ECE 269 Homework 2

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Problem 1

a)

The matrix T will be of the form:

$$\begin{pmatrix} h(0) & 0 & 0 & 0 & 0 \\ h(1) & h(0) & 0 & 0 & 0 \\ h(2) & h(1) & h(0) & 0 & 0 \\ h(3) & h(2) & h(1) & h(0) & 0 \\ ... & h(3) & h(2) & h(1) & h(0) \\ h(N) & h(N-1) & ... & ... & ... \\ 0 & h(N) & h(N-1) & ... & ... \end{pmatrix}$$

b)

The elements for each diagonal of matrix T all have the same value. This structure can be represented by the formula that $T_{i,j} = T_{i+1,j+1}$.

Problem 2

a)

$f(\alpha x + \beta y) = A(\alpha x + \beta y) + b$

$= A\alpha x + A\beta y + b$

$= A\alpha x + A\beta y + (\alpha + \beta)b$

$= A\alpha x + A\beta y + \alpha b + \beta b = \alpha(Ax + b) + \beta(Ay + b)$, which is equivalent to: $\alpha f(x) + \beta f(y)$

b)

First, I need to prove that g(x) is linear.

$g(\alpha x) = f(\alpha x) - f(0)$

$= f(\alpha x + \beta 0) - f(0)$ using the fact that f(x) is affine one obtains:

$= \alpha f(x) + \beta f(0) - f(0)$

$= \alpha f(x) + (1-\alpha)f(0) - f(0)$
$= \alpha f(x) + f(0) - \alpha f(0) - f(0)$
$= \alpha f(x) - \alpha f(0)$
$= \alpha(f(x) - f(0)) = \alpha g(x)$

Therefore, $g(\alpha x) = \alpha g(x)$
$g(x+y) = f(x+y) - f(0) = f(\frac{1}{2} \cdot 2x + \frac{1}{2} \cdot 2y) - \frac{1}{2} \cdot 2 \cdot f(0)$
$= \frac{1}{2} \cdot f(2x) + \frac{1}{2} \cdot f(2y) - \frac{1}{2} \cdot 2 \cdot f(0)$
$= \frac{1}{2} \cdot (f(2x) - f(0)) + \frac{1}{2} \cdot (f(2y) - f(0))$
$= \frac{1}{2} \cdot g(2x) + \frac{1}{2} \cdot g(2y)$
Then, using the previously proven fact that $g(\alpha x) = \alpha g(x)$:
$= \frac{1}{2} \cdot g(2x) + \frac{1}{2} \cdot g(2y) = g(x) + g(y)$
Therefore, $g(x+y) = g(x) + g(y)$ and ultimately g(x) is **linear**
This means that g(x) can be represented in the form g(x) = Ax.
Therefore, f(x) = Ax + f(0).
Representing f(0) as b, one obtains **f(x) = Ax + b**

Problem 3
a)
Counter example:
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} * \begin{pmatrix} 0 & 0 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
b)
Counter example:
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} * \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$
c)
If $A^T A = 0$ then it must be true that $trace(A^T A) = 0$ which in sigma notation is $\sum_{i=1}^{n} [A^T A]_{ii} = 0$. Since $A_{ji}^T = A_{ij}$ the sigma notation can be converted to $\sum_{i,j=1}^{n} A_{ij} * A_{ij} = 0$.
This summation is the sum of squares of each index value, and for each sum of squares to be equal to 0 the individual elements MUST be equal to 0. Therefore, A is equal to 0.

Problem 4
a)
Let $x = a_0 + a_1 x + a_2 x^2 + ... a_n x^n$
Let $y = b_0 + b_1 y + b_2 y^2 + ... b_n y^n$

Then, T(x) = $a_1 + 2a_2x + ...na_nx^{n-1}$ and T(y) = $b_1 + 2b_2y + ...nb_ny^{n-1}$
T(x+y) = $T(a_0 + a_1x + a_2x^2 + ...a_nx^n + b_0 + b_1y + b_2y^2 + ...b_ny^n) =$
$a_1 + 2a_2x + ...na_nx^{n-1} + b_1 + 2b_2y + ...nb_ny^{n-1}$ which is equivalent to T(x) + T(y).
T(αx) = T($\alpha(a_0 + a_1x + a_2x^2 + ...a_nx^n)$)
This equals $\alpha a_1 + \alpha 2a_2x + ...\alpha na_nx^{n-1}$
αT(x) = $\alpha T(a_0 + a_1x + a_2x^2 + ...a_nx^n) = \alpha(a_1 + 2a_2x + ...na_nx^{n-1}) =$
$\alpha a_1 + \alpha 2a_2x + ...\alpha na_nx^{n-1}$
Therefore, T(x+y) = T(x) + T(y) and T(αx) = α T(x) which proves T is linear.

