[note]

This document is based on the Spring 2022 B.S. thesis project of Joseph Jeesung Suh.

The part until [Section 3.2. Perturbational analysis] was presented in my B.S. thesis. The part from [Section 3.3. Fluctuation as a small-amplitude dark soliton and its parameter evolution] to [Section 3.5. Perturbation to all orders] are added afterwards.

Here I affirm that this work is an original work of Joseph Jeesung Suh, under the guidance of Professor Yoonchan Jeong, SNU Laser Laboratory, Seoul National University.

The manuscript is in preparation, with Juhwan Kim, Ph. D. candidate of SNU Laser Laboratory, Kyoungyoon Park, Ph. D., under the guidance of Professor Yoonchan Jeong.

Dark Soliton Formation in
Gain-embedded Highly Nonlinear Optical Fiber

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Abstract

In this dissertation, optical dark soliton dynamics in a gain-embedded highly-nonlinear optical fiber with flattened all-normal dispersion is studied. The semi-classical equation of field fluctuation's spatiotemporal evolution is developed in the framework of perturbation theory and inverse-scattering method. Governing equation for the leading order of amplitude fluctuation is obtained. The governing equation, which incorporated Raman response and optical gain as perturbation terms, describe the evolution of dark solitons.

Keywords: Nonlinear fiber optics, Optical soliton, Optical and Raman gain, Perturbation analysis

Contents

1 Introduction

Nonlinear optics originates from a light-matter interaction, interaction of photons with various particles(fields). As its name suggests, a medium responses in a nonlinear manner to electromagnetic field, and for high intensity such response becomes manifest that linear polarization approximation breaks down. In the light of renormalization group theory, high order terms are irrelevant but cannot be ignored for sufficiently high momentum cutoff. Photons' interaction with constituent atoms gives rise to the second and higher order susceptibilities. However, the second order response vanishes due to amorphous structure of silica fiber, and the leading contribution comes from the third order(Kerr nonlinearity).[1] In addition to this photon-electron coupling, photon-phonon couplings are also prominent. Various couplings, which are strictly quantum in principle, give rise to several nonlinear phenomena. These include, but are not limited to, n^{th} harmonic generation, optical parametric amplification, Brillouin scattering, Raman scattering, and four-wave mixing.[2]

When nonlinear effects and dispersions are taken into consideration, evolution of electromagnetic pulse envelope becomes nontrivial. Of particular interest, anomalous(normal) dispersion conditions support optical bright(resp. dark) soliton solutions. Both solitons were analytically predicted and experimentally observed in 1970s.[3] The bright case has a clear physical interpretation: frequency up- and down-chirping incurred by self-phase modulation and dispersion neatly cancel out, forming a soliton. The dark case is rather subtle. Complex nonlinear effects of background waves maintain the dip intact under propagation. It worths mentioning that dark solitons cannot be understood as $\psi \to \psi_0 - \psi$ i.e. $\mathbb{Z}_2$ operation followed by a constant field shift since Lagrangian is not invariant under such transformation. Also, the theoretical boundary condition for dark solitons, continuous wave at infinity, is in practice replaced by a pulse whose temporal scale is sufficiently larger than that of dark solitons.

K. Park, in his Ph.D. dissertation originally aimed at supercontinuum generation, discovered the numerical evidence of dark soliton generation in a gain-embedded highly-nonlinear photonic crystal fiber.[4] With sufficiently high optical gain, dark solitons emerge; Gaussian pulse with an initial sub-ps width serves as a background wave as it undergoes supercontinuum broadening. This phenomena strongly implies that the combination of optical gain and nonlinear response is more than a mere field amplification and temporal broadening · frequency shift. One naive analogy would be a sand pyramid: optical gain pours sand on the top of the pyramid, and nonlinear response with dispersion spread sand so that the balance of two generates overall shape of the pyramid, but 'somehow' there emerges a dip structure.

This dissertation aims to develop the correct theory. The criterion for dark soliton generation would be discovered as a corollary. There are numerous numerical simulations conducted by the author, in an attempt to verify a theory-simulation consistency and to obtain the soliton generation criterion in an empirical manner. However, those results will not be presented here, since the data and simulation methodologies itself require a number of pages to elaborate. Still, numerical values sometimes appear in the sections hereafter to give a feeling of physical scales. The values are based on following parameters that are taken from [4].

$$\beta_2 = 5.2494 \times 10^{-12}\ [\mathrm{ps}^2\mathrm{nm}^{-1}] \qquad \beta_3 = 4.3701 \times 10^{-16}\ [\mathrm{ps}^3\mathrm{nm}^{-1}]$$

$$\beta_4 = 2.0343 \times 10^{-16}\ [\mathrm{ps}^4\mathrm{nm}^{-1}] \qquad \gamma = 37\ [\mathrm{W}^{-1}\mathrm{km}^{-1}]$$

$$g_0 = 30\ [\mathrm{dB/m}] \qquad \lambda_0 = 1064\ [\mathrm{nm}] \qquad n_0 = 1.45\ [1]$$

$$T_p = 0.2\ [\mathrm{ps}] \qquad T_d = 4.2812 \times 10^4\ [\mathrm{ps}] \qquad T_1 = 8 \times 10^8\ [\mathrm{ps}] \qquad T_2 = 1.592 \times 10^{-2}\ [\mathrm{ps}]$$

$$P_{pump} = 10\ [\mathrm{mW}] \qquad \lambda_{pump} = 976\ [\mathrm{nm}] \qquad \sigma_s = 1.7669 \times 10^{-6}\ [\mathrm{nm}^{-2}]$$

Symbols denote: β_i i^{th} order GVD, γ a nonlinear coefficient, g_0 an unsaturated gain, λ_0 pulse central frequency, n_0 silica refractive index at the central frequency, T_p pulse intensity FWHM, T_d pulse-to-pulse interval, T_1 spontaneous decay time of gain ions, T_2 decoherence time of gain ions, P_{pump} optical pump power, λ_{pump} optical pump wavelength, σ_s absorption cross section of gain ions at the pump wavelength. The numerical integration of Nonlinear Schrodinger equation is based on explicit sixth and eighth order Runge-Kutta methods with constant mesh and propagation grid size, Butcher tableau taken from [5], [6].

2 Background

2.1 Physical setup

The optical medium considered in this dissertation is a highly-nonlinear(HNL) optical fiber with active Ytterbium dopants. The fiber is seeded with sub-ps duration Gaussian pulses with a repetition rate in MHz range. Therefore, there are four characteristic time scales, namely

T_p, T_d, T_1, T_2, representing pulse duration, pulse repetition interval, spontaneous decay time, and decoherence time. $T_2(\sim 100fs) < T_p(\sim 200fs) \ll T_d(\sim 1\mu s) \ll T_1(\sim 1ms)$ assumed. The exact time scales of Ytterbium dopants are taken from [1].

Three main mechanisms of pulse evolution are Kerr nonlinearity, (flattened all-normal) dispersion, and optical gain. The pulse propagation is basically a 3+1D problem, but can be reduced to an effective 1+1D problem by integrating out the transverse spatial structure given that the fiber supports single mode and the mode profile is almost invariant of frequency. The photonic crystal fiber provides this property.[7], [8] The 1+1D problem is described by a well-known Nonlinear Schrodinger equation(NLSE).

In the meantime, optical gain for an ultrashort pulse whose time scale is comparable to the characteristic time scale of optical process should be described by Maxwell-Bloch equation(MBE), which reduces to a well-known rate equation in the long time limit.[9] Also, Ytterbium ion's four-level system is can be reduced to an effective two-level system.

Therefore, this section provides a holistic procedure to properly describe the physics of gain-embedded HNL fiber under periodic pulse train. It is organized as follows: the first part introduces MBE for a two-level system. The second part manipulates MBE under periodic pulse train. In the third part, the result of MBE is incorporated into NLSE as an optical gain term. The fourth part briefly describes inverse scattering method, and the final section presents the motivation for applying perturbational analysis.

