ECE 269 Homework 3

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Problem 1

a)

Given $x, y \in V_{perp}$ and $z \in V$

$(x+y)z = xz + yz = 0 + 0 = 0$

Given $u \in V_{perp}, v \in V, \alpha \in F$

$(\alpha \cdot u)v = \alpha(u \cdot v) = \alpha \cdot 0 = 0$

Therefore, closed under addition and scalar multiplication, so V_{perp} is a subspace of R^n

b)

Given $\begin{pmatrix} v_1 & v_2 & v_3 & ... & v_k \end{pmatrix} \in R^{n \times k}, V = Range(A)$

$V^T = \begin{pmatrix} c_1 & c_2 & c_3 & ... & c_k \end{pmatrix}$ s.t. $v_1 c_1 + v_2 c_2 + v_3 c_3 + ... + v_k c_k = 0$ which is represented in matrix form as:

$$\begin{pmatrix} v_1^T \\ v_2^T \\ v_3^T \\ ... \\ v_k^T \end{pmatrix} \cdot \begin{pmatrix} c_1 & c_2 & c_3 & ... & c_k \end{pmatrix}$$

One can see that left side matrix is equivalent to A^T and the right side matrix is equivalent to $N(A^T)$

Therefore, $V = A^T$ and $V_{perp} = N(A^T)$

c)

Using $x \in (V_{perp})_{perp}, v \in V, w \in V_{perp}, andx = v + w$

$x = v + wcomesfromtherulethattheadditionof2subspacesinR^nisinR^n$

Then, $x \cdot w = 0 = (v + w) \cdot w = vw + ww = 0$

This results in $||w||^2 = 0$ as $vw = 0, sincev \in Vandw \in V_{perp}$

Then, $||w||^2 = \sum_{i=1}^{N} (w_i)^2 = 0$ means that w must equal 0.

This means that $x = v + w = v + 0$ and therefore, $x = v$.

Since $v \in V$ this means every points in $(V_{perp})_{perp} \in V$ so $(V_{perp})_{perp} \subseteq V$
Next, using $x \in (V_{perp})_{perp}, v \in V, w \in V_{perp}, and v = w + x$
$v \cdot w = (w + x) \cdot w = w \cdot w + x \cdot w = w \cdot w + 0 = 0.$
This again means that $||w||^2 = 0$ and $||w||^2 = \sum_{i=1}^{N} (w_i)^2 = 0$. So, w = 0 and then $v = 0 + x$ and therefore, $v = x$ shows that $V \subseteq (V_{perp})_{perp}$
Since $(V_{perp})_{perp} \subseteq V$ and $V \subseteq (V_{perp})_{perp}$
this means that $V = (V_{perp})_{perp}$.

d)

From part b I showed that representing a subspace V as a matrix A one obtains that $V = Range(A)$ and $V_{perp} = N(A^T)$.
Then, following the rule that $rank(A) = n - dim(N(A^T))$ and substituting in V and V_{perp} one obtains $dim(V) = n - dim(V_{perp})$ which can be rearranged to $dim(V) + dim(V_{perp}) = n$

e)

First I will prove uniqueness.
Using $v \in V, w \in W, x_1 and x_2 \in V_{perp}, and y \in W_{perp}$
if $V \subseteq W$ and $w \cdot y = 0$ and using that x is any point in V_{perp} s.t. $v \cdot x = 0$
if $v \cdot x_1 = 0$ and $v \cdot x_2 = 0$ then it follows that $v(x_1 + x_2) = 0$. This shows that $(x_1 + x_2) \in V_{perp}$ which means both x_1 and x_2 must be $\in V_{perp}$ which we know to be true.
Then, since every value of V is in W it must be true that $v \cdot x + v \cdot y = 0$ which results in $v(x + y) = 0$. Therefore, $(x + y) \in V_{perp}$ and using the previous rule it must be that both x and y $\in V_{perp}$. Therefore, every point of $W_{perp} \in V_{perp}$ and ultimately $W_{perp} \subseteq V_{perp}$

f)

Let $A_1, A_2 \in V$ and $B_1, B_2 \in V_{perp}$
Then, $X = A_1 + B_1 = A_2 + B_2$ which is rearranged to $A_1 - A_2 = B_1 - B_2$.
Since we know $A_1 - A_2 \in V$, $B_1 - B_2 \in V_{perp}$, and that $V \cap V_{perp} = 0$ it must be true that $A_1 - A_2 = B_1 - B_2 = 0$ which then follows that $A_1 = A_2$ and $B_1 = B_2$.
Next is to prove existence.
If $V \in R^n$, i.e. V is a finite dimensional inner-product space:
Let B_M be an orthonormal basis of M and let B_{M-perp} be an orthonormal basis of M_{perp}
Since $M \cap M_{perp} = 0$, $B_M \cup B_{M-perp}$ forms an orthonormal basis for some new subspace I'll call S. s.t. $S \subseteq V$.
Now, if $S \neq V$ then there must be some other orthonormal basis E s.t. $B_M \cup B_{M-perp} \cup E$ forms an orthonormal basis for V. Since these are or-

thonormal bases it must follow that E is perpendicular to M, which means that $E \subseteq B_{M-perp}$, but this is not possible since these orthonormal bases are independent by definition. Therefore, E = 0 and $V = B_M \cup B_{M-perp}$

