ are dual variables and $1/p_k + 1/q_k = 1$ (Boyd and Vandenberghe, 2004). It is also known that if a cone K is the Cartesian product of two convex cones K_1 and K_2, then the dual cone K^* is the Cartesian product of the two dual cones K_1^* and K_2^* (Dattorro, 2005). Thus, the dual K^* to cone K in (9) is,

$$\begin{aligned} K^* &= K_1^* \times K_2^* \\ &= \left\{ (u_i^{(1)}; v_i^{(1)}; u_i^{(2)}; v_i^{(2)}) \in \mathbb{R}^{2n+2} : \|u_i^{(1)}\|_{q_1} \le v_i^{(1)}, \|u_i^{(2)}\|_{q_2} \le v_i^{(2)} \right\}. \end{aligned} \tag{11}$$

Using these definitions, the dual of (9) is written in problem (12) (Boyd and Vandenberghe, 2004). The dual variables are grouped in w_i such that $w_i := \left(u_i^{(1)}; v_i^{(1)}; u_i^{(2)}; v_i^{(2)} \right)$.

$$\begin{aligned} \min_{w_i} \quad & \gamma^T w_i \\ \text{s.t.} \quad & A w_i = \beta \\ & w_i \in K^* \end{aligned} \tag{12}$$

By substituting β, w_i, A, and γ back into (12), and expanding the constraint $w \in K^*$, problem (13) is gained.

$$\begin{aligned} \min_{u,v} \quad & \Delta_i^{(1)} v_i^{(1)} + \Delta_i^{(2)} v_i^{(2)} \\ \text{s.t.} \quad & -\left(u_{ij}^{(1)} + u_{ij}^{(2)} \right) = \hat{a}_{ij} x_j \ \forall j \in J_i \\ & \|u_i^{(1)}\|_{q_1} \le v_i^{(1)} \\ & \|u_i^{(2)}\|_{q_2} \le v_i^{(2)} \end{aligned} \tag{13}$$

By noting that in this minimization, the norm constraints will always be active, variables $v_i^{(1)}$ and $v_i^{(2)}$ can be removed. Furthermore, the variables $u_{ij}^{(k)}$ are substituted such that $u_{ij}^{(k)} \longleftarrow -u_{ij}^{(k)}$. Thus, we gain the final, general form of the dual which is equivalent to the original inner maximization.

$$\begin{aligned} \min_{u} \quad & \Delta_i^{(1)} \|u_i^{(1)}\|_{q_1} + \Delta_i^{(2)} \|u_i^{(2)}\|_{q_2} \\ \text{s.t.} \quad & u_{ij}^{(1)} + u_{ij}^{(2)} = \hat{a}_{ij} x_j \qquad \forall j \in J_i \end{aligned} \tag{14}$$

When this problem is placed back into the uncertain constraint, the inner minimization operator will no longer be required; solving the problem without the inner minimization operator in the constraint will provide a solution that remains robust and feasible when the inner minimization operator is present. Thus, the general form of any robust counterpart to constraint (5) in which multiple p-norm constraints are utilized is

$$\begin{aligned} \sum_j a_{ij} x_j + \Delta_i^{(1)} \|u_i^{(1)}\|_{q_1} + \Delta_i^{(2)} \|u_i^{(2)}\|_{q_2} &\le b_i \\ u_{ij}^{(1)} + u_{ij}^{(2)} &= \hat{a}_{ij} x_j \ \forall j \in J_i. \end{aligned} \tag{15}$$

By setting $p_1 = \infty$ and $p_2 = 2$ (i.e. $q_1 = 1$ and $q_2 = 2$), and using $\Delta_i^{(1)} = 1$ and $\Delta_i^{(2)} = \Omega_i$, the interval + ellipsoidal counterpart can be gained from (15). Alternatively, by setting $p_1 = \infty$ and $p_2 = 1$ and assigning $\Delta_i^{(1)} = 1$ and $\Delta_i^{(2)} = \Gamma_i$, the interval + polyhedral counterpart can eventually be gained; however, to acquire the traditional formulation for the interval + polyhedral counterpart, a more complicated derivation is required after the form of (15). It will not be shown in full here, but will be discussed when deriving the general interval + polyhedral counterpart in Section 4.3. It is easy to see that the derivation leading to (15) could be expanded to three or more p-norms; the cone K used in the derivation would be the Cartesian product of three or more closed convex cones and parameters A, γ, etc. would simply be expanded with the same procedure used when two p-norm constraints on ξ_i were utilized.

4. Generalized intersecting *p*-norm uncertainty sets

In Section 3, the general derivation was shown for the robust counterpart of an uncertainty set that is defined by the intersection of two different p-norms. This derivation can provide the well-known interval + ellipsoidal and interval + polyhedral uncertainty sets, but it does not deal with a case when bounded and unbounded parameters both exist in a constraint. In this section, the derivation of Section 3 is modified in order to incorporate both types of parameters into the robust counterparts, which will for the first time provide robust counterparts valid for a mixture of bounded and unbounded parameters in a single uncertain constraint.

The new counterparts begin with the fact that it is possible to apply norm constraints to only a subset of random variables ξ_{ij}. For instance, if an interval set is utilized in a counterpart, then the ∞-norm constraint representative of the interval set should not be applied to the unbounded random variables. Thus, we define a vector $d_i^{(k)} \in \{0,1\}^n$, as follows:

$$d_{ij}^{(k)} := \begin{cases} 1 & \text{if } \xi_{ij} \text{ is included in the } p_k - \text{norm constraint,} \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

The uncertainty set U_i would then be defined generally,

$$U_i^{p_1 \cap p_2} = \left\{ \xi_i : \|d_i^{(1)} \circ \xi_i\|_{p_1} \le \Delta_i^{(1)}, \|d_i^{(2)} \circ \xi_i\|_{p_2} \le \Delta_i^{(2)} \right\}. \tag{17}$$

With this uncertainty set, the original maximization in the uncertain constraint (5) becomes equivalent to Problem (18):

$$\begin{aligned} \max_{\xi_i} \quad & \sum_{j \in J_i} \xi_{ij} \hat{a}_{ij} x_j \\ \text{s.t.} \quad & \|d_i^{(1)} \circ \xi_i\|_{p_1} \le \Delta_i^{(1)} \\ & \|d_i^{(2)} \circ \xi_i\|_{p_2} \le \Delta_i^{(2)} \end{aligned} \tag{18}$$

This new problem can be represented in the same syntax as (9) above, with the same definitions for γ, K, and β, if the definition of A is adjusted slightly. Specifically, the matrix A is redefined, such that

$$A := (-D_{n\times n}^{(1)}, \mathbf{0}_{1\times n}, -D_{n\times n}^{(2)}, \mathbf{0}_{1\times n}) \in \mathbb{R}^{n\times(2n+2)}, \tag{19}$$

and the identity matrix has been replaced with diagonal matrix $D^{(k)}$ with the vector $d_i^{(k)}$ on the diagonal. Defining A as such and using this new definition in problem (9) will give an equivalent reformulation of (18) with the same definitions of K, β, and γ as before. It can be reformulated into the general form of (12), and expanding this gives the new minimization problem

$$\begin{aligned} \min_{u,v} \quad & \Delta_i^{(1)} v_i^{(1)} + \Delta_i^{(2)} v_i^{(2)} \\ \text{s.t.} \quad & -\left(d_{ij}^{(1)} u_{ij}^{(1)} + d_{ij}^{(2)} u_{ij}^{(2)} \right) = \hat{a}_{ij} x_j \ \forall j \in J_i \\ & \|u_i^{(1)}\|_{q_1} \le v_i^{(1)} \\ & \|u_i^{(2)}\|_{q_2} \le v_i^{(2)} \end{aligned} \tag{20}$$

The final general dual that can be placed in the robust counterpart takes the form,

$$\begin{aligned} \min_{u_i^{(1)}, u_i^{(2)}} \quad & \Delta_i^{(1)} \|u_i^{(1)}\|_{q_1} + \Delta_i^{(2)} \|u_i^{(2)}\|_{q_2} \\ \text{s.t.} \quad & d_{ij}^{(1)} u_{ij}^{(1)} + d_{ij}^{(2)} u_{ij}^{(2)} = \hat{a}_{ij} x_j \ \forall j \in J_i \end{aligned} \tag{21}$$

using the same previous arguments regarding the active constraints involving $v_i^{(1)}$ and $v_i^{(2)}$, as well as through redefining $u_{ij}^{(k)} \longleftarrow -u_{ij}^{(k)}$.

Before proceeding to produce generalized robust counterparts that are specifically utilized for the case of simultaneous bounded and unbounded parameters, it is noted that by placing the (21) into the robust constraint, a generalized robust counterpart for intersection of any two p-norms is shown in (22).

$$\begin{aligned} \sum_j a_{ij} x_j + \Delta_i^{(1)} \|u_i^{(1)}\|_{q_1} + \Delta_i^{(2)} \|u_i^{(2)}\|_{q_2} & \le b_i \\ d_{ij}^{(1)} u_{ij}^{(1)} + d_{ij}^{(2)} u_{ij}^{(2)} & = \hat{a}_{ij} x_j \ \forall j \in J_i \end{aligned} \tag{22}$$

The new proposed uncertainty sets to handle bounded and unbounded uncertain parameters in the same constraint will all include one fixed norm constraint and one modifiable norm constraint. The modifiable norm constraint (i.e. box, polyhedral, or ellipsoidal) will be applied to both the bounded and unbounded uncertain parameters and is intersected with the fixed interval set, which is applied only to bounded parameters to guard against parameter realizations beyond their worst case values. Generally, the first norm in (22) will represent the fixed interval contribution, and the second will represent the contribution from the modifiable norm constraint that determines the overall size of the uncertainty set. This will be achieved by setting $d_{ij}^{(2)} = 1, \quad \forall j \in J_i$ since bounded and unbounded random variables are considered in this norm, and by defining $d_i^{(1)}$ such that

$$d_{ij}^{(1)} := \begin{cases} 1 & \text{if } \xi_{ij} \text{ is bounded,} \\ 0 & \text{otherwise.} \end{cases} \tag{23}$$

Proceeding from (21), $q_1 = 1$ and $\Delta_i^{(1)} = 1$ for the interval uncertainty set. The equality constraint is also rewritten to incorporate the values of $d_{ij}^{(k)}$ using disjoint subsets of J_i,

$$\begin{aligned} J_i^{(b)} & := \left\{ j \in J_i : \xi_{ij} \text{ bounded} \right\} \text{ and} \\ J_i^{(u)} & := \left\{ j \in J_i : \xi_{ij} \text{ unbounded} \right\}, \end{aligned}$$

where $J_i^{(b)} \cup J_i^{(u)} = J_i$. Applying these changes to (21) yields

$$\begin{aligned} \min_{u_i^{(1)}, u_i^{(2)}} \quad & \sum_{j \in J_i} |u_{ij}^{(1)}| + \Delta_i^{(2)} \|u_i^{(2)}\|_{q_2} \\ \text{s.t.} \quad & u_{ij}^{(1)} + u_{ij}^{(2)} = \hat{a}_{ij} x_j \ \forall j \in J_i^{(b)} \\ & u_{ij}^{(2)} = \hat{a}_{ij} x_j \ \forall j \in J_i^{(u)} \end{aligned} \tag{24}$$

While $u_i^{(1)}$ is present in the first summation of the objective function and the first equality constraint over $J_i^{(b)}$, it does not appear in the equality constraint over set $J_i^{(u)}$. Thus, from absolute values in the summation in the objective function and the nature of the minimization problem, $u_{ij}^{(1)} = 0, \quad \forall j \in J_i^{(u)}$, and the first summation can be rewritten only over $j \in J_i^{(b)}$. Furthermore, $u_i^{(1)}$ can be removed from the problem completely by using its definition from the first equality constraint in the summation of the objective function. The problem (24) can then be written equivalently as

$$\begin{aligned} \min_{u_i^{(2)}} \quad & \sum_{j \in J_i^{(b)}} |\hat{a}_{ij} x_j - u_{ij}^{(2)}| + \Delta_i^{(2)} \|u_i^{(2)}\|_{q_2} \\ \text{s.t.} \quad & u_{ij}^{(2)} = \hat{a}_{ij} x_j \ \forall j \in J_i^{(u)}. \end{aligned} \tag{25}$$

At this point, the derivation will split to focus on three particular uncertainty sets: the generalized interval + box, generalized interval + ellipsoidal, and generalized interval + polyhedral sets. These uncertainty sets and their robust counterparts are finalized in the following subsections.