2.2 Maxwell-Bloch equation for a two-level system

The equation of motion for the density matrix is as follows.[9]

$$\dot{\rho} = -\frac{i}{\hbar}[H, \rho] \tag{1}$$

$$\dot{\rho}_{nm} = -\frac{i}{\hbar}\sum_{v}(H_{nv}\rho_{vm} - \rho_{nv}H_{vm}) \tag{2}$$

Let's consider a two-level system in particular. Denote a(b) be the upper(resp. lower) state. Hamiltonian can also be split into two terms, H_0 and H_I where $H_0\psi_i = \hbar\omega_i\psi_i$ denotes unperturbed Hamiltonian of the system and $H_I = -\hat{\mu} \cdot \hat{E}$ denotes interaction with the classical electromagnetic field. It is reasonable to assume that diagonal elements of H_I vanish by parity.

$$\begin{aligned}\dot{\rho}_{aa} &= -\frac{i}{\hbar}\sum_v ((H_{0,av}+H_{I,av})\rho_{va} - \rho_{av}(H_{0,va}+H_{I,va})) \\ &= -\frac{i}{\hbar}((H_{0,aa}\rho_{aa} + H_{I,ab}\rho_{ba} - \rho_{aa}H_{0,aa} - \rho_{ab}H_{I,ba}) = -\frac{iE}{\hbar}\mu_{ba}\rho_{ab} + c.c. \end{aligned} \tag{3}$$

Here, $\mu_{ba} = \langle b|\,\mu\,|a\rangle$. The equation is written in 1D for simplicity. Incorporating characteristic time scales of spontaneous decay and decoherence the complete version of the equation is

$$\dot{\rho}_{aa} = -\gamma_{ab}\rho_{aa} - \frac{iE}{\hbar}(\mu_{ba}\rho_{ab} - \mu_{ab}\rho_{ba}) = -\dot{\rho_{bb}} \tag{4}$$

$$\dot{\rho}_{ab} = -\gamma_{\perp}\rho_{ab} - i\omega_{ab}\rho_{ab} - \frac{iE}{\hbar}\mu_{ab}(\rho_{aa} - \rho_{bb}) \tag{5}$$

$$\dot{\rho}_{ba} = -\gamma_{\perp}\rho_{ba} - i\omega_{ba}\rho_{ba} - \frac{iE}{\hbar}\mu_{ba}(\rho_{bb} - \rho_{aa}) \tag{6}$$

where γ_{ab} and $\gamma_{\perp}$ are inverse of spontaneous decay time and decoherence time, respectively. Three physical quantities on the Poincare sphere can be defined: transition dipole moment, its quadrature component, and normalized population inversion. These quantities are

$$p = Tr(\rho\mu) = \rho_{ab}\mu_{ba} + \mu_{ab}\rho_{ba} \tag{7}$$

$$q = i(\rho_{ab}\mu_{ba} - \mu_{ab}\rho_{ba}) \tag{8}$$

$$n = \rho_{aa} - \rho_{bb} \tag{9}$$

The equation of motion can be expressed of three quantities.

$$\begin{aligned}\dot{p} &= \dot{\rho}_{ab}\mu_{ba} + \dot{\rho}_{ba}\mu_{ab} = \mu_{ba}\left[-\gamma_{\perp}\rho_{ab} - i\omega_{ab}\rho_{ab} - \frac{iE}{\hbar}\mu_{ab}(\rho_{aa}-\rho_{bb})\right] \\ &+ \mu_{ab}\left[-\gamma_{\perp}\rho_{ba} - i\omega_{ba}\rho_{ba} - \frac{iE}{\hbar}\mu_{ba}(\rho_{bb}-\rho_{aa})\right] = -\gamma_{\perp}p - \omega_{ab}q \end{aligned} \tag{10}$$

$$\begin{aligned}\dot{q} &= i(\dot{\rho}_{ab}\mu_{ba} - \dot{\rho}_{ba}\mu_{ab}) = i\mu_{ba}\left[-\gamma_{\perp}\rho_{ab} - i\omega_{ab}\rho_{ab} - \frac{iE}{\hbar}\mu_{ab}(\rho_{aa}-\rho_{bb})\right] \\ &- i\mu_{ab}\left[-\gamma_{\perp}\rho_{ba} - i\omega_{ba}\rho_{ba} - \frac{iE}{\hbar}\mu_{ba}(\rho_{bb}-\rho_{aa})\right] = -\gamma_{\perp}q + \omega_{ab}p + \frac{2E\,|\mu_{ab}|^2}{\hbar} \end{aligned} \tag{11}$$

$$\begin{aligned}\dot{n} &= \dot{\rho}_{aa} - \dot{\rho}_{bb} = -\gamma_{ab}\left((\rho_{aa}-\rho_{bb}) - (\rho_{aa}-\rho_{bb})^{eq}\right) - \frac{2iE}{\hbar}(\mu_{ba}\rho_{ab} - \mu_{ab}\rho_{ba}) \\ &= -\gamma_{ab}(n - n^{eq}) - \frac{2E}{\hbar}q \end{aligned} \tag{12}$$

Let the driving field oscillates with frequency ω_0, which may not necessarily be equal to ω_{ab}. p and q would also oscillate with the driving frequency. We can adopt the envelope function $E = Ee^{-i\omega_0 t} + c.c.$. The left-hand side is an electric field, while E on the right-hand side is an envelope; similarly for p, q. With the rotating wave approximation, above equation of motion can be written as follows.

$$\dot{p} = -\left(\gamma_\perp - i\omega_0\right)p - \omega_{ab}q \tag{13}$$

$$\dot{q} = -\left(\gamma_\perp - i\omega_0\right)q + \omega_{ab}p + \frac{2E\left|\mu_{ab}\right|^2}{\hbar}n \tag{14}$$

$$\dot{n} = -\gamma_{ab}(n - n^{eq}) - \frac{2}{\hbar}\left(Eq^* + E^*q\right) \tag{15}$$

2.3 Maxwell-Bloch equation under physical setup

This section covers dynamics of the transition dipole moment under the physical setup. The main idea is that in the steady state, population depletion by a single pulse is equivalent to inter-pulse population replenishment.[10] The derivation starts with the equation of motion for transition dipole moment.

$$\begin{aligned}\ddot{p} &= -(\gamma_\perp - i\omega_0)\dot{p} - \omega_{ab}\dot{q} \\ &= -(\gamma_\perp - i\omega_0)\dot{p} - \omega_{ab}\left[-\frac{(\gamma_\perp - i\omega_0)}{\omega_{ab}}\left(-\dot{p} - (\gamma_\perp - i\omega_0)p\right) + \omega_{ab}p + \frac{2E\left|\mu_{ab}\right|^2}{\hbar}n\right]\end{aligned} \tag{16}$$

$$\therefore \ddot{p} + 2\left(\gamma_\perp - i\omega_0\right)\dot{p} + \left(\omega_{ab}^2 + \left(\gamma_\perp - i\omega_0\right)^2\right)p = -\frac{2E\omega_{ab}\left|\mu_{ab}\right|^2}{\hbar}n \tag{17}$$

The expression for p in the frequency domain becomes

$$p(z, \omega - \omega_0) = -\frac{2\omega_{ab}\left|\mu_{ab}\right|^2 \mathrm{FT}(En)(z, \omega - \omega_0)}{\hbar(-w^2 + w_{ab}^2 + \gamma_\perp^2 - i2\gamma_\perp w)} \tag{18}$$

with Fourier transform convention

$$p(z, t) = \int \frac{dw}{2\pi} p(z, w)e^{-iwt} \tag{19}$$

The representation includes convolution of E and n. However, we can simplify it by setting the time scale of interest comparable to T_2. In the equation 15, inverse decay rate of n is $\gamma_{ab}^{-1}(\ll T_2)$.

Therefore $n(t)$ remain constant during T_2. In this case, $\mathrm{FT}(E(t)n(t))$ may be approximated as $E(w)n$. The (macroscopic) polarization is:

$$P_{macroscopic}(w) = Np(w-w_0) = \epsilon_0 \chi(w) E(w-w_0) \qquad \text{(SI)} \tag{20}$$

$$\therefore \chi(w) = \frac{Np(w-w_0)}{\epsilon_0 E(w-w_0)} = \frac{2Nnw_{ab}\left|\mu_{ab}\right|^2}{\epsilon_0 \hbar (w^2 - w_{ab}^2 - \gamma_\perp^2 + i2\gamma_\perp w)} \tag{21}$$

where N is the density of active ions. In the above equation we assumed n be constant in time and limited the time scale to T_2; we may expand it to T_p provided that the driving field is not intense. Now we can calculate the population depletion by a single pulse from the equation 15. Since $\gamma_{ab} \ll T_p^{-1}$, the first term's contribution is negligible in T_p.

$$\frac{dn}{dt} = -\frac{2}{\hbar}\mathrm{Re}(E^* q) = \frac{2}{\hbar w_{ab}} \mathrm{Re}\left(E^*((\gamma_\perp - iw_0)p + \dot{p})\right) \tag{22}$$

$$\begin{aligned} \int_0^{T_p} \frac{dn}{n} &= \int_0^{T_p} dt \frac{1}{n}\frac{2}{\hbar w_{ab}} \mathrm{Re}\left(E^*((\gamma_\perp - iw_0)p + \dot{p})\right) \\ &\simeq \frac{2}{\hbar w_{ab}} \mathrm{Re}\left[\int_{-\infty}^{+\infty} dt \frac{1}{n}\left(E^*((\gamma_\perp - iw_0)p + \dot{p})\right)\right] \end{aligned} \tag{23}$$

