ECE 269: Linear Algebra and Applications
Fall 2019

Homework # 4
Due: Thursday, November 21 11:59pm,
via Gradescope

Collaboration Policy: This homework set allows limited collaboration. You are expected to try to solve the problems on your own. You may discuss a problem with other students to clarify any doubts, but you must fully understand the solution that you turn in and write it up entirely on your own. Blindly copying results from any resource (such as your friend, or the internet) will be considered a violation of academic integrity. *

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1. **Suggested Readings**. Review Summary Slides 2, 3, video lectures and Discussion Problems.

2. **Problem 1: Moore–Penrose pseudoinverse.** A pseudoinverse of $\mathbf{A} \in \mathbb{R}^{m \times n}$ is defined as a matrix $\mathbf{A}^+ \in \mathbb{R}^{n \times m}$ that satisfies

$$\mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{A}, \quad \mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+$$

   and $\mathbf{A}\mathbf{A}^+$ and $\mathbf{A}^+\mathbf{A}$ are symmetric.

   (a) Show that $\mathbf{A}^+$ is unique.

   (b) Show that $(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$ is the pseudoinverse and a left inverse of a full-rank tall matrix $\mathbf{A}$

   (c) Show that $\mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}$ is the pseudoinverse and a right inverse of a full-rank fat matrix $\mathbf{A}$.

   (d) Show that $\mathbf{A}^{-1}$ is the pseudoinverse of a full-rank square matrix $\mathbf{A}$.

   (e) Show that $\mathbf{A}$ is the pseudoinverse of itself for a projection matrix $\mathbf{A}$.

   (f) Show that $(\mathbf{A}^T)^+ = (\mathbf{A}^+)^T$.

   (g) Show that $(\mathbf{A}\mathbf{A}^T)^+ = (\mathbf{A}^+)^T\mathbf{A}^+$ and $(\mathbf{A}^T\mathbf{A})^+ = \mathbf{A}^+(\mathbf{A}^+)^T$.

   (h) Show that $\mathcal{R}(\mathbf{A}^+) = \mathcal{R}(\mathbf{A}^T)$ and $\mathcal{N}(\mathbf{A}^+) = \mathcal{N}(\mathbf{A}^T)$.

   (i) Show that $\mathbf{P} = \mathbf{A}\mathbf{A}^+$ and $\mathbf{Q} = \mathbf{A}^+\mathbf{A}$ are projection matrices.

   (j) Show that $\mathbf{y} = \mathbf{P}\mathbf{x}$ and $\mathbf{z} = \mathbf{Q}\mathbf{x}$ are the projections of $\mathbf{x}$ onto $\mathcal{R}(\mathbf{A})$ and $\mathcal{R}(\mathbf{A}^T)$, respectively, where $\mathbf{P}$ and $\mathbf{Q}$ are defined as previously.

   (k) Show that $\mathbf{x}^* = \mathbf{A}^+\mathbf{b}$ is a least-squares solution to the linear equation $\mathbf{A}\mathbf{x} = \mathbf{b}$, i.e., $\|\mathbf{A}\mathbf{x}^* - \mathbf{b}\| \leq \|\mathbf{A}\mathbf{x} - \mathbf{b}\|$ for every other $\mathbf{x}$.

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(l) Show that $\mathbf{x}^* = \mathbf{A}^+\mathbf{b}$ is the least-norm solution to the linear equation $\mathbf{A}\mathbf{x} = \mathbf{b}$, i.e., $\|\mathbf{x}^*\| \leq \|\mathbf{x}\|$ for every other solution $\mathbf{x}$, (assuming that a solution exists).

3. **Problem 2: Eigenvalues.** Suppose that $\mathbf{A}$ has $\lambda_1, \lambda_2, \ldots, \lambda_n$ as its eigenvalues.

   (a) Show that $\det(\mathbf{A}) = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$.

   (b) Show that the eigenvalues of $\mathbf{A}^T$ are $\lambda_1, \lambda_2, \ldots, \lambda_n$, that is, $\mathbf{A}$ and $\mathbf{A}^T$ have the same set of eigenvalues.

   (c) Show that the eigenvalues of $\mathbf{A}^k$ are $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$ for $k = 1, 2, \ldots$.

   (d) Show that $\mathbf{A}$ is invertible if and only if it does not have a zero eigenvalue.

   (e) Suppose that $\mathbf{A}$ is invertible. Show that the eigenvalues of $\mathbf{A}^{-1}$ are $\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_n^{-1}$.

   (f) Show that $\mathbf{A}$ and $\mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ have the same set of eigenvalues, that is, eigenvalues are invariant under a similarity transformation $\mathbf{A} \mapsto \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$.

4. **Problem 3: Trace.** We define the *trace* of $\mathbf{A} \in \mathbb{R}^{n\times n}$ as

$$\mathrm{tr}(\mathbf{A}) = \mathbf{A}_{11} + \mathbf{A}_{22} + \cdots + \mathbf{A}_{nn},$$

   that is, the sum of its diagonal entries.

   (a) Suppose that $\mathbf{A}$ has $\lambda_1, \lambda_2, \ldots, \lambda_n$ as its eigenvalues. Show that

$$\mathrm{tr}(\mathbf{A}) = \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$

   (b) Show that

$$\mathrm{tr}(\mathbf{A}^k) = \lambda_1^k + \lambda_2^k + \cdots + \lambda_n^k$$

   for $k = 1, 2, \ldots$.

5. **Problem 4: More on Eigenvalues.** Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be the eigenvalues of $\mathbf{A} \in \mathbb{C}^{n\times n}$, repeated according to their algebraic multiplicities. Show that

$$\sum_{i=1}^{n} |\lambda_i|^2 \leq \sum_{i=1}^{n}\sum_{j=1}^{n} |\mathbf{A}_{i,j}|^2$$

6. **Problem 5: Limit.** Let

$$\mathbf{A} = \begin{pmatrix} 5 & -8/5 \\ 12 & -19/5 \end{pmatrix}$$

   compute $\lim_{n\to\infty} \mathbf{A}^n$.