4.1. Generalized interval + box uncertainty set

An interval + box uncertainty set has not appeared in literature before due to the fact that it would be redundant if all parameters are bounded, and ineffective if all parameters were unbounded. However, through the introduction of parameter $d_i^{(1)}$, these issues are removed, and a modifiable box geometry may be applied to all random variables in a constraint while those that are bounded remain guarded by the interval set. The formal definition of the generalized interval + box uncertainty set is,

$$U_i^{GIB} = \left\{ \xi_i : \|d_i^{(1)} \circ \xi_i\|_\infty \le 1, \|\xi_i\|_\infty \le \Psi_i \right\}. \tag{GIB}$$

Continuing from problem (25), the problem (26) is gained by setting $q_2 = 1$ and introducing $\Delta_i^{(2)} = \Psi_i$.

$$\begin{aligned} \min_{u_i^{(2)}} \quad & \sum_{j \in J_i^{(b)}} |\hat{a}_{ij} x_j - u_{ij}^{(2)}| + \Psi_i \sum_{j \in J_i} |u_{ij}^{(2)}| \\ \text{s.t.} \quad & u_{ij}^{(2)} = \hat{a}_{ij} x_j \ \forall j \in J_i^{(u)}. \end{aligned} \tag{26}$$

The second summation in the objective function of (26) may be split into two summations, one over $J_i^{(u)}$ and another over $J_i^{(b)}$. Then, the equality constraint may be removed by substituting it into the new summation over $j \in J_i^{(u)}$, yielding problem (27).

$$\min_{u_i^{(2)}} \sum_{j \in J_i^{(b)}} |\hat{a}_{ij}x_j - u_{ij}^{(2)}| + \Psi_i \left[\sum_{j \in J_i^{(b)}} |u_{ij}^{(2)}| + \sum_{j \in J_i^{(u)}} |\hat{a}_{ij}x_j| \right] \tag{27}$$

To gain a form similar to the existing interval + ellipsoidal and interval + polyhedral counterparts, variables z_{ij} are introduced such that $\hat{a}_{ij}z_{ij} = u_{ij}^{(2)}$. Then, noting that $\hat{a}_{ij} > 0, \forall j \in J_i$, the inner minimization is

$$\min_{z} \sum_{j \in J_i^{(b)}} \hat{a}_{ij}|x_j - z_{ij}| + \Psi_i \left[\sum_{j \in J_i^{(b)}} \hat{a}_{ij}|z_{ij}| + \sum_{j \in J_i^{(u)}} \hat{a}_{ij}|x_j| \right] \tag{28}$$

Thus, by placing this into the uncertain constraint (5), the final form of the generalized interval + box uncertainty set is gained and shown in Eq. (GIB-RC).

$$\sum_{j} a_{ij}x_j + \sum_{j \in J_i^{(b)}} \hat{a}_{ij}|x_j - z_{ij}| + \Psi_i \left[\sum_{j \in J_i^{(b)}} \hat{a}_{ij}|z_{ij}| + \sum_{j \in J_i^{(u)}} \hat{a}_{ij}|x_j| \right] \le b_i \tag{GIB-RC}$$

4.2. New generalized interval + ellipsoidal uncertainty set

The interval + ellipsoidal uncertainty set is a very popular uncertainty set for which very tight probabilistic bounds have been developed recently (Guzman et al., 2016, 2017a). Originally developed by Ben-Tal and Nemirovski (2000), the traditional interval + ellipsoidal counterpart takes the form of

$$\sum_{j} a_{ij}x_j + \sum_{j \in J_i} \hat{a}_{ij}|x_j - z_{ij}| + \Omega_i \sqrt{\sum_{j \in J_i} \hat{a}_{ij}^2 z_{ij}^2} \le b_i. \tag{29}$$

The new generalized robust counterpart will take a very similar form that will be more widely applicable but will also reduce to (29) in the case of all bounded uncertain parameters, as the following derivation will show.

The formal definition of the generalized interval + ellipsoidal uncertainty sets is

$$U_i^{GIE} = \left\{ \xi_i : \|d_i^{(1)} \circ \xi_i\|_\infty \le 1, \|\xi_i\|_2 \le \Omega_i \right\}, \tag{GIE}$$

where $d_i^{(1)}$ is defined according to (23). By setting $q_2 = 2$ and assigning $\Omega_i = \Delta_i^{(2)}$, problem (25) can be modified into the form of (30).

$$\begin{aligned} \min_{u_i^{(2)}} \quad & \sum_{j \in J_i^{(b)}} |\hat{a}_{ij}x_j - u_{ij}^{(2)}| + \Omega_i \sqrt{\sum_{j \in J_i} \left(u_{ij}^{(2)}\right)^2} \\ \text{s.t.} \quad & u_{ij}^{(2)} = \hat{a}_{ij}x_j \ \forall j \in J_i^{(u)} \end{aligned} \tag{30}$$

The summation inside the square root in the objective function may be split into summations over $J_i^{(u)}$ and $J_i^{(b)}$, and $u_i^{(2)}$ may be substituted out of the new summation over $J_i^{(u)}$ using the equality constraint. This yields the new minimization,

$$\min_{u_i^{(2)}} \sum_{j \in J_i^{(b)}} |\hat{a}_{ij}x_j - u_{ij}^{(2)}| + \Omega_i \sqrt{\sum_{j \in J_i^{(b)}} \left(u_{ij}^{(2)}\right)^2 + \sum_{j \in J_i^{(u)}} \left(\hat{a}_{ij}x_j\right)^2}. \tag{31}$$

Eq. (31) is already close in form to the original interval + ellipsoidal counterpart, but as with the generalized interval + box uncertainty set, one more substitution is made by introducing variables z_{ij} such that $\hat{a}_{ij}z_{ij} = u_{ij}^{(2)}, \quad \forall j \in J_i^{(b)}$. Thus, the minimization problem becomes

$$\min_{z} \sum_{j \in J_i^{(b)}} \hat{a}_{ij}|x_j - z_{ij}| + \Omega_i \sqrt{\sum_{j \in J_i^{(b)}} \hat{a}_{ij}^2 z_{ij}^2 + \sum_{j \in J_i^{(u)}} \hat{a}_{ij}^2 x_j^2} \tag{32}$$

Placing this minimization into the uncertain constraint yields the generalized interval + ellipsoidal counterpart:

$$\sum_{j} a_{ij}x_j + \sum_{j \in J_i^{(b)}} \hat{a}_{ij}|x_j - z_{ij}| + \Omega_i \sqrt{\sum_{j \in J_i^{(b)}} \hat{a}_{ij}^2 z_{ij}^2 + \sum_{j \in J_i^{(u)}} \hat{a}_{ij}^2 x_j^2} \le b_i \tag{GIE-RC}$$

When comparing this generalized counterpart to the original interval + ellipsoidal counterpart, it is obvious that if there are only bounded parameters, then the set $J_i^{(u)}$ will be empty and the generalized counterpart will reduce to the original interval + ellipsoidal counterpart.

4.3. New generalized interval + polyhedral uncertainty set

The interval + polyhedral uncertainty set developed originally by Bertsimas and Sim (2004) has been popular due to its linear counterpart. The form of the counterpart is shown in Eq. (33),

$$\begin{aligned} \sum_{j} a_{ij}x_j + \sum_{j \in J_i} p_{ij} + \Gamma_i z_i &\le b_i \\ z_i + p_{ij} &\ge \hat{a}_{ij}|x_j| \quad \forall j \in J_i \\ p_{ij} &\ge 0 \quad \forall j \in J_i \\ z_i &\ge 0 \end{aligned} \tag{33}$$

Variable z_i here represents the contribution from the polyhedral portion of the counterpart, and auxiliary variables p_{ij} represent the interval contributions from each uncertain parameter.

In contrast to the traditional interval + polyhedral uncertainty set, the formal definition of the generalized interval + polyhedral set is

$$U_i^{GIP} = \left\{ \xi_i : \|d_i^{(1)} \circ \xi_i\|_\infty \le 1, \|\xi_i\|_1 \le \Gamma_i \right\}. \tag{GIP}$$

To proceed from problem (25), parameter $\Delta_i^{(2)}$ is replaced with Γ_i and $q_2 = \infty$. Thus, the form of (34) is gained.

$$\begin{aligned} \min_{u_i^{(2)}} \quad & \sum_{j \in J_i^{(b)}} |\hat{a}_{ij}x_j - u_{ij}^{(2)}| + \Gamma_i \max_{j \in J_i} \left(|u_{ij}^{(2)}|\right) \\ \text{s.t.} \quad & u_{ij}^{(2)} = \hat{a}_{ij}x_j \ \forall j \in J_i^{(u)} \end{aligned} \tag{34}$$

Next, an auxiliary variable $w_i \in \mathbb{R}$ is introduced to represent the inner maximization through an inequality constraint:

$$\begin{aligned} \min_{u_i^{(2)}, w_i} \quad & \sum_{j \in J_i^{(b)}} |\hat{a}_{ij}x_j - u_{ij}^{(2)}| + \Gamma_i w_i \\ \text{s.t.} \quad & u_{ij}^{(2)} = \hat{a}_{ij}x_j \ \forall j \in J_i^{(u)} \\ & w_i \ge |u_{ij}^{(2)}| \ \forall j \in J_i \end{aligned} \tag{35}$$

The equality constraint can be removed by writing the inequality over the two disjoint sets $J_i^{(b)}$ and $J_i^{(u)}$.

$$\begin{aligned} \min_{u_i^{(2)}, w_i} \quad & \sum_{j \in J_i^{(b)}} |\hat{a}_{ij} x_j - u_{ij}^{(2)}| + \Gamma_i w_i \\ \text{s.t.} \quad & w_i \geq |u_{ij}^{(2)}| \ \forall j \in J_i^{(b)} \\ & w_i \geq \hat{a}_{ij} |x_j| \ \forall j \in J_i^{(u)} \end{aligned} \tag{36}$$

Finally, variables z_i are introduced such that $\hat{a}_{ij} z_{ij} = u_{ij}^{(2)}, \quad \forall j \in J_i^{(b)}$ to give the formulation for the generalized interval + polyhedral counterpart, shown in (37).

$$\begin{aligned} \min_{z_i, w_i} \quad & \sum_{j \in J_i^{(b)}} \hat{a}_{ij} |x_j - z_{ij}| + \Gamma_i w_i \\ \text{s.t.} \quad & w_i \geq \hat{a}_{ij} |z_{ij}| \ \forall j \in J_i^{(b)} \\ & w_i \geq \hat{a}_{ij} |x_j| \ \forall j \in J_i^{(u)} \end{aligned} \tag{37}$$

At this point, it should be noted that (37) is mathematically equivalent in this context to the traditional interval + polyhedral counterpart when $J_i^{(b)} = J_i$, and is consistent with the derivation and form of the generalized interval + box and generalized interval + ellipsoidal counterparts. However, it takes a different form than the interval + polyhedral counterpart that is well-established in robust optimization literature when all parameters are bounded. Thus, the derivation to arrive at a form similar to the traditional robust counterpart is demonstrated below.