The integration interval is expanded to infinity since p vanishes exponentially without the driving field, giving the integration outside T_p negligible. Time-varying property of $n(t)$ outside T_p is washed out by vanishing p. The integration is (using Parseval's theorem)

$$\begin{aligned} &\int_{-\infty}^{+\infty} \frac{dw}{2\pi}\frac{1}{n} E^*(w_0 - w)\left((\gamma_\perp - iw_0)p(w-w_0) - i(w-w_0)p(w-w_0)\right) \\ &= \int_{-\infty}^{+\infty} \frac{dw}{2\pi} E^*(w_0-w)(\gamma_\perp - iw)\frac{2\omega_{ab}\left|\mu_{ab}\right|^2 E(w-w_0)}{\hbar(w^2 - w_{ab}^2 - \gamma_\perp^2 + i2\gamma_\perp w)} \\ &= \frac{2\omega_{ab}\left|\mu_{ab}\right|^2}{\hbar}\int_{-\infty}^{+\infty}\frac{dw}{2\pi}\frac{\gamma_\perp - iw}{w^2 - w_{ab}^2 - \gamma_\perp^2 + i2\gamma_\perp w}\left|E(w-w_0)\right|^2 \end{aligned} \tag{24}$$

This results in the change of normalized population after a single pulse

$$\begin{aligned} \therefore n(T_p) &= n(0)\exp\left[\mathrm{Re}\left[\int_{-\infty}^{+\infty}\frac{dw}{2\pi}\left|E(w-w_0)\right|^2 \frac{4\left|\mu_{ab}\right|^2}{\hbar^2}\frac{\gamma_\perp - iw}{w^2 - w_{ab}^2 - \gamma_\perp^2 + i2\gamma_\perp w}\right]\right] \\ &= n(0)\exp\left[\int_{-\infty}^{+\infty}\frac{dw}{2\pi}\left|E(w-w_0)\right|^2\frac{\epsilon_0}{\hbar}\chi_I(w)\right], \end{aligned} \tag{25}$$

$$\chi_I(w) \equiv \frac{4\left|\mu_{ab}\right|^2}{\epsilon_0\hbar}\mathrm{Re}\left[\frac{\gamma_\perp - iw}{w^2 - w_{ab}^2 - \gamma_\perp^2 + i2\gamma_\perp w}\right] = \frac{4\left|\mu_{ab}\right|^2}{\epsilon_0\hbar}\frac{-\gamma_\perp(w^2 + w_{ab}^2 + \gamma_\perp^2)}{(w^2 - w_{ab}^2 - \gamma_\perp^2)^2 + 4\gamma_\perp^2 w^2} \tag{26}$$

It should be noted that above expression for n is an first-order approximation: left-hand side of equation 23 accounts for $n(t)$ change during T_p, while right-hand side assumes $n(t)$ constant during T_p. Combining equation 25 and

$$n(T_d) \sim n^{eq} + (n(T_p) - n^{eq})\, e^{-\gamma_{ab} T_d} \tag{27}$$

altogether with the steady-state condition that $n(T_d)$ be equivalent to $n(0)$,

$$n(0) = n^{eq} \frac{1 - e^{-\gamma_{ab} T_d}}{1 - \exp\left[\int_{-\infty}^{+\infty} \frac{dw}{2\pi} \left|E(w-w_0)\right|^2 \frac{\epsilon_0}{\hbar} \chi_I(w)\right] e^{-\gamma_{ab} T_d}} \tag{28}$$

It is the first order approximation of saturated population inversion. Since $e^{-\gamma_{ab} T_d} \sim e^{-1\mu s/1ms} \sim e^{-0.001}$ and the fractional population change is known to be $\sim 10^{-6}$ [10], it can be further simplified as equation 29.

$$\begin{aligned} n(0) &= n^{eq} \frac{1 - (1 - \gamma_{ab} T_d + \text{H.O.T.})}{1 - (1 - \gamma_{ab} T_d + \text{H.O.T.}) \left(1 + \left[\int_{-\infty}^{+\infty} \frac{dw}{2\pi} \left|E(w-w_0)\right|^2 \frac{\epsilon_0}{\hbar} \chi_I(w)\right] + \text{H.O.T.}\right)} \\ &\simeq \frac{n^{eq}}{1 - \frac{\epsilon_0}{\hbar \gamma_{ab} T_d} \int_{-\infty}^{+\infty} \frac{dw}{2\pi} \left|E(w-w_0)\right|^2 \chi_I(w)} \end{aligned} \tag{29}$$

The result can be compared to the set of equations in [4]:

$$n_{sat} = \frac{n_{eq}}{1 + N^* E_{pulse}^{eff} / E_{sat}} \tag{30}$$

$$E_{pulse}^{eff} = \frac{1}{2\pi} \int \frac{S(z, w-w_0)}{1 + (w-w_a)^2 T_2^2} dw = \frac{1}{2\pi} \int \frac{2nc\epsilon_0 |E(z, w-w_0)|^2}{1 + (w-w_a)^2 T_2^2} dw \tag{31}$$

$$1/E_{sat} = \frac{2\mu^2 T_2}{\hbar^2} \frac{1}{nc\epsilon_0} \tag{32}$$

$$N^* = (\text{Number of pulses over the time } T_1^*) = T_1^* / T_R \tag{33}$$

First, $(\gamma_{ab} T_d)^{-1}$ in equation 29 is equivalent to N^*. Second, from equation 25,

$$\begin{aligned} \frac{\epsilon_0}{\hbar} \int_{-\infty}^{+\infty} \frac{dw}{2\pi} \left|E(w-w_0)\right|^2 \chi_I(w) &= -\frac{4|\mu_{ab}|^2}{2\pi\hbar^2} \int_{-\infty}^{+\infty} dw \left|E(w-w_0)\right|^2 \frac{\gamma_\perp (w^2 + w_{ab}^2 + \gamma_\perp^2)}{(w^2 - w_{ab}^2 - \gamma_\perp^2)^2 + 4\gamma_\perp^2 w^2} \\ &= -\frac{|\mu_{ab}|^2}{\hbar^2 \gamma_\perp} \frac{1}{nc\epsilon_0} \frac{1}{2\pi} \int_{-\infty}^{+\infty} dw \cdot 2nc\epsilon_0 \left|E(w-w_0)\right|^2 \frac{2\gamma_\perp^2 (w^2 + w_{ab}^2 + \gamma_\perp^2)}{(w^2 - w_{ab}^2 - \gamma_\perp^2)^2 + 4\gamma_\perp^2 w^2} \end{aligned} \tag{34}$$

The integration has a unit of intensity, and the coefficient in front of it has a unit of inverse intensity. The fraction can be simplified under approximation $\omega_{ab} \gg \gamma_\perp \gg |\omega - \omega_{ab}| \equiv |\Delta|$:

$$\begin{aligned}\frac{2\gamma_\perp^2(w^2+w_{ab}^2+\gamma_\perp^2)}{(w^2-w_{ab}^2-\gamma_\perp^2)^2+4\gamma_\perp^2 w^2} &= 2\frac{(w_{ab}+\Delta)^2+w_{ab}^2+\gamma_\perp^2}{4w^2+((w_{ab}+\Delta)^2-w_{ab}^2-\gamma_\perp^2)^2/\gamma_\perp^2} \\ &\simeq 2\frac{2w_{ab}^2+2w_{ab}\Delta}{(4w_{ab}^2+4w_{ab}\Delta)(1+\Delta^2/\gamma_\perp^2)} = \frac{1}{1+(w-w_{ab})^2T_2^2}\end{aligned} \tag{35}$$

which appears in equation 31. Therefore, the result is consistent with a set of equations from [4]. The remaining analysis utilizes equation 28. Also, it has been shown that the four-level system of Ytterbium dopants can be treated as an effective two-level which an effective T_1, T_2. [4]

2.4 Nonlinear Schrodinger equation with optical gain term

This section covers the optical gain term incorporated into Nonlinear Schrodinger equation. A polarization contribution to the z-derivative of envelope A is [4]

$$\frac{\partial A}{\partial z} \sim \frac{\mu_0}{2i\beta_0}\frac{\partial^2}{\partial t^2}\langle P\rangle_{transverse,-iw_0t}\, e^{-i(\beta_0 z - w_0 t)} = \frac{1}{2i\epsilon_0 c n_0 w_0}N(\ddot{p}-2iw_0\dot{p}-w_0^2 p)e^{-i\beta_0 z} \tag{36}$$

n_0 stands for refractive index. Bracket refers to integrating out the transverse spatial structure and taking only $-iw_0t$ frequency component. Right-hand side of the above equation in frequency domain is (refer to equation 18)

$$\begin{aligned}(\text{RHS}) &= \frac{1}{2i\epsilon_0 c n_0 w_0}N(-(w-w_0)^2-2iw_0\cdot(-i(w-w_0))-w_0^2)p(z,w-w_0)e^{-ik_0z} \\ &= \left[-\frac{Nw^2}{2i\epsilon_0 c n_0 w_0}\right]\left[-\frac{2\omega_{ab}\,|\mu_{ab}|^2\,E(z,\omega-\omega_0)n}{\hbar(-w^2+w_{ab}^2+\gamma_\perp^2-i2\gamma_\perp w)}\right]e^{-ik_0z}\end{aligned} \tag{37}$$

where n now denotes saturated population inversion n_{sat}. Applying the same approximation assumption as equation 35, it becomes

$$(\text{RHS}) \simeq \frac{w^2}{2\epsilon_0 c n_0 w_0}\frac{|\mu_{ab}|^2 Nn}{\hbar\gamma_\perp}\frac{E(w-w_0)}{1-i(w-w_{ab})T_2}e^{-ik_0z} \tag{38}$$

It is consistent with the set of equations in [10]:

$$(\text{Gain term}) = \frac{g}{2}\frac{1}{1 - i(w - w_a)T_2}A(z, w - w_0) \tag{39}$$

$$g = \frac{g_{us}}{1 + N^* E_{pulse}^{eff}/E_{sat}}, \qquad g_{us} = \sigma_s N n_{eq}, \qquad \sigma_s = \frac{w^2}{\epsilon_0 c n_0 w_0}\frac{|\mu_{ab}|^2}{\hbar\gamma_\perp} \tag{40}$$

We note that equations carry $e^{-ik_0 z}$ term. This can be eliminated by transforming $p = pe^{i(k_0 z - w_0 t)} +$ $c.c.$ instead of $p = pe^{-iw_0 t}$+c.c. from the starting point. The equation forms remain unchanged after compensating this spatial evolution term.