Problem 2

Using the theorems that $rank(AB \leq minrank(A), rank(B) andthatrank(A) + rank(B) - n \leq rank(AB)$

It follows that the minimum and maximum rank of AB is 2

a)

Let A = $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and B = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Then AB = $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ which has rank 2

b)

Let A = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and B = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Then AB = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ which has rank 2

Problem 3

$||U^Tx||^2 = (U^Tx)^TU^Tx = x^TUU^Tx = x^TIx = x^Tx = ||x||^2$. So for the case of square matrices, $||U^Tx|| = ||x||$.

However, if k ¡ n for $U^{n \times k}$ there is a matrix Q that has n-k columns s.t. $[U|Q] \in R^{n \times n}$.

It follows that $||[U|Q]^Tx||^2 = ||\frac{U^T}{Q^T}x||^2 = ||\frac{U^Tx}{Q^Tx}||^2$.

Using $||\frac{a}{b}||^2 = ||a||^2 + ||b||^2$ one obtains $||U^Tx||^2 + ||Q^Tx||^2 = ||x||^2$

Since $||Q^Tx||^2 \geq 0$ it follows that $||U^Tx||^2 \leq ||x||^2$ and ultimately:

$||U^Tx|| \leq ||x||$

Problem 4

a)

$(I - 2uu^T)(I - 2uu^T)^T = (I - 2uu^T)(I - 2uu^T)$

$= I - 2uu^T - 2uu^T + 4(uu^T)(uu^T)$
$= I - 4uu^T = 4uu^T uu^T = I - 4uu^T + 4uu^T = I - 0 = I$
Therefore, Q is orthogonal.
b)
$(I - 2uu^T)u = u - 2uu^T u = u - 2u = -u$
$(I - 2uu^T)v = v - 2uu^T v$
Using $u^T v = 0$
$(I - 2uu^T)v = v - 0 = v$
c)
Since $QQ^T = QQ = I$
$Qy = x$
d)
$det(Q) = det(I - 2uu^T) = det(I - 2u^T u) = 1 - 2u^T u = 1 - 2 = -1$
e) If x is perpendicular to u, then x is just a column or set of columns of y.
$I - 2uu^T = I$
$2uu^T = 0$
$2(u^T u)^T = 0$
Therefore, u = 0.
Problem 5
a)
i) $(I - P)^T = I^T - P^T = I - P^T = I - P$
ii) $(I - P)^2 = (I - P)(I - P) = I \cdot I - I \cdot P - P \cdot I + P \cdot P$
$= I - P - P + P = I - P$
b)
i) $(UU^T)^T = (U^T)^T \cdot U^T = UU^T$
ii) $(UU^T)^2 = (UU^T)(UU^T) = UU^T UU^T = UIU^T = UU^T$
c)
i) $(A(A^T A)^{-1} A^T)^T = (A^T)^T ((A^T A)^{-1})^T A^T$
$= A(A^{-1}(A^T)^{-1})^T A^T = A((A^T)^{-1})^T (A^{-1})^T A^T = AA^{-1}(A^{-1})^T A^T = A(A^T A)^{-1} A^T$
ii)
$(A(A^T A)^{-1} A^T)^2 = (A(A^T A)^{-1} A^T)(A(A^T A)^{-1} A^T) = AA^{-1}(A^T)^{-1} A^T AA^{-1}(A^T)^{-1} A^T$
$= I \cdot I \cdot AA^{-1}(A^T)^{-1} A^T = A(A^T A)^{-1} A^T$
d)
R(P) is the span of the columns of P. Need to verify (x-Px) is perpendicular to the columns of P.
$(x - Px)^T P = (x^T - x^T P^T)P =$
$(x^T - x^T P)P = x^T P - x^T PP = x^T P - x^T P = 0$
e)

Must find the matrix P s.t. the columns of u are perpendicular to the error of x - Px.

$u^T(x - Px) = u^T x - u^T Px = 0$

$u^T x = u^T Px$

$u^T = u^T P$

$uu^T = uu^T P$

$uu^T = 1 * P$

$uu^T = P$