To begin this derivation, a key change is made starting from (24). First, the parameter Γ_i is introduced for $\Delta_i^{(2)}$, and $q_2 = \infty$. Then, the traditional interval + polyhedral variable $z_i \in \mathbb{R}$ is introduced to represent the inner maximization from the ∞-norm. Thus, the form of (38) is gained.

$$\begin{aligned} \min_{u_i^{(1)}, u_i^{(2)}} \quad & \sum_{j \in J_i} |u_{ij}^{(1)}| + \Gamma_i z_i \\ \text{s.t.} \quad & u_{ij}^{(1)} + u_{ij}^{(2)} = \hat{a}_{ij} x_j \ \forall j \in J_i^{(b)} \\ & u_{ij}^{(2)} = \hat{a}_{ij} x_j \ \forall j \in J_i^{(u)} \\ & z_i \geq u_{ij}^{(2)} \ \forall j \in J_i \end{aligned} \tag{38}$$

In the derivation of the traditional interval + polyhedral counterpart, the variables $u_{ij}^{(2)}$ are substituted out of the problem first rather than substituting the variables $u_{ij}^{(1)}$. This involves separating the inequality constraint over sets $J_i^{(u)}$ and $J_i^{(b)}$, and substituting $u_{ij}^{(2)}$ using the two equality constraints. Unlike the very simple inequality constraints in (37), this leads to complicated inequality constraints representing the inner maximization for bounded parameters, as seen in (39).

$$\begin{aligned} \min_{u_i^{(1)}, z_i} \quad & \sum_{j \in J_i^{(b)}} |u_{ij}^{(1)}| + \Gamma_i z_i \\ \text{s.t.} \quad & z_i \geq |\hat{a}_{ij} x_j - u_{ij}^{(1)}| \quad \forall j \in J_i^{(b)} \\ & z_i \geq |\hat{a}_{ij} x_j| \quad \forall j \in J_i^{(u)} \end{aligned} \tag{39}$$

Thus, the following steps are required to reformulate the inequality constraint for bounded parameters into its traditional form. From the reverse triangle inequality, the right-hand side of the inequality constraint over $j \in J_i^{(b)}$ has the following relationship,

$$z_i \geq |\hat{a}_{ij} x_j - u_{ij}^{(1)}| \geq ||\hat{a}_{ij} x_j| - |u_{ij}^{(1)}|| \ \forall j \in J_i^{(b)}. \tag{40}$$

If any of the constraints over $j \in J_i^{(b)}$in (39) are active at an optimal solution, then from the form of the minimization it can be seen that

$$|u_{ij}^{(1)}| \leq |\hat{a}_{ij} x_j|,$$

at the optimal solution, since if at any point $|u_{ij}^{(1)}| > |\hat{a}_{ij} x_j|$, moving $|u_{ij}^{(1)}|$ towards $|\hat{a}_{ij} x_j|$ will not hurt the objective function value. It is also seen that at an optimal solution, the sign of $u_{ij}^{(1)}$ will be equivalent to the sign of $\hat{a}_{ij} x_j$ for active constraints; if the signs were different, then bringing $u_{ij}^{(1)}$ closer to zero would only benefit the objective function.

Constraint (40) from the reverse triangle inequality can be expanded into two inequalities,

$$\begin{aligned} & |\hat{a}_{ij} x_j - u_{ij}^{(1)}| \geq |\hat{a}_{ij} x_j| - |u_{ij}^{(1)}| \\ & |\hat{a}_{ij} x_j - u_{ij}^{(1)}| \geq |u_{ij}^{(1)}| - |\hat{a}_{ij} x_j| \end{aligned}$$

The first of these is always tighter than the second, since the right-hand side of the second is always less than or equal to zero because $|u_{ij}^{(1)}| \leq |\hat{a}_{ij} x_j|$ at an optimal solution. However, not only is the first constraint tighter, it is also active at the optimal solution such that

$$|\hat{a}_{ij} x_j - u_{ij}^{(1)}| = |\hat{a}_{ij} x_j| - |u_{ij}^{(1)}| \ \forall j \in J_i^{(b)}. \tag{41}$$

To prove this, consider that if $\hat{a}_{ij} x_j = u_{ij}^{(1)}$, then (41) reduces to 0 = 0. If $\hat{a}_{ij} = 0$, $|u_{ij}| \leq |\hat{a}_{ij} x_j| = 0$, thus (41) reduces to $|u_{ij}^{(1)}| = -|u_{ij}^{(1)}| = 0$. If $u_{ij}^{(1)} = 0$, then (41) reduces to $|\hat{a}_{ij} x_j|$ = $|\hat{a}_{ij} x_j|$. When $\hat{a}_{ij} x_j > 0$ and $u_{ij}^{(1)} > 0$, then $\hat{a}_{ij} x_j \geq u_{ij}^{(1)}$, and

$$\begin{aligned} |\hat{a}_{ij} x_j - u_{ij}^{(1)}| \quad & = \hat{a}_{ij} x_j - u_{ij}^{(1)} \\ & = |\hat{a}_{ij} x_j| - |u_{ij}^{(1)}|. \end{aligned}$$

Finally, if $\hat{a}_{ij} x_j < 0$ and $u_{ij}^{(1)} < 0$, then $\hat{a}_{ij} x_j \leq u_{ij}^{(1)}$. Thus,

$$\begin{aligned} |\hat{a}_{ij} x_j - u_{ij}^{(1)}| \quad & = \left| - \left[|\hat{a}_{ij} x_j| - |u_{ij}^{(1)}| \right] \right| \\ & = \left| |\hat{a}_{ij} x_j| - |u_{ij}^{(1)}| \right|. \end{aligned}$$

At this point, all allowable signs and magnitudes of $\hat{a}_{ij} x_j$ and $u_{ij}^{(1)}$ have been enumerated, and thus (41) can be substituted into the inequality constraint in the robust counterpart.

With the inequality constraint modified by (41), the robust counterpart takes the form:

$$\begin{aligned} \min_{u_i^{(1)}, z_i} \quad & \sum_{j \in J_i^{(b)}} |u_{ij}^{(1)}| + \Gamma_i z_i \\ \text{s.t.} \quad & z_i \geq |\hat{a}_{ij} x_j| - |u_{ij}^{(1)}| \quad \forall j \in J_i^{(b)} \\ & z_i \geq |\hat{a}_{ij} x_j| \quad \forall j \in J_i^{(u)} \end{aligned} \tag{42}$$

At this point, the traditional variables $p_{ij} \geq 0, \forall j \in J^{(b)}$ are introduced such that $p_{ij} = |u_{ij}^{(1)}|$, and the more traditional formulation is gained in (43).

$$
\begin{aligned}
\min_{p_i, z_i} \quad & \sum_{j \in J_i^{(b)}} p_{ij} + \Gamma_i z_i \\
\text{s.t.} \quad & z_i + p_{ij} \geq \hat{a}_{ij}|x_j| \quad \forall j \in J_i^{(b)} \\
& z_i \geq \hat{a}_{ij}|x_j| \quad \forall j \in J_i^{(u)} \\
& p_{ij} \geq 0 \quad \forall j \in J_i^{(b)}
\end{aligned} \tag{43}
$$

To generate the final generalized interval + polyhedral counterpart that is consistent in form with the traditional interval + polyhedral counterpart, the minimization problem of (43) replaces the inner maximization of the general robust constraint.

$$
\begin{aligned}
\sum_j a_{ij}x_j + \sum_{j \in J_i^{(b)}} p_{ij} + \Gamma_i z_i \quad & \leq b_i \\
z_i + p_{ij} \quad & \geq \hat{a}_{ij}|x_j| \quad \forall j \in J_i^{(b)} \\
z_i \quad & \geq \hat{a}_{ij}|x_j| \quad \forall j \in J_i^{(u)} \\
p_{ij} \quad & \geq 0 \quad \forall j \in J_i^{(b)}
\end{aligned} \tag{44}
$$

As with the other generalized counterparts, this formulation reduces to the polyhedral counterpart if all parameters are unbounded, and the traditional interval + polyhedral counterpart if all parameters are bounded.

However, due to the ease and intuitiveness of derivation to arrive at (37), and the consistency with the other generalized counterparts regarding auxiliary variables, we propose the new generalized interval + polyhedral counterpart with the formulation below.

$$
\begin{aligned}
\sum_j a_{ij}x_j + \sum_{j \in J_i^{(b)}} \hat{a}_{ij}|x_j - z_{ij}| + \Gamma_i w_i \quad & \leq b_i \\
w_i \quad & \geq \hat{a}_{ij}|z_{ij}| \; \forall j \in J_i^{(b)} \\
w_i \quad & \geq \hat{a}_{ij}|x_j| \; \forall j \in J_i^{(u)}
\end{aligned} \tag{GIP-RC}
$$

5. Generalized robust counterparts and probabilistic bounds

A considerable amount of research in modern robust optimization focuses on gaining probabilistic guarantees of the feasibility of robust solutions. This traces back to the probabilistic bounds presented with the original interval + ellipsoidal (Ben-Tal and Nemirovski, 2000) and interval + polyhedral (Bertsimas and Sim, 2004) uncertainty sets. A recent growth in probabilistic guarantees has focused on distributionally robust optimization, which differs from traditional robust optimization in that probability distributions themselves are subject to uncertainty (Wiesemann et al., 2014; Hanasusanto et al., 2015). Studies have also considered probabilistic guarantees on robust solutions when uncertainty sets are constructed from available data and statistical hypothesis tests (Bertsimas et al., 2017). These investigations address different cases than the new generalized uncertainty sets presented in Section 4, but the imperative desire for probabilistic information still stands for these new uncertainty sets. Recently, Guzman et al. (2016, 2017a,b) derived considerably tighter *a priori* and *a posteriori* probabilistic bounds on constraint violation for traditional robust counterparts with box, polyhedral, and ellipsoidal geometries. For the new generalized robust counterparts to be widely applicable for use with robust optimization, these probabilistic bounds must remain applicable for each appropriate counterpart.

A posteriori bounds, which simply characterize one specific solution to an uncertain optimization problem, are independent of the uncertainty set used to gain that solution, and thus a proof of applicability is not required. However, it must be shown that existing *a priori* probabilistic bounds are applicable when using the new robust counterparts, and this requires the definition of parameters δ_{ij} for these counterparts. Proofs similar to that of Proposition 1 were conducted for the first probabilistic bounds involving the interval + ellipsoidal and interval + polyhedral uncertainty sets by Ben-Tal and Nemirovski (2000) and Bertsimas and Sim (2004) respectively, and parameters δ_{ij} were defined explicitly in the context of new, tighter probabilistic bounds for all of the traditional uncertainty sets in Guzman et al. (2016). It is shown below that parameters δ_{ij} exist and may be defined for these new generalized robust counterparts so that Proposition 1 holds.