This section provided background for the later analytical developments by incorporating Maxwell-Bloch equation into Nonlinear Schrodinger equation under current physical situation in a rigorous manner. Main results are: equation 18, 25, 29, 37.

2.5 Inverse scattering method

In the framework of quantum mechanics, scattering problem is mathematically formulated as matching an asymptotic far-field wavefunctions to an exact near-potential wavefunction via a boundary condition. The information of scattering potential is reflected on the scattering data e.g. reflection or transmission coefficient. This map(inverse-map) of scattering potential to scattering data plays a crucial role in inverse scattering method.

An evolution of $u(x, t)$ in accordance to a nonlinear equation is hard to obtain directly. However, it is often mathematically tractable to understand $u(x, t)$ as a scattering potential, obtain the scattering data via potential $\rightarrow$ data map, and find the evolution of the scattering data. By using an inverse map data $\rightarrow$ potential, $u(x, t)$ can be obtained indirectly. This is the inverse scattering method. Detailed method specific to NLSE can be found in [11], [12].

2.6 Motivation for the perturbational analysis

As mentioned in section 1, the ultimate goal of this study is to describe the onset criterion for dark solitons in terms of physical parameters. Solitons are not generated instantaneously; rather, they develop gradually from an almost-smooth envelope. Therefore it is of a primary interest to describe the physics in the fluctuation limit. If we properly describe the fluctuation certain criterion must emerge, which, when satisfied, allows field fluctuation to further develop into dark solitons observable by numerical simulations or experiments. This is not to argue that dynamics of developed dark solitons are of no importance: there may also be important physics in the evolution of developed dark solitons. However, fluctuation limit is the physically right spot to develop the first theory.

3 Perturbation theory to NLSE

3.1 Manipulating equation for perturbational analysis

NLSE (equation 41, written in dimensionless units) has a single($N = 1$) dark soliton solution (equation 42) where v is the blackness parameter. [15]

$$i\frac{\partial u}{\partial x} - \frac{1}{2}\frac{\partial^2 u}{\partial t^2} + |u|^2 u = 0, \qquad \text{boundary condition:} \qquad |u(x \to \pm\infty)| = u_0 \tag{41}$$

$$u(x,t) = u_0 \frac{(\lambda - iv)^2 + \exp(z)}{1 + \exp(z)} e^{iu_0^2 x}, \qquad z = 2vu_0(t - \lambda u_0 x), \qquad \lambda^2 = 1 - v^2 \tag{42}$$

On the other hand, NLSE of current interest, written in physical units, has the following form.

$$\frac{\partial A}{\partial Z} = -\frac{\alpha}{2}A - \left(\sum_{m=2}^{\infty} \frac{i^{m-1}\beta_m}{m!}\frac{\partial^m}{\partial T^m}\right) A + i\gamma\left(1 + \frac{i}{w_0}\frac{\partial}{\partial T}\right) A \left[\int_{-\infty}^{+\infty} R(T') \left|A(T - T')\right|^2 dT' + i\tau_R\right]$$
$$+ \int_{-\infty}^{+\infty} \frac{dw}{2\pi} g(w - w_0) A(w - w_0) e^{-i(w-w_0)T} \tag{43}$$

Four terms in the right-hand side denote attenuation, second and higher order dispersion, Kerr nonlinearity, and optical gain, respectively. By considering only the second-order dispersion and treating Raman response $R(T')$ as a local function, it reduces to equation 44. Rescaling of variables (equation 45) with an introduction of arbitrary time scale t_0 transforms equation 44 to 41. Correspondingly, the single dark soliton solution 46 transforms to equation 42.

$$\frac{\partial A}{\partial Z} = -i\frac{\beta_2}{2}\frac{\partial^2 A}{\partial T^2} + i\gamma |A|^2 A \tag{44}$$

$$Z \to \frac{t_0^2}{\beta_2}x, \qquad T \to t_0 t, \qquad A(Z,T) \to \left(\frac{\beta_2}{t_0^2}\frac{1}{\gamma}\right)^{1/2} u(x,t) \tag{45}$$

$$A(Z,T) = \left(\frac{\beta_2}{t_0^2}\frac{1}{\gamma}\right)^2 u_0 \frac{(\lambda - iv)^2 + \exp(z)}{1 + \exp(z)} e^{iu_0^2\beta_2 Z/t_0^2}, \qquad z = 2vu_0(\frac{T}{t_0} - \lambda u_0 \frac{\beta_2}{t_0^2} Z) \tag{46}$$

Perturbation analysis hereafter adheres to equations with dimensionless units. The outcome will be transformed back into physical units. In order to apply perturbation theory, terms included in equation 43 but omitted from 44 would be regarded as perturbations. This section presents a method of treating omitted terms and a proof that perturbation terms are indeed small compared to original terms. To begin with, Yuri S. Kivshar suggested a method to treat (delayed) Raman response term as a perturbation. [13]

$$|u|^2 \to (1-f_R)|u|^2 + f_R \int_{-\infty}^{t} f(t-t')|u(t')|^2 dt' \tag{47}$$

Corresponding physical version is $R(T') = (1-f_R)\delta(T') + f_R h_R(T')$ in equation 43. $f(t-t')$ is a dimensionless counterpart of (delayed) Raman response function $h_R(T')$, and f_R is the strength of Raman response whose approximate value is known to be 0.2. [2] $f(t-t')$(resp. $h_R(T')$) is an exponentially decaying function: the contribution of $f(t-t')|u(t')|^2$ to the integration is negligible for $(t-t') \gg$ (time scale of f). Restricting the integration domain to an interval $\sim (t-t')$, Taylor series expansion of $u(t')$ up to the first order (equation 48) is valid provided that $u(t')$ is varying over time scale greater than $(t-t')$.

$$\begin{aligned}
&\int_{-\infty}^{t} f(t-t')|u(t')|^2 dt' = \int_{0}^{\infty} f(t')|u(t-t')|^2 dt' \\
&= \int_0^\infty f(t')\left(u(t) - t'\frac{\partial u}{\partial t} + \frac{1}{2}t'^2\frac{\partial^2 u}{\partial t^2} + \text{H.O.T.}\right)\left(u^* - t'\frac{\partial u^*}{\partial t} + \frac{1}{2}t'^2\frac{\partial^2 u^*}{\partial t^2} + \text{H.O.T.}\right) dt' \\
&= |u(t)|^2 \int_0^\infty f(t')dt' - \frac{\partial}{\partial t}|u(t)|^2 \int_0^\infty t' f(t')dt' + \frac{1}{2}\frac{\partial^2}{\partial t^2}|u(t)|^2 \int_0^\infty t'^2 f(t')dt' + \text{H.O.T.}
\end{aligned} \tag{48}$$

Thus, up to the first order Taylor series, equation 47 becomes

$$u|u|^2 \to u|u|^2 - f_R \int_0^\infty t' f(t')dt' \cdot u\frac{\partial}{\partial t}|u(t)|^2 + \text{H.O.T.} \tag{49}$$

Note that one of the most widely used Raman function models is [1]

$$h_R(T>0) = \frac{\tau_1^2+\tau_2^2}{\tau_1\tau_2^2}\exp\left(-\frac{T}{\tau_2}\right)\sin\left(\frac{T}{\tau_1}\right), \qquad \tau_1 \sim 12.2\text{fs}, \qquad \tau_2 \sim 32\text{fs} \tag{50}$$

Similar approach can be taken for the optical gain term. Frequency dependent optical gain $g(w-w_0)$ in equation 43 was obtained in the previous section (equation 37) with the saturated population inversion n_{sat} (equation 29).

