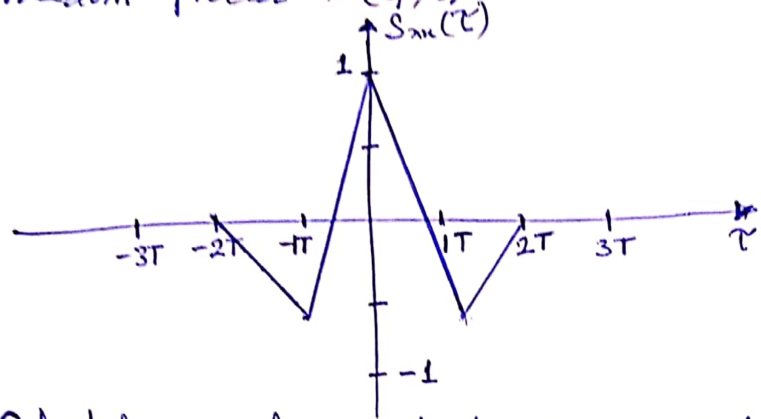
# Task 4.5

This is the auto correlation function $S_{xx}(\tau)$ of the stationary random process $x(\zeta, t)$:




Calculate and sketch the power spectral density, $S_{xx}(\omega)$.

Power Spectral Density = ?

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} S_{xx}(\tau) e^{-j\omega\tau} d\tau$$

NB: $S_{xx}(\omega) = S_{xx}(-\omega)$

$$\because \int_0^T \frac{\left(\frac{-\tau}{2} - \tau + 1\right)}{T} d\tau + \int_T^{2T} \frac{0 + \frac{\tau}{2}}{2T - T} - 1$$

Looking at the negative side of the plot above.

$$2\left[\int_0^T \left(1 - \frac{3\tau}{2T}\right) e^{-j\omega\tau} d\tau + \int_T^{2T} \left(-1 + \frac{\tau}{2T}\right) e^{-j\omega\tau} d\tau\right]$$

$$\Rightarrow 2\left[\int_0^T \cos\omega\tau\, d\tau + \frac{3}{2T}\int_0^T \tau\cos\omega\tau\, d\tau - \int_T^{2T} \cos\omega\tau\, d\tau + \frac{1}{2T}\int_T^{2T} \cos\omega\tau\, d\tau\right]$$

$$\Rightarrow 2\left[\frac{\sin\omega\tau}{\omega}\Big|_0^T - \frac{3}{2T}\left(\frac{T\sin\omega T}{\omega} + \frac{\cos\omega T}{\omega^2} - 0 - \frac{\cos(\omega)T}{\omega^2}\right) - \left[\frac{\sin 2\omega T}{\omega} - \frac{\sin\omega T}{\omega}\right] + \frac{1}{2T}\left[\frac{2T\sin 2\omega T}{\omega} + \frac{\cos\omega 2T}{\omega} - \frac{T\sin\omega T}{\omega} - \frac{\cos\omega T}{\omega^2}\right]\right]$$

$$\Rightarrow 2\left[\frac{\sin\omega T}{\omega} - \frac{3T\sin\omega T}{2\omega T} - \frac{3\cos\omega T}{2T\omega^2} + \frac{3}{2T\omega^2} - \frac{\sin\omega(2T)}{\omega} + \frac{\sin\omega T}{\omega} + \frac{\sin 2\omega T}{\omega} + \frac{\cos\omega(2T)}{2T\omega^2} - \frac{\sin\omega T}{2\omega} - \frac{\cos\omega T}{2T\omega^2}\right]$$

$$\Rightarrow \frac{2}{\omega}\left[\frac{\sin\omega T}{\cancel{\omega}} - \frac{3}{2}\sin\omega T - \frac{3\cos\omega T}{2T\omega} + \frac{3}{2T\omega} - \sin\omega(2T) + \sin\omega T + \sin 2\omega T + \frac{\cos\omega(2T)}{2T\omega} - \frac{\sin\omega T}{2} - \frac{\cos\omega T}{2T\omega}\right]$$

$$= \frac{2}{\omega}\left[\frac{3}{2T\omega} - \frac{2\cos\omega T}{T\omega} + \frac{\cos\omega(2T)}{2\omega T}\right]$$

$$= \frac{2}{\omega^2}\left[\frac{3}{2T} - \frac{2\cos\omega T}{T} + \frac{\cos\omega(2T)}{2T}\right]$$

$$\Rightarrow \frac{2}{\omega^2 T}\left[\frac{3}{2} - 2\cos\omega T + \frac{\cos 2\omega T}{2}\right] \quad \text{——— (1)}$$

NB! $\cos 2x = \cos^2 x - \sin^2 x$

$\& \sin^2 x = 1 - \cos^2 x$

$\therefore \cos 2x = 2\cos^2 x - 1)$

$$\Rightarrow \frac{2}{\omega^2 T}\left[\frac{3}{2} - 2\cos\omega T + \cos^2\omega T - \frac{1}{2}\right]$$

$$= \frac{2}{\omega^2 T}\left[1 - 2\cos\omega T\right]^2$$

| $\omega$ | $S_{xx}$ |
|---|---|
| $0$ | $0$ |
| $\pi/T$ | $8T/\pi^2$ |
| $2\pi/T$ | $0$ |
| $3\pi/T$ | $8T/9\pi^2$ |
| $4\pi/T$ | $0$ |
| $\frac{5\pi}{T}$ | $8T/25\pi^2$ |
| $\frac{6\pi}{T}$ | $0$ |