Table 1
Definition of δ_{ij} for the existing and generalized uncertainty sets.[a]

Type	δ_{ij}
Box	$\delta_{ij} := \frac{\hat{a}_{ij}x_j}{\sum_j \hat{a}_{ij}\|x_j\|}$
Generalized interval + box	$\delta_{ij} := \begin{cases} \frac{\hat{a}_{ij}x_j}{\sum_{j \in J_i^{(u)}} \hat{a}_{ij}\|x_j\| + \sum_{j \in J_i^{(b)}} \hat{a}_{ij}\|z_{ij}\|} & \text{if } j \in J_i^{(u)} \\ \frac{\hat{a}_{ij}z_{ij}}{\sum_{j \in J_i^{(u)}} \hat{a}_{ij}\|x_j\| + \sum_{j \in J_i^{(b)}} \hat{a}_{ij}\|z_{ij}\|} & \text{if } j \in J_i^{(b)} \end{cases}$
Ellipsoidal	$\delta_{ij} := \frac{\hat{a}_{ij}x_j}{\sqrt{\sum_j \hat{a}_{ij}^2 x_j^2}}$
Interval + ellipsoidal[b]	$\delta_{ij} := \frac{\hat{a}_{ij}z_{ij}}{\sqrt{\sum_j \hat{a}_{ij}^2 z_{ij}^2}}$
Generalized interval + ellipsoidal	$\delta_{ij} := \begin{cases} \frac{\hat{a}_{ij}x_j}{\sqrt{\sum_{j \in J_i^{(u)}} \hat{a}_{ij}^2 x_j^2 + \sum_{j \in J_i^{(b)}} \hat{a}_{ij}^2 z_{ij}^2}} & \text{if } j \in J_i^{(u)} \\ \frac{\hat{a}_{ij}z_{ij}}{\sqrt{\sum_{j \in J_i^{(u)}} \hat{a}_{ij}^2 x_j^2 + \sum_{j \in J_i^{(b)}} \hat{a}_{ij}^2 z_{ij}^2}} & \text{if } j \in J_i^{(b)} \end{cases}$
Polyhedral	$\delta_{ij} := \frac{\hat{a}_{ij}x_j}{\max_j \hat{a}_{ij}\|x_j\|}$
Interval + polyhedral[b]	$\delta_{ij} := \frac{\hat{a}_{ij}z_{ij}}{\max_j \hat{a}_{ij}\|z_{ij}\|}$
Generalized interval + polyhedral	$\delta_{ij} := \begin{cases} \frac{\hat{a}_{ij}x_j}{\max\left(\max_{j \in J_i^{(u)}}(\hat{a}_{ij}\|x_j\|), \max_{j \in J_i^{(b)}}(\hat{a}_{ij}\|z_{ij}\|)\right)} & \text{if } j \in J_i^{(u)} \\ \frac{\hat{a}_{ij}z_{ij}}{\max\left(\max_{j \in J_i^{(u)}}(\hat{a}_{ij}\|x_j\|), \max_{j \in J_i^{(b)}}(\hat{a}_{ij}\|z_{ij}\|)\right)} & \text{if } j \in J_i^{(b)} \end{cases}$

[a] Note that all $j \in J_i$, and that for all uncertainty sets, $\delta_{ij} \in [-1, 1]$.
[b] Assumes that probability distributions are bounded.

Proposition 1. *Given that, for all $j \in J_i$, random variable ξ_{ij} is independent, and the generalized uncertainty set U_i can be enforced on an uncertain constraint through problem (24), the following relations hold for the a priori probability of constraint violation of the ith constraint and $\theta > 0$:*

$$
Pr\left\{\sum_j a_{ij}x_j + \sum_{j \in J_i} \xi_{ij}\hat{a}_{ij}x_j > b_i\right\} \leq Pr\left\{\sum_{j \in J_i} \delta_{ij}\xi_{ij} > \Delta_i\right\} \tag{45}
$$

$$
\leq e^{-\theta\Delta_i} \prod_{j \in J_i} \mathbf{E} e^{\theta\delta_{ij}\xi_{ij}}, \tag{46}
$$

where the definitions of δ_{ij} are specific to the generalized uncertainty set U_i and are defined in Table 1 .

Proof. The proof is broad to encompass each of the three uncertainty sets presented above and will keep the modifiable norm constraint general through the use of $||\cdot||_q$. The derivation is as follows:

Table 2
Properties of δ_i for the generalized uncertainty sets.[a]

Type	$\|\delta_{ij}\|_1 \le \delta_i^{(1)}$	$\|\delta_{ij}\|_2 \le \delta_i^{(2)}$
Generalized interval + box	$\sum_{j\in J_i}\delta_{ij}=1$	$\sqrt{\sum_{j\in J_i}\delta_{ij}^2}\le 1$
Generalized interval + ellipsoidal	$\sum_{j\in J_i}\delta_{ij}\le\sqrt{\|J_i\|}$	$\sqrt{\sum_{j\in J_i}\delta_{ij}^2}=1$
Generalized interval + polyhedral	$\sum_{j\in J_i}\delta_{ij}\le \|J_i\|$	$\sqrt{\sum_{j\in J_i}\delta_{ij}^2}\le\sqrt{\|J_i\|}$

[a] Note that these properties are identical to those of the previous uncertainty sets, as highlighted in the reference tables of Guzman et al. (2016).

$$\begin{aligned}
&Pr\left\{\sum_j a_{ij}x_j + \sum_{j\in J_i}\xi_{ij}\hat{a}_{ij}x_j > b_i\right\}\\
&\overset{1}{\le} Pr\left\{\sum_{j\in J_i}\xi_{ij}\hat{a}_{ij}x_j > \sum_{j\in J_i^{(b)}}\hat{a}_{ij}|x_j-z_{ij}| + \Delta_i^{(2)}\|\zeta_i\|_q\right\}\\
&= Pr\left\{\sum_{j\in J_i^{(u)}}\xi_{ij}\hat{a}_{ij}x_j + \sum_{j\in J_i^{(b)}}\xi_{ij}\hat{a}_{ij}(x_j-z_{ij}+z_{ij}) - \sum_{j\in J_i^{(b)}}\hat{a}_{ij}|x_j-z_{ij}| > \Delta_i^{(2)}\|\zeta_i\|_q\right\}\\
&= Pr\left\{\sum_{j\in J_i^{(b)}}\xi_{ij}\hat{a}_{ij}z_{ij} + \sum_{j\in J_i^{(u)}}\xi_{ij}\hat{a}_{ij}x_j + \sum_{j\in J_i^{(b)}}\left[\xi_{ij}\hat{a}_{ij}(x_j-z_{ij}) - \hat{a}_{ij}|x_j-z_{ij}|\right] > \Delta_i^{(2)}\|\zeta_i\|_q\right\}\\
&\overset{2}{\le} Pr\left\{\sum_{j\in J_i^{(b)}}\xi_{ij}\hat{a}_{ij}z_{ij} + \sum_{j\in J_i^{(u)}}\xi_{ij}\hat{a}_{ij}x_j > \Delta_i^{(2)}\|\zeta_i\|_q\right\}\\
&= Pr\left\{\frac{\sum_{j\in J_i^{(b)}}\xi_{ij}\hat{a}_{ij}z_{ij} + \sum_{j\in J_i^{(u)}}\xi_{ij}\hat{a}_{ij}x_j}{\|\zeta_i\|_q} > \Delta_i^{(2)}\right\}\\
&\overset{3}{=} Pr\left\{\sum_{j\in J_i}\delta_{ij}\xi_{ij} > \Delta_i^{(2)}\right\}
\end{aligned}$$

where 1 is from the definition of the robust counterparts, and through defining ζ_i such that

$$\zeta_{ij} := \begin{cases}\hat{a}_{ij}x_j\ \forall j\in J_i^{(u)}\\ \hat{a}_{ij}z_{ij}\ \forall j\in J_i^{(b)},\end{cases} \tag{47}$$

2 is due to the fact that $\xi_{ij}\in[-1,1]$ for all $j\in J_i^{(b)}$, and 3 is based on the definition of δ_i such that

$$\delta_{ij} := \begin{cases}\dfrac{\hat{a}_{ij}x_j}{\|\zeta_i\|_q} & \forall j\in J_i^{(u)}\\[2ex] \dfrac{\hat{a}_{ij}z_{ij}}{\|\zeta_i\|_q} & \forall j\in J_i^{(b)}.\end{cases}$$

Having reached relation (45) in the derivation, relation (46) follows from Markov's inequality and independence of ξ_{ij}. □

The explicit forms of δ_{ij} for each generalized robust counterpart are shown in Table 1, with δ_{ij} from traditional uncertainty sets shown for reference. It is obvious upon review of the form of δ_{ij} that for a case where all parameters are unbounded, δ_{ij} will become the non-interval definition, while with all bounded parameters it will become identical to the "interval +" definitions. The key properties of δ_{ij} required for recent advances in *a priori* probabilistic bounds by Guzman et al. (2016, 2017a) are also applicable, as shown in Propositions 2 and 3, and are highlighted in Table 2.

Proposition 2. *The values of $||\delta_i||_1$ for the generalized interval + box, generalized interval + ellipsoidal, and generalized interval + polyhedral uncertainty sets are bounded by* 1, $\sqrt{|J_i|}$, *and* $|J_i|$ *respectively.*

Proof. *Generalized Interval + Box*:

$$\begin{aligned}
\|\delta_i\|_1 &= \sum_{j\in J_i}|\delta_{ij}|\\
&= \sum_{j\in J_i^{(u)}}\frac{|\hat{a}_{ij}x_i|}{\sum_{j\in J_i^{(u)}}\hat{a}_{ij}|x_j| + \sum_{j\in J_i^{(b)}}\hat{a}_{ij}|z_{ij}|}\\
&\quad+\sum_{j\in J_i^{(b)}}\frac{|\hat{a}_{ij}z_{ij}|}{\sum_{j\in J_i^{(u)}}\hat{a}_{ij}|x_j| + \sum_{j\in J_i^{(b)}}\hat{a}_{ij}|z_{ij}|}\\
&= \frac{\sum_{j\in J_i^{(u)}}\hat{a}_{ij}|x_j| + \sum_{j\in J_i^{(b)}}\hat{a}_{ij}|z_{ij}|}{\sum_{j\in J_i^{(u)}}\hat{a}_{ij}|x_j| + \sum_{j\in J_i^{(b)}}\hat{a}_{ij}|z_{ij}|}\\
&= 1
\end{aligned}$$

Generalized Interval + Ellipsoidal:

$$\begin{aligned}
\|\delta_i\|_1 &= \sum_{j\in J_i}|\delta_{ij}|\\
&= \sum_{j\in J_i^{(u)}}\frac{|\hat{a}_{ij}x_i|}{\sqrt{\sum_{j\in J_i^{(u)}}\hat{a}_{ij}^2x_j^2 + \sum_{j\in J_i^{(b)}}\hat{a}_{ij}^2z_{ij}^2}}\\
&\quad+\sum_{j\in J_i^{(b)}}\frac{|\hat{a}_{ij}z_{ij}|}{\sqrt{\sum_{j\in J_i^{(u)}}\hat{a}_{ij}^2x_j^2 + \sum_{j\in J_i^{(b)}}\hat{a}_{ij}^2z_{ij}^2}}\\
&\overset{1}{=}\frac{\sum_{j\in J_i}|\zeta_{ij}|}{\sqrt{\sum_{j\in J_i^{(u)}}\hat{a}_{ij}^2x_j^2 + \sum_{j\in J_i^{(b)}}\hat{a}_{ij}^2z_{ij}^2}}\\
&= \frac{|\sum_{j\in J_i}|\zeta_{ij}||}{\sqrt{\sum_{j\in J_i^{(u)}}\hat{a}_{ij}^2x_j^2 + \sum_{j\in J_i^{(b)}}\hat{a}_{ij}^2z_{ij}^2}}\\
&\overset{2}{\le}\frac{\sqrt{\sum_{j\in J_i}\zeta_{ij}^2}\sqrt{\sum_{j\in J_i}1^2}}{\sqrt{\sum_{j\in J_i^{(u)}}\hat{a}_{ij}^2x_j^2 + \sum_{j\in J_i^{(b)}}\hat{a}_{ij}^2z_{ij}^2}}\\
&= \frac{\sqrt{\sum_{j\in J_i^{(u)}}\hat{a}_{ij}^2x_j^2 + \sum_{j\in J_i^{(b)}}\hat{a}_{ij}^2z_{ij}^2}\sqrt{|J_i|}}{\sqrt{\sum_{j\in J_i^{(u)}}\hat{a}_{ij}^2x_j^2 + \sum_{j\in J_i^{(b)}}\hat{a}_{ij}^2z_{ij}^2}}\\
&= \sqrt{|J_i|}
\end{aligned}$$

where 1 is through the definition of ζ_i in (47) and 2 is via the Cauchy-Schwarz inequality.