$$g(w-w_0) = \left[-\frac{Nw^2}{2i\epsilon_0 cn_0 w_0}\right]\left[-\frac{2\omega_{ab}\left|\mu_{ab}\right|^2 n_{sat}}{\hbar(-w^2+w_{ab}^2+\gamma_\perp^2-i2\gamma_\perp w)}\right] \tag{51}$$

n_{sat} in general depends on an envelope profile, and tends to decrease along propagation due to gain saturation. Numerically calculated n_{sat} initially starts at 0.9 and drops to 0.4 after a propagation of 3.6m under a certain initial condition, described in section 1. However, we assume it to be constant over the propagation length of the interest, so that $g(w-w_0)$ is simply a function of frequency. Now $g(w-w_0)A(w-w_0)$ is inverse Fourier transformed as

$$-\frac{Nw_{ab}|\mu_{ab}|^2 n_{sat}}{i\epsilon_0 cn_0 w_0\hbar}\cdot \mathrm{IFT}\left(\frac{-w^2}{-w^2+w_{ab}^2+\gamma_\perp^2-i2\gamma_\perp w}\right)\Bigg|_{w-w_0} * A(Z,T) \tag{52}$$

The coefficient has an unit of inverse length, value $1.4115\cdot 10^{-26}N_{tot}n_{sat}[\mathrm{m}^{-1}]$ when N_{tot} is written in the unit $[\mathrm{m}^{-3}]$. Meanwhile, the inverse Fourier transform part can be easily calculated. From equation 53 and equation 54, ($h(T)$ is a unit step function)

$$\frac{-w^2}{-w^2+w_{ab}^2+\gamma_\perp^2-i2\gamma_\perp w} = 1+\frac{1}{2}\frac{-w_{ab}+\gamma_\perp^2/w_{ab}-2i\gamma_\perp}{w_{ab}+(w+i\gamma_\perp)}+\frac{1}{2}\frac{-w_{ab}+\gamma_\perp^2/w_{ab}+2i\gamma_\perp}{w_{ab}-(w+i\gamma_\perp)} \tag{53}$$

$$\mathrm{FT}\left(e^{(-\gamma_\perp \pm iw_{ab})T}h(T)\right)(w-w_0) = \frac{\pm i}{w_{ab}\pm(w-w_0+i\gamma_\perp)} \tag{54}$$

IFT part in equation 52 becomes equation 55. The gain response respects causality.

$$\delta(T)+\zeta(T) \equiv \delta(T)+\left[\left(-w_{ab}+\frac{\gamma_\perp^2}{w_{ab}^2}\right)sin(w_{ab}T)-2\gamma_\perp cos(w_{ab}T)\right]e^{(iw_0-\gamma_\perp)T}h(T) \tag{55}$$

We note that $\zeta(T)$ has an exponential decay, as does Raman response model (equation 50). $\gamma_\perp \sim (10fs)^{-1}, \tau_2 \sim (32fs)^{-1}$. Therefore, we can apply the procedure identical to equation 49.

$$(\delta(T)+\zeta(T)) * A(Z,T) = A(Z,T)+\int_0^\infty \zeta(T')\left(A(Z,T)-t'\frac{\partial A}{\partial T}+\frac{1}{2}t'^2\frac{\partial^2 A}{\partial T^2}+\mathrm{H.O.T.}\right)dt'$$
$$=\left(1+\int_0^\infty \zeta(t')dt'\right)A(z,t)+\left(\int_0^\infty -t'\zeta(t')\right)\frac{\partial A}{\partial T}+\left(\int_0^\infty \frac{1}{2}t'^2\zeta(t')dt'\right)\frac{\partial^2 A}{\partial T^2}+\mathrm{H.O.T.} \tag{56}$$

We truncate terms of order equal or greater than 3. Each terms bounded with () can be analytically evaluated. For example,

$$
\begin{aligned}
\int_0^{\infty} \zeta(t')t'^2 dt' &= -\left(w_{ab} - \frac{\gamma_\perp^2}{w_{ab}}\right) \frac{2w_{ab}(3(\gamma_\perp - iw_0)^2 - w_{ab}^2)}{(w_{ab}^2 + (\gamma_\perp - iw_0)^2)^3} - 2\gamma_\perp \frac{2(\gamma_\perp - iw_0)((\gamma_\perp - iw_0)^2 - 3w_{ab}^2)}{(w_{ab}^2 + (\gamma_\perp - iw_0)^2)^3} \\
&= \frac{-4\gamma_\perp(\gamma_\perp - iw_0)^3 + 6(\gamma_\perp^2 - w_{ab}^2)(\gamma_\perp - iw_0)^2 + 12\gamma_\perp w_{ab}^2(\gamma_\perp - iw_0) - 2w_{ab}^2(\gamma_\perp^2 - w_{ab}^2)}{(w_{ab}^2 + (\gamma_\perp - iw_0)^2)^3} \\
&= \frac{2(\gamma_\perp^4 + w_{ab}^4) + 4\gamma_\perp^2 w_{ab}^2 + 6w_{ab}^2 w_0^2 + 6w_0^2\gamma_\perp^2 - 4i\gamma_\perp w_0^3}{(w_{ab}^2 + (\gamma_\perp - iw_0)^2)^3} \qquad [(\text{time})^2] \qquad (57)
\end{aligned}
$$

The above calculations can be embedded into a unified equation. First, ignore higher order dispersions and attenuation terms from equation 43. (convention for Fourier transform: 19)

$$
\frac{\partial A}{\partial Z} = -\frac{i\beta_2}{2}\frac{\partial^2 A}{\partial T^2} + i\gamma A \int_{-\infty}^{+\infty} R(T') \left|A(T - T')\right|^2 dT' + \text{IFT}(g(w - w_0)A(w - w_0)) \qquad (58)
$$

Rewrite the optical gain term with equation 52, 55.

$$
\frac{\partial A}{\partial Z} = -\frac{i\beta_2}{2}\frac{\partial^2 A}{\partial T^2} + i\gamma A \int_{-\infty}^{+\infty} R(T') \left|A(T - T')\right|^2 dT' + \eta\left(A + \int_0^{\infty} \zeta(T')A(T - T')dT'\right) \qquad (59)
$$

$$
\eta \equiv -\frac{Nw_{ab}|\mu_{ab}|^2 n_{sat}}{i\epsilon_0 c n_0 w_0 \hbar} \qquad [(\text{length})^{-1}] \qquad (60)
$$

Using transformation rules (equation 45) and eliminating co-factors, we get

$$
\frac{\partial u}{\partial x} = -\frac{i}{2}\frac{\partial^2 u}{\partial t^2} + iu \int_{-\infty}^{+\infty} t_0 R(t_0 t')|u(t - t')|^2 dt' + \eta\frac{t_0^2}{\beta_2}\left(u + \int_0^{\infty} t_0\zeta(t_0 t')u(t - t')dt'\right) \qquad (61)
$$

Application of equation 49, 56 approximates the second and last term in right-hand side:

$$
iu \int_{-\infty}^{+\infty} t_0 R(t_0 t')|u(t - t')|^2 dt' \to iu|u|^2 - \frac{if_R}{t_0}\left[\int_0^{+\infty} t_0 t' h_R(t_0 t')t_0 dt'\right] \cdot u\frac{\partial}{\partial t}|u(t)|^2 \qquad (62)
$$

$$
\begin{aligned}
\eta\frac{t_0^2}{\beta_2}\left(u + \int_0^{\infty} t_0\zeta(t_0 t')u(t - t')dt'\right) &\to \eta\frac{t_0^2}{\beta_2}\left[1 + \int_0^{\infty} \zeta(t_0 t')t_0 dt'\right] \cdot u \\
&+ \frac{\eta t_0}{\beta_2}\left[\int_0^{\infty} -t_0 t'\zeta(t_0 t')t_0 dt'\right] \cdot \frac{\partial u}{\partial t} + \frac{\eta}{\beta_2}\left[\int_0^{\infty} \frac{1}{2}(t_0 t')^2\zeta(t_0 t')t_0 dt'\right] \cdot \frac{\partial^2 u}{\partial t^2} \qquad (63)
\end{aligned}
$$

The dimensionless approximated NLSE is equation 64. Coefficients $\bar{\alpha}$ to $\bar{\delta}$ can be easily traced.

$$
i\frac{\partial u}{\partial x} - \frac{1}{2}\frac{\partial^2 u}{\partial t^2} + u|u|^2 = \bar{\alpha} \cdot u\frac{\partial}{\partial t}|u(t)|^2 + i\bar{\beta} \cdot u + i\bar{\gamma} \cdot \frac{\partial u}{\partial t} + i\bar{\delta} \cdot \frac{\partial^2 u}{\partial t^2} \qquad (64)
$$

It worths mentioning that the rescaled coefficients depend on the choice of arbitrary time scale t_0. Different choice of t_0 gives different time scale to the dimensionless equation. Therefore, the direct comparison of magnitudes of coefficients is invalid, as is well-known in the renormalization group theory. The valid comparison must be performed in the physical dimension. Coefficients η and α through δ, physical counterparts of bar coefficients, are as follows. γ' corresponds to $\bar{\gamma}$, γ saved for nonlinear coefficient. $\beta_2 = 5.2494 \cdot 10^{-12}[\mathrm{ps}^2/\mathrm{nm}]$. f_R is purposefully left as a variable although the known value of f_R for silica is ~ 0.2.[1]

$$\eta \sim 1.40 \times 10^{-10} i \ [\mathrm{nm}^{-1}], \qquad \alpha \sim 8.12 \times 10^{-3} f_R \ [\mathrm{ps}], \qquad \beta \sim -2.50 \times 10^{-1} - 14.1i \ [1],$$
$$\gamma' \sim -1.59 \times 10^{-2} - 2.24 \times 10^{-1} i \ [\mathrm{ps}], \qquad \delta \sim -5.07 \times 10^{-4} - 7.13 \times 10^{-3} i \ [\mathrm{ps}^2] \tag{65}$$