b)

$\frac{d(1)}{dx} = 0$ so the basis vector is an n*1 vector of $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ ... \\ 0 \end{pmatrix}$

$\frac{d(x)}{dx} = 1$ so the basis vector is an n*1 vector of $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ ... \\ 0 \end{pmatrix}$

$\frac{d(x^2)}{dx} = 2x$ so the basis vector is an n*1 vector of $\begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \\ ... \\ 0 \end{pmatrix}$

$\frac{d(x^3)}{dx} = 3x^2$ so the basis vector is an n*1 vector of $\begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \\ ... \\ 0 \end{pmatrix}$

$\frac{d(x^4)}{dx} = 4x^3$ so the basis vector is an n*1 vector of $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 4 \\ ... \\ 0 \end{pmatrix}$

This trend continues all the way to $\frac{d(x^n)}{dx} = nx^{n-1}$ with the n*1 basis vector of $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ ... \\ n \\ 0 \end{pmatrix}$

Stacking these columns together to form the transformation matrix one obtains: $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & n \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

This results in an (n+1)*(n+1) matrix with a rank of n. The first column is the only zero column. The rest of the columns continue with the pattern of $A_{i+1,j+1} = A_{i,j} + 1$ starting with $A_{1,2}$ up to $A_{n,n+1}$

Problem 5

a)

Let $x \in N(A)$ and $x \in R(A^T)$. This means $x \in N(A) \cap R(A^T)$ and that $Ax = 0$ and $x = Ay$ for some y.

Then, $x^T x = (A^T y)^T x$ which be can rearranged to the form ${}^T x = y^T Ax$ and since x is in the null space of A: $x^T x = y^T Ax = 0$ which means $\Sigma(x_i)^2 = 0$ and therefore, x = $\{0\}$.

Using the general equation 4.5.1 from the text that states: $rank(AB) = rank(B) - dimN(A) \cap R(B)$ I can input A and A^T s.t. $rank(AA^T) = rank(A^T) - dimN(A) \cap R(A^T)$

Then , since $dimN(A) \cap R(A^T) = 0$ it can be concluded that $rank(AA^T) = rank(A^T)$

Next, to show that $rank(AA^T) = rank(A)$ one must show that $rank(A^T) = rank(A)$ which is shown by the following proof:
To show that $dim(R(A^T)) \leq dim(R(A))$ I construct a basis for $R(A^T)$ and let $dim(R(A^T)) = k$. The basis for $R(A^T)$ is displayed as $\{x_1, x_2, ...x_k\}$.
If one were to multiply the basis by A, one obtains $\{Ax_1, Ax_2, ...Ax_k\}$ which is in $R(A)$. Next, suppose that $\Sigma c_i Ax_i = 0$. This can be rearranged to the form $A\Sigma c_i x_i = 0$, which represents $\Sigma c_i x_i$ to be in the null space of A.
Let $v = \Sigma c_i x_i$. Since v belongs to the null space of A and the range space of A^T one can conclude that v could be the 0 vector. Now, one must show that the only vector that v could be equal to is the 0 vector.
Suppose some vector b is in the null space of A, then $Ab = 0$. This must also be true for v since it is in the null space of A, so $Av = 0$. Representing A as $\begin{pmatrix} (a_1)^T \\ (a_2)^T \\ ... \\ (a_m)^T \end{pmatrix}$ where $(a_i)^T$ is a row vector - it can be represented that
$(a_1)^T v + (a_2)^T v + ...(a_m)^T v = 0$
Using the fact that $v \in R(A^T)$ one can conclude that $v^T v = 0$ by the following:
Let $v = \Sigma c_i a_i$ and $v \cdot \Sigma(a_i)^T = 0$ for $R(A^T)$ and $N(A)$ respectively. Then, $v^T v = (\Sigma c_i a_i)^T \cdot \Sigma c_i a_i$ which can be rearranged to the form $\Sigma(c_i)^T \cdot \Sigma(a_i)^T \cdot v$. The last two terms, as shown before, represent the null space of A. Therefore, $v^T v = 0$ which is equivalent to $\Sigma(v_i)^2 = 0$. Therefore, v = 0.
Therefore, since $v = \Sigma c_i x_i = 0$, and x is a basis it must follow that all values of c equal 0 and Ax forms a basis for R(A). This means k is less than or equal to dim(R(A)) = rank(A). However, to prove that k = rank(A) let b = a^T and substitute.
This results in $dim(R(b^T)) \leq dim((R(b))$
$dim(R(A)) \leq dim(A^T)$ Therefore, Rank(A) $\leq$ k and since this finally results in k $\leq$ Rank(A) $\leq$ k it must be that Rank(A) = k = Rank(A^T).
FINALLY, since Rank(A) = Rank(A^T) and Rank(A^T) = Rank(AA^T) it can be concluded that **Rank(A) = Rank(AA^T)** .