Generalized Interval + Polyhedral:

$$\begin{aligned}
\|\delta_i\|_1 &= \sum_{j\in J_i}|\delta_{ij}|\\
&\overset{1}{=}\sum_{j\in J_i}\frac{|\zeta_{ij}|}{\max\limits_{j\in J_i}\left(|\zeta_{ij}|\right)}\\
&\overset{2}{\le}|J_i|
\end{aligned}$$

where 1 is through the definition of ζ_i in (47) and 2 is due to the fact that

$$\frac{|\zeta_{ij}|}{\max\limits_{j\in J_i}\left(|\zeta_{ij}|\right)} \le 1\ \forall j\in J_i.$$

□

Proposition 3. *The values of $||\delta_i||_2$ for the generalized interval + box, generalized interval + ellipsoidal, and generalized interval + polyhedral uncertainty sets are bounded by* 1, 1, *and* $\sqrt{|J_i|}$ *respectively.*

Proof. *Generalized Interval + Box*:

$$\begin{aligned}\|\delta_i\|_2 &\overset{1}{\le} \|\delta_i\|_1 \\ &\overset{2}{=} 1\end{aligned}$$

Here, 1 comes from the general property of p-norms, $||\phi||_2 \le ||\phi||_1$ for any vector ϕ, and 2 comes from the results of Proposition 2 for the generalized interval + box set.

Generalized Interval + Ellipsoidal:

$$\begin{aligned}\|\delta_i\|_2 &= \sqrt{\sum_{j\in J_i}\delta_{ij}^2} \\ &= \sqrt{\sum_{j\in J_i^{(u)}}\frac{(\hat{a}_{ij}x_i)^2}{\left(\sqrt{\sum_{j\in J_i^{(u)}}(\hat{a}_{ij}x_j)^2+\sum_{j\in J_i^{(b)}}(\hat{a}_{ij}z_{ij})^2}\right)^2}+\sum_{j\in J_i^{(b)}}\frac{(\hat{a}_{ij}z_{ij})^2}{\left(\sqrt{\sum_{j\in J_i^{(u)}}(\hat{a}_{ij}x_j)^2+\sum_{j\in J_i^{(b)}}(\hat{a}_{ij}z_{ij})^2}\right)^2}} \\ &= \sqrt{\frac{\sum_{j\in J_i^{(u)}}(\hat{a}_{ij}x_j)^2+\sum_{j\in J_i^{(b)}}(\hat{a}_{ij}z_{ij})^2}{\left(\sqrt{\sum_{j\in J_i^{(u)}}(\hat{a}_{ij}x_j)^2+\sum_{j\in J_i^{(b)}}(\hat{a}_{ij}z_{ij})^2}\right)^2}} \\ &= \sqrt{\frac{\sum_{j\in J_i^{(u)}}(\hat{a}_{ij}x_j)^2+\sum_{j\in J_i^{(b)}}(\hat{a}_{ij}z_{ij})^2}{\sum_{j\in J_i^{(u)}}(\hat{a}_{ij}x_j)^2+\sum_{j\in J_i^{(b)}}(\hat{a}_{ij}z_{ij})^2}} \\ &= 1\end{aligned}$$

Generalized Interval + Polyhedral:

$$\begin{aligned}\|\delta_i\|_2 &= \sqrt{\sum_{j\in J_i}\delta_{ij}^2} \\ &\overset{1}{=} \sum_{j\in J_i}\frac{\zeta_{ij}^2}{\left(\max_{j\in J_i}\left(|\zeta_{ij}|\right)\right)^2} \\ &\overset{2}{\le} \sqrt{|J_i|}\end{aligned}$$

where 1 is through the definition of ζ_i in (47) and 2 is due to the fact that

$$\frac{|\zeta_{ij}|}{\max_{j\in J_i}\left(|\zeta_{ij}|\right)} \le 1 \quad \forall j \in J_i.$$

□

These three propositions are all that are required to ensure that the probabilistic bounds previously developed may be applied to these new generalized robust counterparts (Guzman et al., 2016).

Regarding practical implementation, these robust counterparts will likely be most useful when the parameter distributions are known, or the unknown distributions are bounded, rather than having parameters that are completely unknown and unbounded. If all distributions are known and symmetric, then the very tight *a priori* bounds GMF6, GMF7, and GMF7′ from Guzman et al. (2017a) may be utilized. If some distributions are known and asymmetric, then bounds GMF8, GMF9, or GMF10 from Guzman et al. (2017a) are available. If there is a case when parameters with known, unbounded distributions are in a constraint with parameters that have unknown but bounded distributions, than a conservative distribution such as a discrete or uniform distribution may be assigned to the parameters with unknown, bounded distributions to allow the usage of *a priori* probabilistic bounds. However, if any of the parameters have distributions that are unknown and unbounded, then there currently exists no probabilistic bounds for these parameters; this lack of probabilistic bounds would exist even if all parameters are unknown and unbounded and one of the traditional uncertainty sets is utilized.

6. Computational examples

In this section, two computational examples are presented that demonstrate the advantage of the proposed generalized uncertainty sets compared to the traditional, existing uncertainty sets. In both examples, either one or multiple constraints will contain uncertain parameters, some of which will be bounded and others unbounded. Thus, only the box, ellipsoidal, and polyhedral uncertainty sets will be implemented and compared computationally to the new generalized interval + box, generalized interval + ellipsoidal, and generalized interval + polyhedral uncertainty sets. These comparisons will demonstrate how the new robust counterparts may be implemented and utilized with probabilistic bounds and will clearly illustrate the reduction of conservatism which is now available due to the new uncertainty sets.

6.1. First computational example

The importance of the new generalized uncertainty sets and robust counterparts is demonstrated through a simple example problem, adapted from Example 2-9 in Vanderbei (2014). This simple problem only contains three variables, and three constraints, one of which is uncertain. The maximization problem is shown in Problem (48).

$$\begin{aligned}\max_x\quad & 2x_1+3x_2+4x_3 \\ \text{s.t.}\quad & 2x_2+3x_3\le 5 \\ & x_1+x_2+2x_3\le 4 \\ & \tilde{1}x_1+\tilde{2}x_2+\tilde{3}x_3\le 7 \\ & x_j\ge 0\,\forall j\end{aligned}\tag{48}$$

The uncertain parameters are designated with a tilde in the third constraint. Basic assumptions about the probability distributions and perturbations of these uncertain parameters will clearly emphasize the necessity of the new robust counterparts.

To generalize the third constraint, it is rewritten in the following problem,

$$\begin{aligned}\max_x\quad & 2x_1+3x_2+4x_3 \\ \text{s.t.}\quad & 2x_2+3x_3\le 5 \\ & x_1+x_2+2x_3\le 4 \\ & \tilde{a}_{31}x_1+\tilde{a}_{32}x_2+\tilde{a}_{33}x_3\le 7 \\ & x_j\ge 0\,\forall j,\end{aligned}\tag{49}$$

where

$$\tilde{a}_{31} = 1 + \xi_{31}\cdot 0.40$$

$$\tilde{a}_{32} = 2 + \xi_{32}\cdot 0.50$$

$$\tilde{a}_{33} = 3 + \xi_{33}\cdot 0.30.$$

The distributions for ξ_{31}, ξ_{32}, ξ_{33} will be varied in the following subsections. It is assumed that ξ_{31} and ξ_{33} have bounded, uniform distributions, while ξ_{32} is subject to a standard normal distribution. Thus, set $J_i^{(u)}$ contains one index of j, $\{2\}$, while $J_i^{(b)}$:=

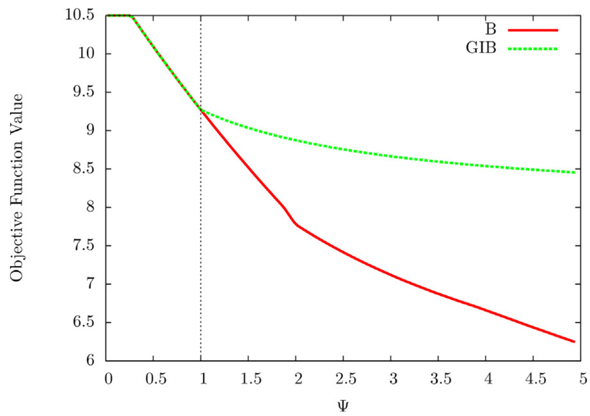

Fig. 1. Objective function vs Ψ value for Problem (48) when ξ_{31} and ξ_{33} are subject to a uniform distribution while ξ_{32} is subject to a standard normal distribution, with the box and generalized interval + box uncertainty sets utilized. The dotted black line at $\Psi = 1$ represents the point when the uncertainty set includes all possible realizations of the bounded parameters.

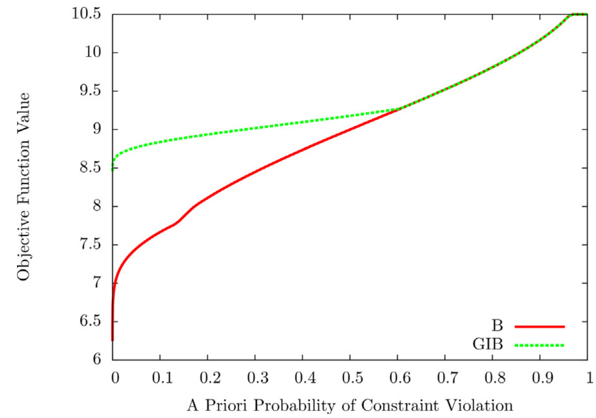

Fig. 2. Objective function vs *a priori* probability of constraint violation for Problem (48) when ξ_{31} and ξ_{33} are subject to a uniform distribution while ξ_{32} is subject to a standard normal distribution, with the box and generalized interval + box uncertainty sets utilized.

$\{1, 3\}$. The resulting robust counterparts are rewritten under these assumptions and computational results are provided to compare the traditional and new uncertainty sets for this simple problem.

6.1.1. Box and generalized interval + box

The box robust counterpart to the uncertain constraint in Problem (48) is simple and is shown in (50).

$$x_1 + 2x_2 + 3x_3 + \Psi(0.40x_1 + 0.50x_2 + 0.30x_3) \le 7 \tag{50}$$

The generalized interval + box robust counterpart, on the other hand, is shown in (51).

$$\begin{aligned} &x_1 + 2x_2 + 3x_3 + (0.40|x_1 - z_{31}| + 0.30|x_3 - z_{33}|) \\ &\quad + \Psi(0.40z_{31} + 0.50x_2 + 0.30z_{33}) \le 7 \end{aligned} \tag{51}$$

These robust counterparts are used along with probabilistic bound GMF6 from Guzman et al. (2017a) in order to generate robust solutions at a variety of *a priori* probabilities of constraint violation and Ψ values. The form of probabilistic bound GMF6 is shown below.

$$Pr\left\{\sum_j a_{ij}x_j + \sum_{j\in J_i}\xi_{ij}\hat{a}_{ij}x_j > b_i\right\} \le \exp\left(\min_{\theta>0}\left\{-\theta\Delta_i + \sum_{k\in K_i}\Lambda_{ik}(\theta)\right\}\right) \tag{GMF6}$$

These results can be seen in Figs. 1 and 2.

It is immediately obvious from Fig. 1 that as the value of Ψ increases, the difference between the objective values increases. The constraint is not initially active, as Ψ values below 0.25 all yield the same objective function values. For Ψ values ranging from 0 to 1, the objective function values are identical between the two sets. This makes sense, as no spurious realizations of the bounded parameters are imposed from the robust counterpart when $\Psi \le 1$. Above this point, however, parameters realizations beyond the worst case for bounded parameters are imposed on the model, and thus the two lines split at this point. The box uncertainty set objective function values continue downward with essentially the same slope, while the slope begins curving for the generalized interval + box robust counterpart.

The results can be viewed from the perspective of probabilities of constraint violation. For this problem, there is a tremendous difference between the box and generalized interval + box uncertainty sets when using GMF6 *a priori* at low probabilities of constraint violation. For instance, at a 50% probability of constraint violation, using the generalized interval + box uncertainty set gives an objective value of 9.1778, whereas the box uncertainty set only gives a value of 8.9986. At a 10% probability of constraint violation, the difference is much larger; the box set gives an objective function value of 7.6643, while the generalized interval + box uncertainty set provides a value of 8.8368. This difference of 1.1725 represents approximately 11% of the nominal value, showing the considerable improvement for this problem when the generalized uncertainty set is used.