3.2 Perturbational analysis

The section aims to obtain the master equation for fluctuation incurred by perturbation terms in equation 64. Multiple steps of field redefinition and rescaling are utilized. First, to remove the field amplification caused mainly by $i\bar{\beta}x$ (DC amplification), redefine field

$$u(x,t) = A(x,t) \exp\left[\bar{\beta}x + i\int^{x} u_0^2(\exp\left(2\bar{\beta}x'\right) - 1)dx'\right] \tag{66}$$

By factoring out exponentially growing amplitude and corresponding phase evolution, $i\bar{\beta}x$ is removed. Now $\bar{\beta}$ resides in the exponential form:

$$i\frac{\partial A}{\partial x} - \frac{1}{2}\frac{\partial^2 A}{\partial t^2} + \left[\exp(2\bar{\beta}x)(|A|^2 - u_0^2) + u_0^2\right] A = \bar{\alpha}\exp(2\bar{\beta}x)A\frac{\partial}{\partial t}|A|^2 + i\bar{\gamma}\frac{\partial A}{\partial t} + i\bar{\delta}\frac{\partial^2 A}{\partial t^2} \tag{67}$$

To observe the effect of perturbation terms, we adopt the fluctuation of amplitude $a(x,t)$ and phase $\phi(x,t)$. It worths mentioning that $a = 0, \phi = 0$ case solves the unperturbed equation.

$$A(x,t) = (u_0 + a(x,t))\exp\left(iu_0^2 x + i\phi(x,t)\right) \tag{68}$$

The resulting equation can be divided into real and imaginary parts.

$$
\begin{aligned}
&-(u_0+a)\frac{\partial\phi}{\partial x}-\frac{1}{2}\frac{\partial^2 a}{\partial t^2}+\frac{1}{2}(u_0+a)\left(\frac{\partial\phi}{\partial t}\right)^2+(u_0+a)(2u_0+a)ae^{2\bar{\beta}x}\\
&=2\bar{\alpha}(u_0+a)^2\frac{\partial a}{\partial t}e^{2\bar{\beta}x}-\bar{\gamma}(u_0+a)\frac{\partial\phi}{\partial t}-2\bar{\delta}\frac{\partial a}{\partial t}\frac{\partial\phi}{\partial t}-\bar{\delta}(u_0+a)\frac{\partial^2\phi}{\partial t^2}\qquad\text{(Real)}
\end{aligned}
\tag{69}
$$

$$
\frac{\partial a}{\partial x}-\frac{\partial a}{\partial t}\frac{\partial\phi}{\partial t}-\frac{1}{2}(u_0+a)\frac{\partial^2\phi}{\partial t^2}=\bar{\gamma}\frac{\partial a}{\partial t}+\bar{\delta}\frac{\partial^2 a}{\partial t^2}-\bar{\delta}(u_0+a)\left(\frac{\partial\phi}{\partial t}\right)^2\qquad\text{(Imaginary)}
\tag{70}
$$

It has been shown that in a no-perturbation limit, leading order amplitude and phase satisfy a wave equation.[13] To see this, suppress all perturbation terms and take the lowest order terms to the front.

$$
-u_0\left[\frac{\partial\phi}{\partial x}-2u_0a\right]-\frac{1}{2}\frac{\partial^2 a}{\partial t^2}-a\frac{\partial\phi}{\partial x}+\frac{1}{2}(u_0+a)\left(\frac{\partial\phi}{\partial t}\right)^2+3u_0a^2+a^3=0\qquad\text{(Real)}
\tag{71}
$$

$$
\left[\frac{\partial a}{\partial x}-\frac{1}{2}u_0\frac{\partial^2\phi}{\partial t^2}\right]-\left[\frac{\partial a}{\partial t}\frac{\partial\phi}{\partial t}+\frac{1}{2}a\frac{\partial^2\phi}{\partial t^2}\right]=0\qquad\text{(Imaginary)}
\tag{72}
$$

Leading terms (the first [ ] in each line), combined together, satisfy a wave equation with phase velocity $1/|u_0|$. Define $C=\pm u_0$, an inverse velocity scale. The fact that fluctuation travels with $1/C$ in the no-perturbation limit motivates to adopt the following retarded small frame.

$$
y=\epsilon^3 x,\qquad \tau=\epsilon(t-(-\bar{\gamma}+Ce^{\bar{\beta}x/2})x)
\tag{73}
$$

The exponential factor compensates temporal width broadening due to optical gain. Under single perturbative term $i\bar{\beta}x$ representing attenuation or amplification, dark soliton's temporal width increases as $\exp(\bar{\beta}x)$.[14] Therefore, rescaling time by multiplying $\exp(-\bar{\beta}x)$ would recover the original temporal width. This is equivalent to scaling propagation length by $\exp(\bar{\beta}x)$ in the expression for τ. Factor of 1/2 appears for a mathematical treatment. Derivative is:

$$
\frac{\partial}{\partial x}=\frac{\partial y}{\partial x}\frac{\partial}{\partial y}+\frac{\partial\tau}{\partial x}\frac{\partial}{\partial\tau}=\epsilon^3\frac{\partial}{\partial y}-\epsilon\left(-\bar{\gamma}+Ce^{\bar{\beta}x/2}(1+\bar{\beta}x/2)\right)\frac{\partial}{\partial\tau}\simeq\epsilon^3\frac{\partial}{\partial y}+\epsilon\left(\bar{\gamma}-Ce^{\bar{\beta}x}\right)\frac{\partial}{\partial\tau}
\tag{74}
$$

The approximation is valid only for $\bar{\beta}x\ll 1$. In other words, this perturbational analysis will only correctly describe short propagation. Following the approach similar to [16], expand fluctuation terms as follows. Our main concern will be leading terms i.e. a_0, ϕ_0.

$$a(x,t) = \sum_{i=1}^{\infty} \epsilon^{2i} a_{i-1}(y,\tau), \qquad \phi(x,t) = \sum_{i=1}^{\infty} \epsilon^{2i-1} \phi_{i-1}(y,\tau) \tag{75}$$

Combination of equation 70, 73, 74, 75, gives equation 76.

$$\begin{aligned}
&\left(\left(\epsilon\bar{\gamma} - \epsilon C e^{\bar{\beta}x}\right)\frac{\partial}{\partial\tau} + \epsilon^3\frac{\partial}{\partial y}\right)(\epsilon^2 a_0 + \epsilon^4 a_1 + ...) - \epsilon^2\frac{\partial}{\partial\tau}(\epsilon^2 a_0 + \epsilon^4 a_1 + ...)\frac{\partial}{\partial\tau}(\epsilon\phi_0 + \epsilon^3\phi_1 + ...) \\
&- \frac{1}{2}(u_0 + \epsilon^2 a_0 + \epsilon^4 a_1 + ...)\epsilon^2\frac{\partial^2}{\partial\tau^2}(\epsilon\phi_0 + \epsilon^3\phi_1 + ...) = \bar{\gamma}\epsilon\frac{\partial}{\partial\tau}(\epsilon^2 a_0 + \epsilon^4 a_1 + ...) \\
&+ \bar{\delta}\epsilon^2\frac{\partial^2}{\partial\tau^2}(\epsilon^2 a_0 + \epsilon^4 a_1 + ...) - \bar{\delta}(u_0 + \epsilon^2 a_0 + \epsilon^4 a_1 + ...)\epsilon^2\left(\frac{\partial}{\partial\tau}(\epsilon\phi_0 + \epsilon^3\phi_1 + ...)\right)^2
\end{aligned} \tag{76}$$

The lowest and second-lowest order terms are of third and fifth. (equation 77, 78)

$$C e^{\bar{\beta}x}\frac{\partial a_0}{\partial\tau} + \frac{1}{2}u_0\frac{\partial^2\phi_0}{\partial\tau^2} = 0 \tag{77}$$

$$-C e^{\bar{\beta}x}\frac{\partial a_1}{\partial\tau} + \frac{\partial a_0}{\partial y} - \frac{\partial a_0}{\partial\tau}\frac{\partial\phi_0}{\partial\tau} - \frac{1}{2}u_0\frac{\partial^2\phi_1}{\partial\tau^2} - \frac{1}{2}a_0\frac{\partial^2\phi_0}{\partial\tau^2} = \frac{\bar{\delta}}{\epsilon}\frac{\partial^2 a_0}{\partial\tau^2} - \frac{\bar{\delta}}{\epsilon}u_0\left(\frac{\partial\phi_0}{\partial\tau}\right)^2 \tag{78}$$

It must be noted that x in exponential terms were not series expanded. Rather, they are understood as scaling factors. Such exponential terms are carried intact throughout the derivation. The same procedure can be applied to equation 69, yielding equation 79 and 80 for second and fourth order, respectively. Equations 77 to 80 forms a system of equations.

