ECE 269: Linear Algebra and Applications
Fall 2019

Homework # 3
Due: Tuesday, November 5 11:59pm,
via Gradescope

Collaboration Policy: This homework set allows limited collaboration. You are expected to try to solve the problems on your own. You may discuss a problem with other students to clarify any doubts, but you must fully understand the solution that you turn in and write it up entirely on your own. Blindly copying results from any resource (such as your friend, or the internet) will be considered a violation of academic integrity. *

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1. **Suggested Readings**. Review Summary Slides 2, video lectures and Discussion Problems.

2. **Problem 1: Orthogonal Complement of a Subspace.** Suppose that $\mathcal{V}$ is a subspace of $\mathbb{R}^n$. Let

$$\mathcal{V}^{\perp} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^T\mathbf{y} = 0, \forall \mathbf{y} \in \mathcal{V}\}$$

be the set of vectors orthogonal to every element in $\mathcal{V}$.

   (a) Verify that $\mathcal{V}^{\perp}$ is a subspace of $\mathbb{R}^n$.

   (b) Suppose that $\mathcal{V} = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ for some $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \mathbb{R}^n$. Express $\mathcal{V}$ and $\mathcal{V}^{\perp}$ as subspaces induced by the matrix $\mathbf{A} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \end{bmatrix} \in \mathbb{R}^{n \times k}$ and its transpose $\mathbf{A}^T$.

   (c) Show that $(\mathcal{V}^{\perp})^{\perp} = \mathcal{V}$.

   (d) Show that $\dim(\mathcal{V}) + \dim(\mathcal{V}^{\perp}) = n$.

   (e) Show that $\mathcal{V} \subseteq \mathcal{W}$ for another subspace $\mathcal{W}$ implies $\mathcal{W}^{\perp} \subseteq \mathcal{V}^{\perp}$.

   (f) Show that every $\mathbf{x} \in \mathbb{R}^n$ can be expressed uniquely as $\mathbf{x} = \mathbf{v} + \mathbf{v}^{\perp}$, where $\mathbf{v} \in \mathcal{V}$ and $\mathbf{v}^{\perp} \in \mathcal{V}^{\perp}$. (Hint: Let $\mathbf{v}$ be the projection of $\mathbf{x}$ on $\mathcal{V}$.)

3. **Problem 2: Rank of a Product.** Let $\mathbf{A} \in \mathbb{R}^{4 \times 3}$ has rank 2 and $\mathbf{B} \in \mathbb{R}^{3 \times 3}$ has rank 3.

   (a) Find the smallest possible value $r_{\min}$ of $\text{rank}(\mathbf{AB})$. Find specific $\mathbf{A}$ and $\mathbf{B}$ such that $\text{rank}(\mathbf{AB}) = r_{\min}$.

   (b) Find the largest possible value $r_{\max}$ of $\text{rank}(\mathbf{AB})$. Find specific $\mathbf{A}$ and $\mathbf{B}$ such that $\text{rank}(\mathbf{AB}) = r_{\max}$.

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4. **Problem 3: An Inequality for Orthonormal Matrices:** Suppose that the columns of $\mathbf{U} \in \mathbb{R}^{n \times k}$ are orthonormal (i.e. they are orthogonal and have unit l_2 norm). Show that

$$\|\mathbf{U^T x}\|_2 \leq \|\mathbf{x}\|_2.$$

5. **Problem 4: Householder Reflections.** A *Householder matrix* is defined as

$$\mathbf{Q} = \mathbf{I} - 2\mathbf{u}\mathbf{u}^T$$

   for a unit vector $\mathbf{u} \in \mathbb{R}^n$.

   (a) Show that $\mathbf{Q}$ is orthogonal.

   (b) Show that $\mathbf{Qu} = -\mathbf{u}$ and that $\mathbf{Qv} = \mathbf{v}$ for every $\mathbf{v} \perp \mathbf{u}$. Thus, the linear transformation $\mathbf{y} = \mathbf{Qx}$ reflects $\mathbf{x}$ through the hyperplane with normal vector $\mathbf{u}$.

   (c) Given $\mathbf{y}$, find $\mathbf{x}$ such that $\mathbf{y} = \mathbf{Qx}$.

   (d) Show that $\det(\mathbf{Q}) = -1$.
   Hint: Use the following properties of a determinant of a matrix: if $\mathbf{AB}$ is square (but $\mathbf{A}$ and $\mathbf{B}$ do not have to be), then

$$\det(\mathbf{I} + \mathbf{AB}) = \det(\mathbf{I} + \mathbf{BA}).$$

   (e) Given nonzero vectors $\mathbf{x}$ and $\mathbf{y}$, find a unit vector $\mathbf{u}$ such that

$$\mathbf{Qx} = (\mathbf{I} - 2\mathbf{u}\mathbf{u}^T)\mathbf{x} \in \text{span}(\mathbf{y}),$$

   in terms of $\mathbf{x}$ and $\mathbf{y}$.

6. **Problem 5: Projection Matrices.** A symmetric matrix $\mathbf{P} = \mathbf{P}^T \in \mathbb{R}^{n \times n}$ is said to be a *projection matrix* if $\mathbf{P} = \mathbf{P}^2$.

   (a) Show that if $\mathbf{P}$ is a projection matrix, then so is $\mathbf{I} - \mathbf{P}$.

   (b) Suppose that the columns of $\mathbf{U} \in \mathbb{R}^{n \times k}$ are orthonormal. Show that $\mathbf{U}\mathbf{U}^T$ is a projection matrix.

   (c) Suppose that $\mathbf{A} \in \mathbb{R}^{n \times k}$ is full-rank with $k \leq n$. Show that $\mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$ is a projection matrix.

   (d) Recall from lectures that the point $\mathbf{y} \in \mathcal{S} \subseteq \mathbb{R}^n$ closest to $\mathbf{x} \in \mathbb{R}^n$ is said to be the *orthogonal projection* (or *projection* in short) of $\mathbf{x}$ onto $\mathcal{S}$. Show that if $\mathbf{P}$ is a projection matrix, then $\mathbf{y} = \mathbf{Px}$ is the projection of $\mathbf{x}$ onto $\mathcal{R}(\mathbf{P})$.

   (e) Let $\mathbf{u}$ be a unit vector. Find the projection matrix $\mathbf{P}$ such that $\mathbf{y} = \mathbf{Px}$ is the projection of $\mathbf{x}$ onto $\text{span}(\mathbf{u})$.