6.1.2. Ellipsoidal and generalized interval + ellipsoidal

The ellipsoidal robust counterpart to the uncertain constraint in Problem (48) is seen in (52).

$$x_1 + 2x_2 + 3x_3 + \Omega\sqrt{0.40^2x_1^2 + 0.50^2x_2^2 + 0.30^2x_3^2} \le 7 \tag{52}$$

Alternatively, the generalized interval + ellipsoidal counterpart is shown in (53).

$$\begin{aligned} &x_1 + 2x_2 + 3x_3 + (0.40|x_1 - z_{31}| + 0.30|x_3 - z_{33}|) \\ &\quad + \Omega\sqrt{0.40^2z_{31}^2 + 0.50^2x_2^2 + 0.30^2z_{31}^2} \le 7 \end{aligned} \tag{53}$$

Results are generated to compare these two counterparts for their impact on the objective function, relative to the values of Ω and the *a priori* probability of constraint violation. The probability of constraint violation is calculated using bound GMF7′ from Guzman et al. (2017a), which is the tightest possible bound for the ellipsoidal based sets. Its form is shown below.

$$\begin{aligned} &Pr\left\{\sum_j a_{ij}x_j + \sum_{j\in J_i}\xi_{ij}\hat{a}_{ij}x_j > b_i\right\} \\ &\le \exp\left(\min_{\theta>0}\left\{-\theta\Delta_i + |J_i|\max_{j\in J_i}\Lambda_{ij}\left(\theta/\sqrt{|J_i|}\right)\right\}\right) \end{aligned} \tag{GMF7′}$$

As with the box counterparts, there is a clear deviation between the ellipsoidal and generalized interval + ellipsoidal uncertainty sets, which increases at higher levels of Ω. The deviation begins well before $\Omega = \sqrt{3}$, which is the point at which the ellipsoidal set completely envelopes all possible values for the bounded parameters. This is because even before this point, the ellipsoidal set can contain some realizations for bounded parameters outside of their worst case. The intersection of the interval set for bounded parameters prevents this, causing higher objective values (Fig. 3).

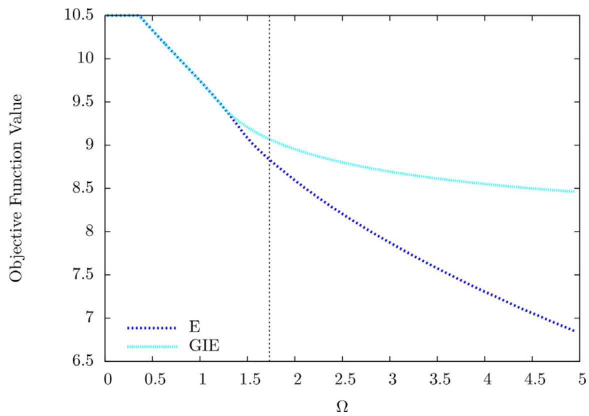

Fig. 3. Objective function vs Ω value for Problem (48) when ξ_{31} and ξ_{33} are subject to a uniform distribution while ξ_{32} is subject to a standard normal distribution, with the ellipsoidal and generalized interval + ellipsoidal uncertainty sets utilized. The dotted black line at $\Omega=\sqrt{3}$ represents the point when the uncertainty set includes all possible realizations of the bounded parameters.

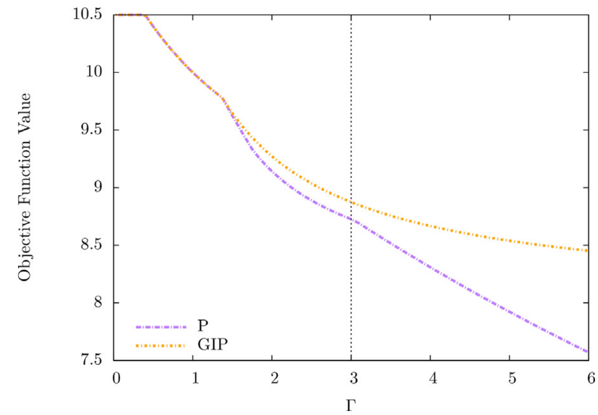

Fig. 5. Objective function vs Γ for Problem (48) when ξ_{31} and ξ_{33} are subject to a uniform distribution while ξ_{32} is subject to a standard normal distribution, with the polyhedral and generalized interval + polyhedral uncertainty sets utilized. The dotted black line at Γ = 3 represents the point when the uncertainty set includes all possible realizations of the bounded parameters.

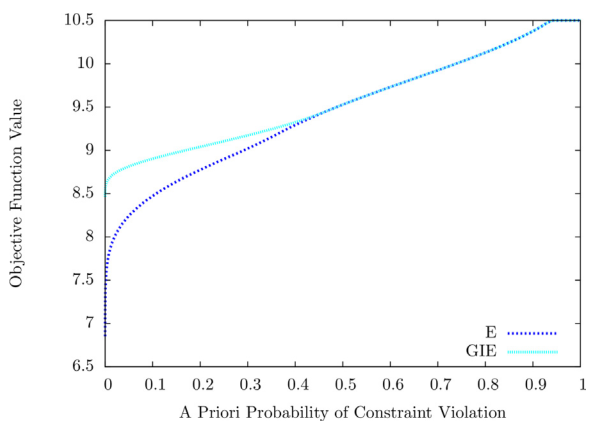

Fig. 4. Objective function vs *a priori* probability of constraint violation for Problem (48) when ξ_{31} and ξ_{33} are subject to a uniform distribution while ξ_{32} is subject to a standard normal distribution, with the ellipsoidal and generalized interval + ellipsoidal uncertainty sets utilized.

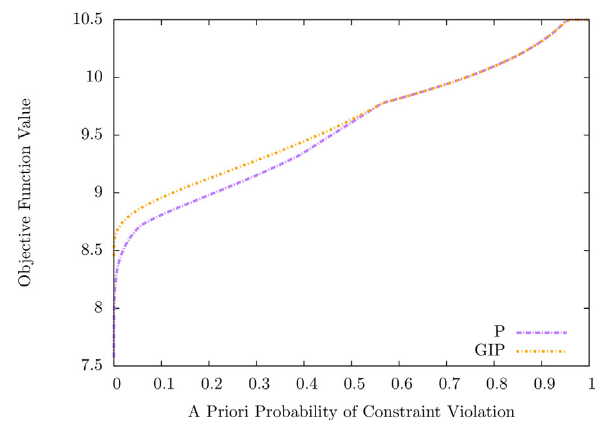

Fig. 6. Objective function vs *a priori* probability of constraint violation for Problem (48) when ξ_{31} and ξ_{33} are subject to a uniform distribution while ξ_{32} is subject to a standard normal distribution, with the polyhedral and generalized interval + polyhedral uncertainty sets utilized.

High values of Ω correspond to low probabilities of constraint violation, and thus it is not surprising that there is significant separation between the objective function values for the ellipsoidal and generalized interval + ellipsoidal sets at low probabilities of constraint violation in Fig. 4. At a 1% probability of constraint violation, there is a 10.8% difference in objective function values; at a 10% probability of constraint violations, the gap between the objective function values closes to 5.1%. The solutions become effectively identical by a probability of constraint violation of 46%; below this probability, it is clearly beneficial to use the generalized interval + ellipsoidal uncertainty set for this problem.

6.1.3. Polyhedral and generalized interval + polyhedral

Finally, the polyhedral robust counterpart to the uncertain constraint in Problem (48) is shown in (54).

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + \Gamma w_3 &\le 7 \\ w_3 &\ge 0.40|x_1| \\ w_3 &\ge 0.50|x_2| \\ w_3 &\ge 0.30|x_3| \end{aligned} \tag{54}$$

In contrast, the generalized interval + polyhedral robust counterpart is shown in (55).

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + (0.40|x_1 - z_{31}| + 0.30|x_3 - z_{33}|) + \Gamma w_3 &\le 7 \\ w_3 &\ge 0.40|z_{31}| \\ w_3 &\ge 0.50|x_2| \\ w_3 &\ge 0.30|z_{33}| \end{aligned} \tag{55}$$

These counterparts are used to generate results for Figs. 5 and 6, for which GMF6 from Guzman et al. (2017a) is used to calculate Γ at known probabilities of constraint violation.

Again, there is a considerable deviation between the traditional counterpart and the new generalized counterpart when the polyhedral geometry is utilized. The deviation is clearly seen well before Γ = 3, as even before then the polyhedral geometry contains realizations of the bounded parameters that are outside of their bounds, which the generalized interval + polyhedral set prevents. From a probabilistic perspective, the deviation between the objective function values produced by the polyhedral and generalized interval + polyhedral counterparts is not as drastic as in the ellipsoidal based sets. At a 10% probability of constraint violation, the gap between these solutions is only 1.7%. However, the gap between the solutions exists for this problem until the probability of constraint violation is 56%. As low levels of risk are generally

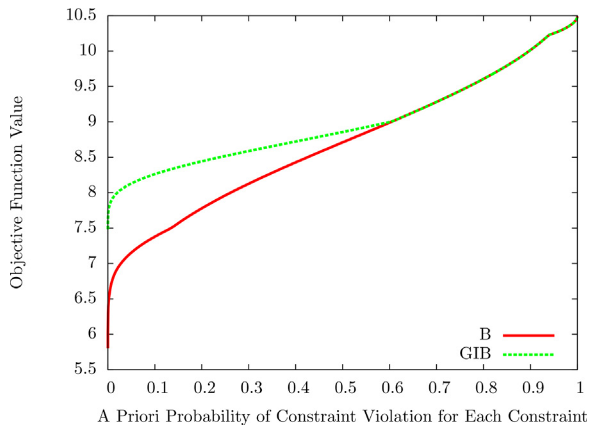

Fig. 7. Objective function vs *a priori* probability of constraint violation for Problem (56) when each constraint is subject to the same probability of constraint violation and each constraint contains three uncertain parameters, one of which is subject to a normal distribution while the other two are subject to uniform distributions.

desired, the generalized interval + polyhedral set is an important improvement over the existing traditional polyhedral counterpart.

6.1.4. Multiple uncertain constraints

While Problem (48) was defined with one uncertain constraint for the ease of demonstrating the creation of robust counterparts, the scope of these methods is certainly not limited to problems with only one uncertain constraint. To demonstrate this, we consider the same general problem as in (48) but with the second constraint also uncertain. Thus, rather than coefficients 1, 1, and 2, the second constraint now contains parameters $\tilde{a}_{21}$, $\tilde{a}_{22}$, and $\tilde{a}_{23}$ with nominal values of 1, 1, and 2. For each parameter in the second constraint, $\hat{a}_{2j}$ is defined as 10% of the nominal value, and it is assumed that the random variables ξ_{21}, ξ_{22}, and ξ_{23} are subject to uniform, standard normal, and uniform distributions respectively. With these characteristics defined, it is possible to formulate robust counterparts in a manner similar to each of the previous subsections, depending on the desired shape of the uncertainty set. Assuming a box geometry, the reformulation with the generalized interval + box robust counterpart is seen below.

$$\begin{array}{lrl}
\max\limits_{x,z} & 2x_1 + 3x_2 + 4x_3 & \\
\text{s.t.} & 2x_2 + 3x_3 & \leq 5 \\
& x_1 + x_2 + 2x_3 + (0.10|x_1 - z_{21}| + 0.20|x_3 - z_{23}|) + \Psi_2(0.10z_{21} + 0.10x_2 + 0.20z_{23}) & \leq 4 \\
& x_1 + 2x_2 + 3x_3 + (0.40|x_1 - z_{31}| + 0.30|x_3 - z_{33}|) + \Psi_3(0.40z_{31} + 0.50x_2 + 0.30z_{33}) & \leq 7 \\
& x_j & \geq 0 \quad \forall j
\end{array} \tag{56}$$

Fig. 7 provides a comparison of the box and generalized interval + box counterpart for this problem when each constraint is subject to the same *a priori* probability of constraint violation. As expected, the addition of more uncertainty lowers the objective function value at low probabilities of constraint violation compared to the case with only one uncertain constraint, and there remains a clear advantage to using the generalized interval + box robust counterpart compared to the box robust counterpart. Note that since each uncertain constraint in this example contains three uncertain parameters, two of which are subject to uniform distributions and one subject to a normal distribution, the values of Ψ_2 and Ψ_3 will be the same for each constraint when using *a priori* probabilistic bounds. This was constructed for the simplicity of the example, but this convenient consistency of the number of uncertain parameters per constraint and corresponding probability distributions is not required for these robust counterparts to be applied. It would be entirely feasible to apply these robust counterparts to Problem (48) if all of the parameters in the objective function and each constraint were uncertain, simply by formulating a robust counterpart for each constraint individually and replacing it in the original model. Because robust counterparts are created for each individual constraint, it is also possible for the same uncertain parameter to appear in multiple constraints. In this case, as in the current example, an upper bound on each constraint's individual probability of constraint violation could also be calculated.