$$C e^{\bar{\beta}x}\frac{\partial\phi_0}{\partial\tau} + 2u_0 e^{2\bar{\beta}x} a_0 = 0 \tag{79}$$

$$\begin{aligned}
&C e^{\bar{\beta}x} a_0\frac{\partial\phi_0}{\partial\tau} - u_0\left(\frac{\partial\phi_0}{\partial y} - C e^{\bar{\beta}x}\frac{\partial\phi_1}{\partial\tau}\right) - \frac{1}{2}\frac{\partial^2 a_0}{\partial\tau^2} + \frac{1}{2}u_0\left(\frac{\partial\phi_0}{\partial\tau}\right)^2 + (3u_0 a_0^2 + 2u_0^2 a_1)e^{2\bar{\beta}x} \\
&= 2\frac{\bar{\alpha}}{\epsilon}u_0^2 e^{2\bar{\beta}x}\frac{\partial a_0}{\partial\tau} - \frac{\bar{\delta}}{\epsilon}u_0\frac{\partial^2\phi_0}{\partial\tau^2}
\end{aligned} \tag{80}$$

Rearranging equation 77 and differentiating 79 with respect to τ, we see:

$$C e^{\bar{\beta}x}\frac{\partial a_0}{\partial\tau} + \frac{1}{2}u_0\frac{\partial^2\phi_0}{\partial\tau^2} = 0 \quad \text{and} \quad 2u_0 e^{2\bar{\beta}x}\frac{\partial a_0}{\partial\tau} + C e^{\bar{\beta}x}\frac{\partial^2\phi_0}{\partial\tau^2} = 0 \quad \therefore \frac{\partial\phi_0}{\partial\tau} = -2\frac{C}{u_0}e^{\bar{\beta}x}a_0 \tag{81}$$

On the other hand, multiply $2Ce^{\bar{\beta}x}$ to equation 78 and differentiate 80 with respect to τ. The addition of two equations neatly cancels higher order fluctuations i.e. a_1, ϕ_1,

$$
\begin{aligned}
&2(Ce^{\bar{\beta}x})\frac{\partial a_0}{\partial y} - 2(Ce^{\bar{\beta}x})\frac{\partial a_0}{\partial \tau}\frac{\partial \phi_0}{\partial \tau} - (Ce^{\bar{\beta}x})a_0\frac{\partial^2 \phi_0}{\partial \tau^2} + (Ce^{\bar{\beta}x})\left(a_0\frac{\partial^2 \phi_0}{\partial \tau^2} + \frac{\partial a_0}{\partial \tau}\frac{\partial \phi_0}{\partial \tau}\right) \\
&- u_0\frac{\partial^2 \phi_0}{\partial y \partial \tau} - \frac{1}{2}\frac{\partial^3 a_0}{\partial \tau^3} + u_0\frac{\partial \phi_0}{\partial \tau}\frac{\partial^2 \phi_0}{\partial \tau^2} + 6u_0 e^{2\bar{\beta}x} a_0 \frac{\partial a_0}{\partial \tau} \\
&= 2(Ce^{\bar{\beta}x})\frac{\bar{\delta}}{\epsilon}\frac{\partial^2 a_0}{\partial \tau^2} - 2(Ce^{\bar{\beta}x})\frac{\bar{\delta}}{\epsilon}u_0\left(\frac{\partial \phi_0}{\partial \tau}\right)^2 + 2\frac{\bar{\alpha}}{\epsilon}u_0^2 e^{2\bar{\beta}x}\frac{\partial^2 a_0}{\partial \tau^2} - \frac{\bar{\delta}}{\epsilon}u_0\frac{\partial^3 \phi_0}{\partial \tau^3}
\end{aligned}
\tag{82}
$$

Equation 81 can be used to rewrite equation 82 as 83

$$
\begin{aligned}
&2(Ce^{\bar{\beta}x})\frac{\partial a_0}{\partial y} + \frac{4}{u_0}(Ce^{\bar{\beta}x})^2 a_0\frac{\partial a_0}{\partial \tau} + \frac{2}{u_0}(Ce^{\bar{\beta}x})^2 a_0\frac{\partial a_0}{\partial \tau} - \frac{4}{u_0}(Ce^{\bar{\beta}x})^2 a_0\frac{\partial a_0}{\partial \tau} \\
&+ 2(Ce^{\bar{\beta}x})\frac{\partial a_0}{\partial y} - \frac{1}{2}\frac{\partial^3 a_0}{\partial \tau^3} + \frac{4}{u_0}(Ce^{\bar{\beta}x})^2 a_0\frac{\partial a_0}{\partial \tau} + 6u_0 e^{2\bar{\beta}x}a_0\frac{\partial a_0}{\partial \tau} \\
&= 2(Ce^{\bar{\beta}x})\frac{\bar{\delta}}{\epsilon}\frac{\partial^2 a_0}{\partial \tau^2} - \frac{8}{u_0^2}(Ce^{\bar{\beta}x})^3\frac{\bar{\delta}}{\epsilon}u_0 a_0^2 + 2\frac{\bar{\alpha}}{\epsilon}u_0^2 e^{2\bar{\beta}x}\frac{\partial^2 a_0}{\partial \tau^2} + 2\frac{\bar{\delta}}{\epsilon}(Ce^{\bar{\beta}x})\frac{\partial^2 a_0}{\partial \tau^2}
\end{aligned}
\tag{83}
$$

Canceling out terms gives a simple form, which is actually Korteweg-de Vries(KdV) equation.

$$
4(Ce^{\bar{\beta}x})\frac{\partial a_0}{\partial y} + \frac{12}{u_0}(Ce^{\bar{\beta}x})^2 a_0\frac{\partial a_0}{\partial \tau} - \frac{1}{2}\frac{\partial^3 a_0}{\partial \tau^3} = \left[4(Ce^{\bar{\beta}x})\frac{\bar{\delta}}{\epsilon} + 2\frac{\bar{\alpha}}{\epsilon}u_0^2 e^{2\bar{\beta}x}\right]\frac{\partial^2 a_0}{\partial \tau^2} - \frac{8}{u_0}(Ce^{\bar{\beta}x})^3\frac{\bar{\delta}}{\epsilon}a_0^2 \tag{84}
$$

To manifestly see KdV equation, another transformation is adopted. This is a simple rescaling. Equation 86 is the master equation for leading amplitude fluctuation.

$$
y' = -\frac{1}{8}\left(\frac{4}{u_0}\right)^{\frac{3}{2}}(Ce^{\bar{\beta}x})^2 y, \qquad \tau' = \left(\frac{4}{u_0}\right)^{\frac{1}{2}}(Ce^{\bar{\beta}x})\tau \tag{85}
$$

$$
\frac{\partial a_0}{\partial y'} - 6a_0\frac{\partial a_0}{\partial \tau'} + \frac{\partial^3 a_0}{\partial \tau'^3} = 2u_0^{1/2}\left[-\left(2\frac{\bar{\delta}}{\epsilon} + \frac{\bar{\alpha}}{\epsilon}(Ce^{\bar{\beta}x})\right)\frac{\partial^2 a_0}{\partial \tau'^2} + \frac{\bar{\delta}}{\epsilon}a_0^2\right] \tag{86}
$$

This equation is the governing equation for leading amplitude fluctuation. When perturbation terms originating from optical gain terms are suppressed, the result reduces to the equation presented in [13]. KdV equation describes the fluctuation a_0 with respect to background continuous wave amplitude u_0.

3.3 Fluctuation as a small-amplitude dark soliton and its parameter evolution

This subsection identifies fluctuation governed by equation 86 with a small-amplitude dark soliton and determines its blackness parameter evolution. The procedure taken here is similar to that of [13]. Left-hand side (KdV equation) has a single soliton solution (equation 87) and the KdV soliton parameter κ would evolve upon perturbation. [17]

$$u_s(y', \tau') = -2\kappa^2 sech^2(z) = -2\kappa^2 sech^2\left(\kappa(\tau' - 4\kappa^2 y')\right) \tag{87}$$

Thus, mathematical description for fluctuation is as follows. (Equation 66, 68, 73, 75, 85, 87)

$$u(x,t) = [u_0 + a(x,t)] \exp(iu_0^2 x + i\phi(x,t)) \cdot \exp(\bar{\beta}x + iu_0^2 \int^x (e^{2\bar{\beta}x'} - 1)dx')$$

$$= \left[u_0 - \epsilon^2 2\kappa^2 sech^2\left[\kappa(\tau' - 4\kappa^2 y')\right] + O(\epsilon^4)\right] \exp\left(\bar{\beta}x + i\int^x u_0^2 e^{2\bar{\beta}x'} dx' + i\phi(x,t)\right) \tag{88}$$

$$|u(x,t)| \simeq \left[u_0 - \epsilon^2 2\kappa^2 sech^2\left[\left(\frac{4}{u_0}\right)^{\frac{1}{2}} \kappa(Ce^{\bar{\beta}x})\tau + \frac{1}{2}\kappa^3\left(\frac{4}{u_0}\right)^{\frac{3}{2}}(Ce^{\bar{\beta}x})^2 y\right]\right] \exp\left(\bar{\beta}x\right) \tag{89}$$

$$= \left[u_0 - \epsilon^2 2\kappa^2 sech^2\left[\left(\frac{4}{u_0}\right)^{\frac{1}{2}}(Ce^{\bar{\beta}x})\kappa\epsilon(t - (-\bar{\gamma} + Ce^{\bar{\beta}x/2})x) + \frac{1}{2}\left(\frac{4}{u_0}\right)^{\frac{3}{2}}(Ce^{\bar{\beta}x})^2\kappa^3\epsilon^3 x\right]\right] \exp\left(\bar{\beta}x\right)$$

