ECE 269: Linear Algebra and Applications
Fall 2019

Homework # 1
Due: Tuesday, October 15, 11:59pm, via Gradescope

Collaboration Policy: This homework set allows limited collaboration. You are expected to try to solve the problems on your own. You may discuss a problem with other students to clarify any doubts, but you must fully understand the solution that you turn in and write it up entirely on your own. Blindly copying another student's result will be considered a violation of academic integrity. *

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1. **Suggested Reading**. Sections 4.1 – 4.4 of Carl D. Meyer's book "Matrix Analysis and Applied Linear Algebra".

2. **Problem 1: Vectors Spaces other than $\mathbb{R}^N$.**
   Show that the following sets with operations defined over the given fields are **not** valid vector fields.

   (a) The set of rational numbers defined over $\mathbb{R}$.

   (b) The set of polynomials $\{a_0 + a_1x + a_2x^2 \mid a_0, a_1, a_2 \in \mathbb{R}^+\}$ defined over $\mathbb{R}$, where $\mathbb{R}^+$ is the set of positive real numbers

   (c) The set of vectors in $\mathbb{R}^2$ defined over $\mathbb{R}$ with vector addition and scalar multiplication defined as follows. Here $a, b, c, d, r \in \mathbb{R}$

   i. $\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix}$ and $r. \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r.a \\ b \end{bmatrix}$

   ii. $\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a+c \\ b+d \end{bmatrix}$ and $r. \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r.a \\ 0 \end{bmatrix}$

   iii. $\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $r. \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r.a \\ r.b \end{bmatrix}$

   iv. $\begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a-c \\ b-d \end{bmatrix}$ and $r. \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r.a \\ r.b \end{bmatrix}$

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3. **Problem 2: Adjacency graph.**
Consider a simple graph (an undirected graph with no self loops or multiple edges) with n vertices. Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be the adjacency graph, defined by

$$\mathbf{A}_{ij} = \begin{cases} 1 & \text{if there is an edge between vertex } i \text{ and vertex } j, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\mathbf{A} = \mathbf{A}^T$ and $\mathbf{A}_{ii} = 0$, $i = 1, 2, \ldots, n$, since there are no self loops. Let $\mathbf{B} = \mathbf{A}^k$, where $k \in \mathbb{Z}$. Give a simple interpretation of $\mathbf{B}_{ij}$ in terms of the original graph and k, and justify your answer. (Hint: Use the concept of a *path* between two nodes.)

4. **Problem 3: Vector Spaces of Polynomials.**
Consider the set $\mathbb{P}_n(\mathbb{R})$ of all real valued polynomials of degree $\leq n$ with real coefficients:

$$\mathbb{P}_n(\mathbb{R}) = \{f(x) = \sum_{k=0}^{n} c_k x^k, c_0, c_1, \cdots, c_n \in \mathbb{R}\} \tag{1}$$

   (a) Show that $\mathbb{P}_n(\mathbb{R})$ is a vector space. What is the dimension of this vector space?
   (b) Is the union $\cup_{n=1}^{m} \mathbb{P}_n$ a vector space? Does this contradict or comply with something you learned in class?
   (c) Find a basis for $\mathbb{P}_4$ containing $\{x^2 + 1, x^2 - 1\}$
   (d) Find a basis for $\mathbb{P}_2$ from the set $\{1 + x, x + x^2, x + 2x^2, 2x + 3x^2, 1 + 2x + x^2\}$

5. **Problem 4: Symmetric and Hermitian matrices.**
A square matrix A is said to be *symmetric* if its transpose A^T satisfies $A^T = A$, and a complex-valued square matrix A is said to be *Hermitian* if its conjugate transpose $A^H = (\overline{A})^T = \overline{A^T}$ satisfies $A^H = A$. Thus, a real-valued square matrix A is symmetric if and only if it is Hermitian. Which of the following is a vector space?

   (a) The set of all $n \times n$ real-valued symmetric matrices over $\mathbb{R}$.
   (b) The set of all $n \times n$ complex-valued symmetric matrices over $\mathbb{C}$.
   (c) The set of all $n \times n$ complex-valued Hermitian matrices over $\mathbb{R}$.
   (d) The set of all $n \times n$ complex-valued Hermitian matrices over $\mathbb{C}$.

   For each case, either verify that it is a vector space or prove otherwise.

6. **Problem 5: Properties of Vector Spaces.**

   (a) Prove that the additive inverse of an element in a vector space is unique.
   (b) Prove that adding a vector $\mathbf{v}$ to a set of vectors S=$\{\mathbf{w}_1, \mathbf{w}_2, ..., \mathbf{w}_N\}$ will not add new vectors to span(S), if and only if $\mathbf{v} \in$ span(S).

7. **Problem 6: Linear Independence.**

   (a) Consider the stacked vectors

$$\mathbf{z}_1 = \begin{bmatrix} \mathbf{x}_1 \\ \\ \mathbf{y}_1 \end{bmatrix}, \cdots, \mathbf{z}_n = \begin{bmatrix} \mathbf{x}_n \\ \\ \mathbf{y}_n \end{bmatrix},$$

      i. Suppose $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly independent (no assumption is made on $\mathbf{y}_1, \ldots, \mathbf{y}_n$). Can we conclude that the vectors $\mathbf{z}_1, \ldots, \mathbf{z}_n$ are linearly independent? If yes, provide a proof. If no, give a counterexample.

      ii. Suppose $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are linearly dependent (no assumption is made on $\mathbf{y}_1, \ldots, \mathbf{y}_n$). Can we conclude that the vectors $\mathbf{z}_1, \ldots, \mathbf{z}_n$ are linearly dependent? If yes, prove the result. If no, give a counterexample.

8. **Problem 7: Finding Basis.**
   Find a basis for each of the following subspaces of $\mathbb{R}^n$.

   (a) Subspace S, which is the intersection of U and V, where $U = \text{span}\{(2, -1, 3, 0)^T, (1, 0, -1, 0)^T\}$ and $V = \text{span}\{(0, 1, 0, 0)^T, (0, 0, 1, 0)^T, (0, 0, 0, 1)^T\}$.

   (b) All vectors whose components are equal.

   (c) All vectors whose components sum to zero.

   (d) All vectors spanned by $\begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T, \begin{bmatrix} 0 & 1 & 1 & 0 \end{bmatrix}^T, \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}^T$, and $\begin{bmatrix} 1 & 0 & 0 & 1 \end{bmatrix}^T$