6.2. Second computational example

A second computational example is provided in Problem (57) to provide more insight into the importance of the generalized interval uncertainty sets by analyzing the behavior of the variables as the sizes of the uncertainty sets change.

$$\begin{array}{lrl}
\max\limits_{x} & 3x_1 + 4x_2 + 5x_3 + 8x_4 + 6x_5 & \\
\text{s.t.} & \tilde{1}x_1 + \tilde{2}x_2 + \tilde{3}x_3 + \tilde{6}x_4 + \tilde{4}x_5 & \leq 30 \\
& 2x_1 + 3x_2 + 4x_3 + 9x_4 + 5x_5 & \leq 45 \\
& 3x_1 + 4x_3 + 5x_5 & \leq 16 \\
& 2x_2 + 2x_4 & \leq 12 \\
& x_1 + x_2 + x_3 & \leq 10 \\
& 2x_4 + x_5 & \leq 15 \\
& x_j & \geq 0 \ \forall j
\end{array} \tag{57}$$

This problem contains five variables, all of which appear with uncertain coefficients in the first constraint. These parameters are all defined such that $\hat{a}_{1j} = 0.20 \cdot a_{1j}, \forall j$. It is assumed that the random variables (ξ_{11}, ξ_{13}, and ξ_{15}) associated with the first, third, and fifth parameters ($\tilde{1}$, $\tilde{3}$, and $\tilde{4}$) in the constraint will be subject to a standard normal distribution. The random variables (ξ_{12} and ξ_{14}) associated with the second and fourth parameters ($\tilde{2}$ and $\tilde{6}$) will be subject to the uniform distribution. Thus, three of the parameters are unbounded, and two of the parameters are bounded, necessitating the use of the generalized uncertainty sets.

Results for the box, ellipsoidal, polyhedral, generalized interval + box, generalized interval + ellipsoidal, and generalized interval + polyhedral uncertainty sets were generated for a variety of Δ values by varying the *a priori* probability from 1E−10 — essentially zero, although a true probability of zero cannot be reached with unbounded parameters — to 0.9999 (essentially the nominal case). The bound GMF6 was utilized for the box and polyhedral based uncertainty sets, while bound GMF7′ was used for the ellipsoidal based sets. The results for the objective function values, as well as the variable values at the optimal solutions, are shown for each uncertainty set in Figs. 8–16.

The deviation between the results of the box and generalized interval + box uncertainty sets begins when $\Psi > 1$, which is the point when spurious realizations beyond the worst case would be included for bounded parameters when a traditional box uncertainty set is used. The objective function is higher for the generalized interval + box uncertainty set after this point, and interesting behavior is seen regarding the variable values at the optimal

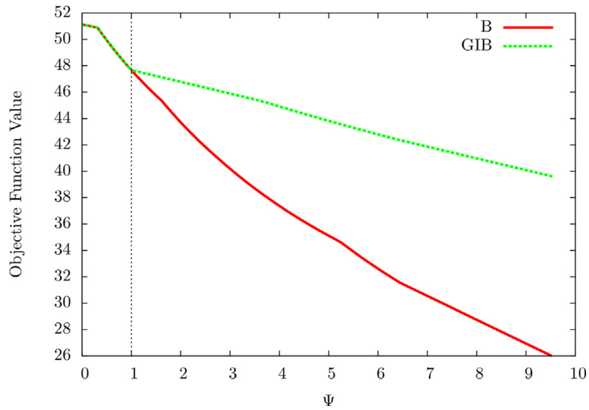

Fig. 8. Objective function vs Ψ value for Problem (57) with the box and generalized interval + box uncertainty sets utilized. Uncertain parameters $\tilde{a}_{11}$, $\tilde{a}_{13}$, and $\tilde{a}_{15}$ are subject to normal distributions, and parameters $\tilde{a}_{12}$ and $\tilde{a}_{14}$ are subject to uniform distributions. The dotted black line at $\Psi = 1$ represents the point when the uncertainty set includes all possible realizations of the bounded parameters.

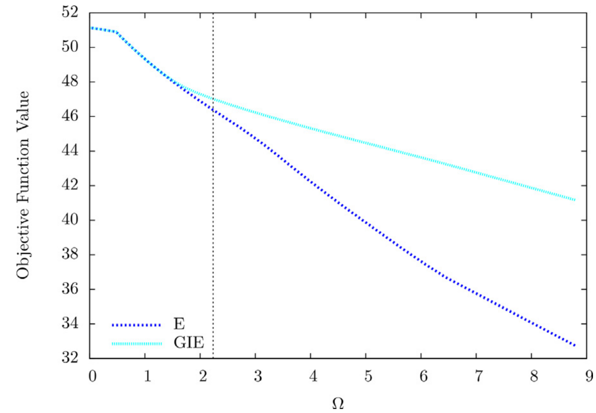

Fig. 11. Objective function vs Ω value for Problem (57) with the ellipsoidal and generalized interval + ellipsoidal uncertainty sets utilized. Uncertain parameters $\tilde{a}_{11}$, $\tilde{a}_{13}$, and $\tilde{a}_{15}$ are subject to normal distributions, and parameters $\tilde{a}_{12}$ and $\tilde{a}_{14}$ are subject to uniform distributions. The dotted black line at $\Omega = \sqrt{5}$ represents the point when the uncertainty set includes all possible realizations of the bounded parameters.

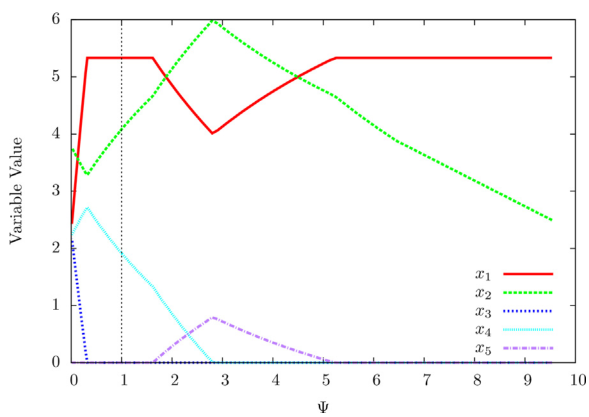

Fig. 9. Values of variables x_j at the optimal solutions at varying levels of Ψ when the uncertain parameters $\tilde{a}_{11}$, $\tilde{a}_{13}$, and $\tilde{a}_{15}$ are subject to normal distributions, and parameters $\tilde{a}_{12}$ and $\tilde{a}_{14}$ are subject to uniform distributions and a box uncertainty set is utilized. The dotted black line at $\Psi = 1$ represents the point when the uncertainty set includes all possible realizations of the bounded parameters.

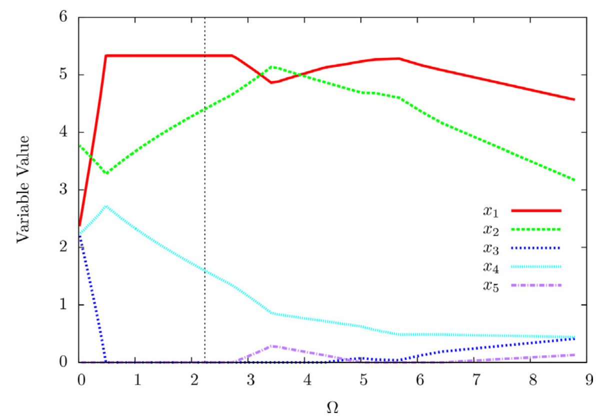

Fig. 12. Values of variables x_j at the optimal solutions at varying levels of Ω when the uncertain parameters $\tilde{a}_{11}$, $\tilde{a}_{13}$, and $\tilde{a}_{15}$ are subject to normal distributions, and parameters $\tilde{a}_{12}$ and $\tilde{a}_{14}$ are subject to uniform distributions and an ellipsoidal uncertainty set is utilized. The dotted black line at $\Omega = \sqrt{5}$ represents the point when the uncertainty set includes all possible realizations of the bounded parameters.

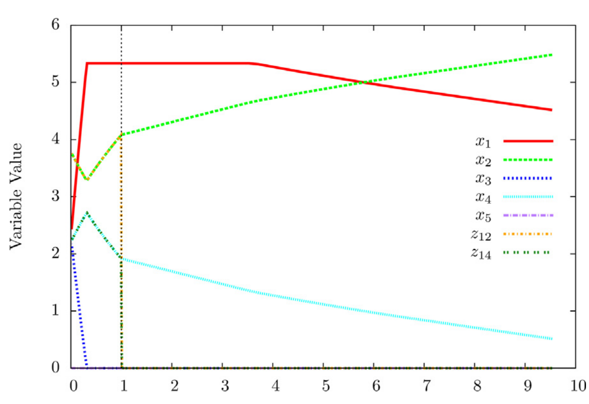

Fig. 10. Values of variables $x_j, \forall j$, and $z_{1j}, \forall j \in J_i^{(b)}$ at the optimal solutions at varying levels of Ψ when the uncertain parameters $\tilde{a}_{11}$, $\tilde{a}_{13}$, and $\tilde{a}_{15}$ are subject to normal distributions, and parameters $\tilde{a}_{12}$ and $\tilde{a}_{14}$ are subject to uniform distributions and a generalized interval + box uncertainty set is utilized. The dotted black line at $\Psi = 1$ represents the point when the uncertainty set includes all possible realizations of the bounded parameters.

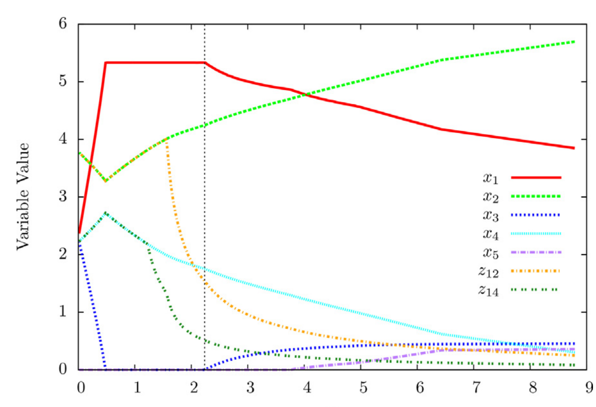

Fig. 13. Values of variables $x_j, \forall j$, and $z_{1j}, \forall j \in J_i^{(b)}$ at the optimal solutions at varying levels of Ω when the uncertain parameters $\tilde{a}_{11}$, $\tilde{a}_{13}$, and $\tilde{a}_{15}$ are subject to normal distributions, and parameters $\tilde{a}_{12}$ and $\tilde{a}_{14}$ are subject to uniform distributions and a generalized interval + ellipsoidal uncertainty set is utilized. The dotted black line at $\Omega = \sqrt{5}$ represents the point when the uncertainty set includes all possible realizations of the bounded parameters.