b)
COUNTER EXAMPLE:
Given A = $\begin{pmatrix} i & 1 \\ 0 & 0 \end{pmatrix}$ and $A^T = \begin{pmatrix} i & 0 \\ 1 & 0 \end{pmatrix}$. It follows that that $AA^T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Therefore, rank(A) = 1 and rank(AA^T) = 0

c)
Same as part a except substitute A^H for A^T. Also, it can be shown that since $A^H = A$ then the equation $Rank(AA^H) = rank(A^H - N(A) \cap R(A^H)$ can be rewritten as $Rank(AA) = rank(A) - N(A) \cap R(A)$ and since the only vector in the range and null space of a matrix is the 0 vector - Rank(AA) = Rank(A) = Rank(AA^H)
Problem 6
a)
Since matrix A is full rank and tall it's rank must be equal to n. Using the equation that $rank(A) = n - dim(N(A)))$ it must then be true that $dim(N(A)) = 0$
Let $x \in N(A^TA)$ and assume $x \neq 0$
If we have $A^TA \cdot x = 0$ it must be true that A^TA is singular as multiplying by the inverse would result in $x = 0$.
Then, $x^TA^TAx = 0$ which can be simplified to $(Ax)^TAx = 0$ which is simplified more to $||Ax||^2 = 0$ which concludes that $Ax = 0$.
Therefore, x is in the null space of A. However, since A is full rank the null space must be equal to 0. So reverting back to the beginning, if x is equal to 0 in the equation $A^TA \cdot x = 0$ it must be true that A^TA is non-singular.
b)
If $A \in R^{m \times n}$ then $A^T \cdot A$ is a square matrix of $n \times n$. Since A has rank n, $A^T \cdot A$ also has rank n (proven in problem 5). Therefore, $A^T \cdot A$ is a non-singular matrix.
Since A^TA is non-singular then $(A^TA)^{-1} \cdot (A^TA) = I$ where I is the identity matrix. This can be simplified to $(A^TA)^{-1}A^T(A) = I$
Therefore, $(A^TA)^{-1}A^T$ is a left inverse of A.
c)
It does not have a unique left inverse.

COUNTER EXAMPLE: Let A = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Both matrices $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ when multiplied on the left side of A result in the identity matrix of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

d)

Since A is full rank and fat it follows that rank(A) = m. Therefore, using the equation that $rank(A) = m - dim(N(A^T))$ it follows that $dim(N(A^T)) = 0$

Let $x \in N(AA^T)$ and assume $x \neq 0$

If we have $AA^T \cdot x = 0$ it must be true that AA^T is singular as multiplying by the inverse would result in $x = 0$.

Then, $x^T AA^T \cdot x = 0$

This is equivalent to $(A^T x)^T \cdot A^T x = 0$ which is equivalent to $||A^T x||^2 = 0$ and therefore, $A^T x = 0$

Therefore, $x \in N(A^T)$ and x must be equal to 0. That means AA^T is non-singular.

e)

If $A \in R^{m \times n}$ then $A \cdot A^T$ is a square matrix of $m \times m$. Since A has rank m, $A \cdot A^T$ also has rank m (proven in problem 5). Therefore, $A \cdot A^T$ is a non-singular matrix.

Since AA^T is non-singular then $(AA^T) \cdot (AA^T)^{-1} = I$ where I is the identity matrix. This can be simplified to $(A)A^T(AA^T)^{-1} = I$

Therefore, $A^T(AA^T)^{-1}$ is a right inverse of A.

f)

It does not have a unique right inverse.

COUNTER EXAMPLE: Let A = Let A = $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ Both matrices $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ when multiplied on the right side of A result in the identity matrix of $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$