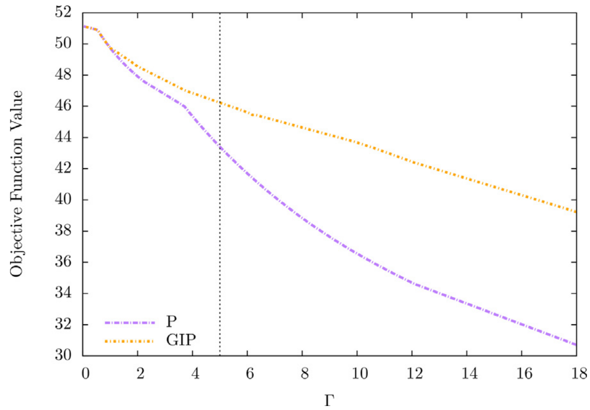

Fig. 14. Objective function vs Γ value for Problem (57) with the polyhedral and generalized interval + polyhedral uncertainty sets utilized. Uncertain parameters $\tilde{a}_{11}$, $\tilde{a}_{13}$, and $\tilde{a}_{15}$ are subject to normal distributions, and parameters $\tilde{a}_{12}$ and $\tilde{a}_{14}$ are subject to uniform distributions. The dotted black line at Γ = 5 represents the point when the uncertainty set includes all possible realizations of the bounded parameters.

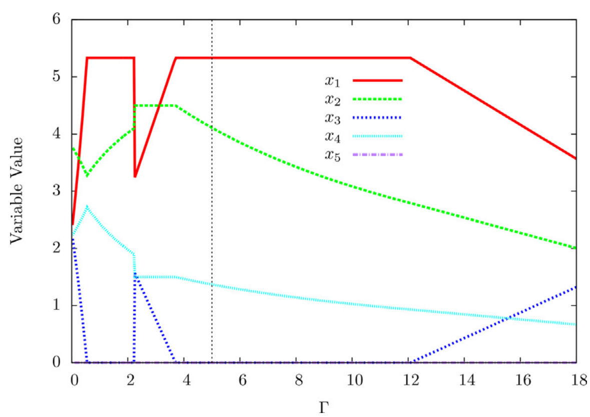

Fig. 15. Values of variables x_j at the optimal solutions at varying levels of Γ when the uncertain parameters $\tilde{a}_{11}$, $\tilde{a}_{13}$, and $\tilde{a}_{15}$ are subject to normal distributions, and parameters $\tilde{a}_{12}$ and $\tilde{a}_{14}$ are subject to uniform distributions and a polyhedral uncertainty set is utilized. The dotted black line at Γ = 5 represents the point when the uncertainty set includes all possible realizations of the bounded parameters.

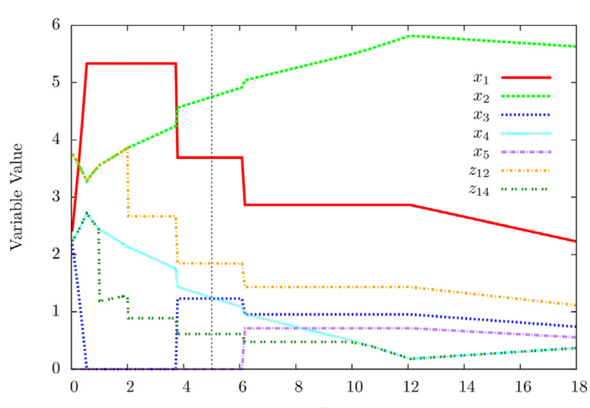

Fig. 16. Values of variables $x_j, \forall j$, and $z_{1j}, \forall j \in J_i^{(b)}$ at the optimal solutions at varying levels of Γ when the uncertain parameters $\tilde{a}_{11}$, $\tilde{a}_{13}$, and $\tilde{a}_{15}$ are subject to normal distributions, and parameters $\tilde{a}_{12}$ and $\tilde{a}_{14}$ are subject to uniform distributions and a generalized interval + polyhedral uncertainty set is utilized. The dotted black line at Γ = 5 represents the point when the uncertainty set includes all possible realizations of the bounded parameters.

solutions, shown in Figs. 9 and 10. There is not a clear shift in behavior for the variables at Ψ = 1 with the traditional box set, yet there is for the generalized interval + box set. The comparison of x_2 to z_{12}, and x_4 to z_{14} demonstrates that the interval portion of the generalized counterpart is active after Ψ = 1. Based on the formulation, when $x_2 = z_{12}$ and $x_4 = z_{14}$, the interval geometry is not impacting the optimal solution; however, after Ψ = 1, $z_{12} = z_{14} = 0$, meaning that the interval portion is completely active. It is seen that in both the box and generalized box + interval counterparts, the values of x_4 decrease while the values of x_2 increase. Yet due to the interval portion of the generalized interval + box set, the values of x_4 does not need to decrease as fast in Fig. 10, because the effective Ψ value remains at one for the bounded uncertain parameters.

Looking at the behavior of the variables as Ψ continues to increase, another clear distinction is seen between the box and generalized interval + box uncertainty sets. While x_1 is effectively becoming the dominant variable in the box case, due to the smaller perturbation value, the generalized interval + box uncertainty set eventually devalues x_1 because its coefficient is unbounded. In fact, at Ψ = 20, the optimal solution only contains non-zero values for x_2 and x_4, the bounded parameters. Of course, it is unnecessary to have a Ψ value this high, as an *a priori* probability of constraint violation of 1E-10 is reached by a Ψ value of 9.535.

Similar trends are seen for the ellipsoidal-based sets regarding variable values and the overall objective function. The deviation in objective function value actually begins before $\Omega = \sqrt{5}$, which is the point when the ellipsoidal shape includes all possible realizations of the bounded parameters. This can be seen by the fact that variables z_{12} and z_{14} begin to deviate from x_2 and x_4 respectively before $\Omega = \sqrt{5}$, and is due to the shape of the ellipsoidal set, which contains some spurious realizations of the bounded parameters before this point is reached. Changes in behavior of the variables can be seen after Ω = 1.2, the same point at which the objective function values begin to deviate. A more drastic deviation begins when $\Omega = \sqrt{5}$ however, as at this point the generalized interval + ellipsoidal uncertainty set begins to devalue x_1, which has an unbounded uncertain coefficient, and begins to more rapidly increase x_2, whose coefficient is bounded. The same solution is gained with generalized interval + ellipsoidal set as with the generalized interval + box set when Δ = 20; only x_2 and x_4 are chosen due to their bounded coefficients, and the values for z_{12} and z_{14} decrease to zero meaning only the interval portion of the counterpart is contributing to the constraint.

Finally, these trends may be observed from the polyhedral and generalized interval + polyhedral uncertainty sets. A deviation in objective function value and variable behavior begins very early, at Γ > 1, based on the geometry of the polyhedral set which unnecessarily enforces realizations of the bounded parameters that are outside their bounds after this point. The inner maximization in the polyhedral counterparts causes sharp changes in solutions, and also provides incentive to prevent the z_{ij} variables from decreasing all the way to zero. Essentially, an interval contribution (i.e. decreasing z_{1j} for $j \in J_i^{(b)}$) for a parameter is only necessary until its $\hat{a}_{1j}z_{1j}$ term is equivalent to the maximum $\hat{a}_{1j}z_{1j}$ or $\hat{a}_{1j}x_j$ terms of the other variables with bounded or unbounded parameters, respectively. Thus, in the range of Γ considered here, neither of z_{12} or z_{14} decreases to zero, because x_1 (which has an unbounded coefficient) is still non-zero in this range. However, the interval portion for bounded parameters is still having a clear impact on the objective function value, vastly reducing conservatism as seen in Fig. 14. And while variables with unbounded coefficients (x_1 and x_3) are prominent in solutions of the pure polyhedral problem, they are devalued in the generalized interval + polyhedral sets relative to x_2, and eventually x_4, based on these variables' bounded coefficients. At very high Γ, the z_{11}, z_{13}, and z_{15} values will go to zero

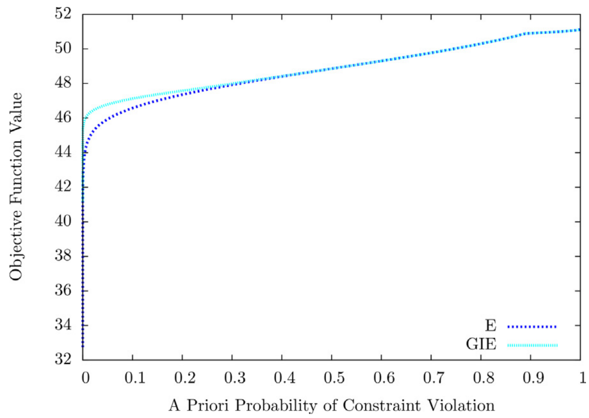

Fig. 17. Objective function vs *a priori* probability of constraint violation for Problem (57) with the ellipsoidal and generalized interval + ellipsoidal uncertainty sets utilized and probabilities based on bound GMF7′. Uncertain parameters $\tilde{a}_{11}$, $\tilde{a}_{13}$, and $\tilde{a}_{15}$ are subject to normal distributions, and parameters $\tilde{a}_{12}$ and $\tilde{a}_{14}$ are subject to uniform distributions.

in the generalized interval + polyhedral counterpart and the solution becomes equivalent to that of the generalized interval + box and generalized interval + ellipsoidal solutions. Thus, despite their different geometries, all of the generalized "interval +" sets behave appropriately by preventing unnecessary realizations of bounded parameters beyond their worst case, as reflected in the behavior of the variables, thus limiting conservatism and giving better objective function values.

The differences in uncertainty sets utilized have a large impact from a probabilistic perspective as well. The ellipsoidal based sets provide the best results for this problem, primarily due to the tightness of the *a priori* bound GMF7′ (Guzman et al., 2017a). Relative to the other geometries, the difference between the objective function values is also the smallest when comparing the ellipsoidal and generalized interval + ellipsoidal sets, as seen in Fig. 17. That is, the gap between the generalized "interval +" sets and the traditional sets would be even larger for similar figures with the box and polyhedral sets. Yet, even with the ellipsoidal based sets, there is a 3.3% increase in objective function value using the generalized interval + ellipsoidal set at a 1% probability of constraint violation. For the box sets, the increase is 14.4% at a 1% probability of constraint violation, while the increase is 8.8% for the polyhedral based sets. Thus, as already shown in the previous computational example, using the new generalized sets has a large impact on reducing conservatism from a probabilistic view, especially at low probabilities of constraint violation.

7. Conclusions

The robust counterparts provided in this paper increase the applicability and reduce the conservatism of robust optimization when an uncertain constraint contains both bounded and unbounded uncertain parameters. After demonstrating how robust counterparts can be derived for uncertainty sets based on any number of *p*-norm constraints, this derivation was generalized for the case when some random variables are included in certain norm constraints, but not others. Three specific robust counterparts are provided as analogues to the traditional box, ellipsoidal, and polyhedral robust counterparts for the case when uncertain constraints contain both bounded and unbounded uncertain parameters. The generalized interval + box, generalized interval + ellipsoidal, and generalized interval + polyhedral counterparts are less conservative than the traditional counterparts, which would impose parameter values outside of the worst-case values for bounded parameters at high Δ values. They are also more widely applicable than the traditional interval + ellipsoidal and interval + polyhedral uncertainty sets, which require bounds on all uncertain parameters in a constraint. It is shown that established *a priori* probabilistic bounds remain valid for these new counterparts, and the impact of the new formulations is demonstrated through two optimization problems. These new counterparts are especially important at low probabilities of constraint violation, when Δ values are high. Altogether, these theoretical developments widen the scope of the already powerful methodologies of robust optimization.

Acknowledgments

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