When we suppress perturbation coefficients, the $|u(x,t)|$ should reduce to a small-amplitude limit of exact single dark soliton solution, equation 90. v is the blackness parameter.

$$u(x,t) = \left[u_0 - \frac{1}{2}u_0 v^2 sech^2\left(\frac{z}{2}\right)^2\right] \exp\left(iu_0^2 x - \frac{2iv}{1 + \exp(z)}\right), z = 2vu_0(t \mp u_0 x \pm u_0 v^2 x) \tag{90}$$

Comparing equation 89 and 90, identification $\epsilon\kappa \sim v\sqrt{u_0}/2$ can be made, and equation 89 can be written as equation 91. This identification is particularly important since it relates KdV soliton parameter κ to dark soliton blackness parameter v. v would evolve due to perturbations. Its evolution equation can be found by inverse scattering method based κ evolution equation [17] , and the identification that relates v with κ.

$$|u| = u_0\left[1 - 2v^2 sech^2\left((Ce^{\bar{\beta}x})v(t - (-\bar{\gamma} + Ce^{\bar{\beta}x/2})x) + \frac{1}{2}(Ce^{\bar{\beta}x})^2 v^3 x\right)\right] \exp(\bar{\beta}x) \tag{91}$$

The evolution of κ is given from equation 86 by

$$\begin{aligned}\frac{d\kappa}{dy'} &= -\frac{1}{4\kappa}\int_{-\infty}^{+\infty} P[u_s(z)]sech^2(z)dz \\ &= -\frac{\kappa^3 u_0^{1/2}}{\epsilon}\left[-\frac{16}{15}\left(2\bar{\delta}+\bar{\alpha}(Ce^{\bar{\beta}x})\right)+\frac{32}{15}\bar{\delta}\right] = \frac{16}{15}\kappa^3 u_0^{1/2}\frac{\bar{\alpha}}{\epsilon}(Ce^{\bar{\beta}x}) \qquad (92)\end{aligned}$$

where P is a perturbation terms, right-hand side of equation 86, and $u_s(z)$ is a substitution of equation 87 into a_0 site. Surprisingly, the second order optical gain coefficient $\bar{\delta}$ vanishes. Since this equation is written in a rescaled frame, it should be recovered back to the first dimensionless units, and ultimately to the physical version.

$$\frac{dv}{dx} = \epsilon^3\frac{d}{dy}\left(\frac{2\epsilon\kappa}{\sqrt{u_0}}\right) = \frac{2\epsilon^4}{\sqrt{u_0}}\cdot\left[-\frac{1}{8}\left(\frac{4}{u_0}\right)^{\frac{3}{2}}(Ce^{\bar{\beta}x})^2\right]\frac{d\kappa}{dy'} = -\frac{4}{15}\bar{\alpha}\left(Ce^{\bar{\beta}x}\right)^3 v^3 \qquad (93)$$

Integration yields the v evolution equation in the first dimensionless units, equation 94. Using equation 45, 64, and 65, equation 94 transforms to 95, equation 89 transforms to 96. It must be mentioned that the arbitrary time scale t_0 appears only in the form of u_0/t_0. Therefore, the physics is invariant upon a choice of t_0, since u_0/t_0 is fully expressed in terms of physical quantities, namely $\sqrt{\gamma/\beta_2}|A|$. Substituting the initial amplitude $|A|$ into this relation and using equation 96, we can obtain the amplitude evolution of (physical) fluctuation.

$$\begin{aligned}\frac{1}{v^2} &= \frac{1}{v_0^2}+\frac{8}{45}\frac{\bar{\alpha}}{\bar{\beta}}u_0^3\mathrm{sgn}(C)(e^{3\bar{\beta}x}-e^{3\bar{\beta}x_0}) \qquad (94)\\ \frac{1}{v^2} &= \frac{1}{v_0^2}+\frac{8}{45}\frac{\beta_2}{\eta}\frac{\alpha}{\beta}\left(\frac{u_0}{t_0}\right)^3\mathrm{sgn}(C)(e^{3\eta\beta Z}-e^{3\eta\beta Z_0}) \qquad (95)\\ |A(Z,T)| &= \left(\frac{\beta_2}{\gamma}\right)^{1/2}\frac{u_0}{t_0}\left[1-2v^2 sech^2\left[\left\{e^{\eta\beta Z}v\left(\frac{u_0}{t_0}\right)\mathrm{sgn}(C)\right\}T\right.\right.\\ &\left.\left.+\left\{e^{\eta\beta Z}v\left(\frac{u_0}{t_0}\right)\mathrm{sgn}(C)\eta\gamma' - e^{\frac{3}{2}\eta\beta Z}v\left(\frac{u_0}{t_0}\right)^2\beta_2+\frac{1}{2}e^{2\eta\beta Z}v^3\left(\frac{u_0}{t_0}\right)^2\beta_2\right\}Z\right]\right]e^{\eta\beta Z} \qquad (96)\end{aligned}$$

$$v^2 = \frac{v_0^2}{1+8/15\bar{\alpha}u_0^3\mathrm{sgn}(C)x} \qquad (97)$$

This subsection changed our view of fluctuation to a small-amplitude single dark soliton. Based on the mathematical framework so far, the next subsection describes a new phenomenon, emergence of critical field amplitude for oscillatory wave behavior.

3.4 Theory of critical field amplitude

We substitute equation 98 into equation 84 to search for a oscillatory solution.

$$a_0(\tau, y) = a_0(\zeta) = a_0(\tau - Wy) \tag{98}$$

The physical meaning of this substitution is as follows: (y, τ) is a second retarded frame in a sense that it is a retarded frame of an already retarded frame (x, t). In other words, to describe fluctuation we have adopted a frame moving respect to a background wave with a speed of leading order phase velocity $1/C$. Fluctuation is superimposed on the background, and the above substitution is an ansatz for oscillatory waves superimposed on the that fluctuation.

$$\text{Background wave} \underset{\text{second retardation}}{\overset{\text{superimposed on}}{\rightleftharpoons}} \text{Fluctuation} \underset{\text{wavelike solution}}{\overset{\text{superimposed on}}{\rightleftharpoons}} \text{oscillatory wave}$$

Resulting equation has several terms including $a_0^{(3)}$. For simplicity, integrate both sides with respect to ζ' with integration range $[-\infty, \zeta]$. It is important to impose a boundary condition $a_0(\zeta \to -\infty) \to 0$ to respect our perturbation assumption, $|a_0| \ll u_0$. Also, write a_0 in the form of a_0/u_0 to observe only fractional amplitude.

$$\begin{aligned}\left(\frac{a_0}{u_0}\right)'' &= -8|W|\mathrm{sgn}(CW)u_0 e^{\bar{\beta}x}\left(\frac{a_0}{u_0}\right) + 12(u_0 e^{\bar{\beta}x})^2\left(\frac{a_0}{u_0}\right)^2 \\ &\quad - 4\left[2\mathrm{sgn}(C)u_0 e^{\bar{\beta}x}\frac{\bar{\delta}}{\epsilon} + (u_0 e^{\bar{\beta}x})^2\frac{\bar{\alpha}}{\epsilon}\right]\left(\frac{a_0}{u_0}\right)' + 16\mathrm{sgn}(C)(u_0 e^{\bar{\beta}x})^3\frac{\bar{\delta}}{\epsilon}\int_{-\infty}^{\zeta}\left(\frac{a_0(\zeta')}{u_0}\right)^2 d\zeta'\end{aligned} \tag{99}$$

As mentioned in [13], this equation can be viewed as a particle in the potential, albeit it contains a non-local term that depends on a history of the trajectory. Defining the fractional amplitude X, the first two terms in right-hand side are $-\partial U/\partial X$ where U is equation 100. The third term depends on X' and so is a linear friction term.

$$U = 4|W|\mathrm{sgn}(CW)u_0 e^{\bar{\beta}x}X^2 - 4(u_0 e^{\bar{\beta}x})^2 X^3 \tag{100}$$

Depending on the sign of CW, the 'potential' has different characteristics. Although we have not decided value of W, nor the sign of C and exponential growth, for moderate range of coefficients the potential would look like Fig. 1. The initial 'particle' at the center would oscillate in the negative a_0 region for negative sign, but around the center for positive sign. Taking the friction term into account, depending on the strength of friction, the 'particle' may undergo different types of damping towards the potential minima.

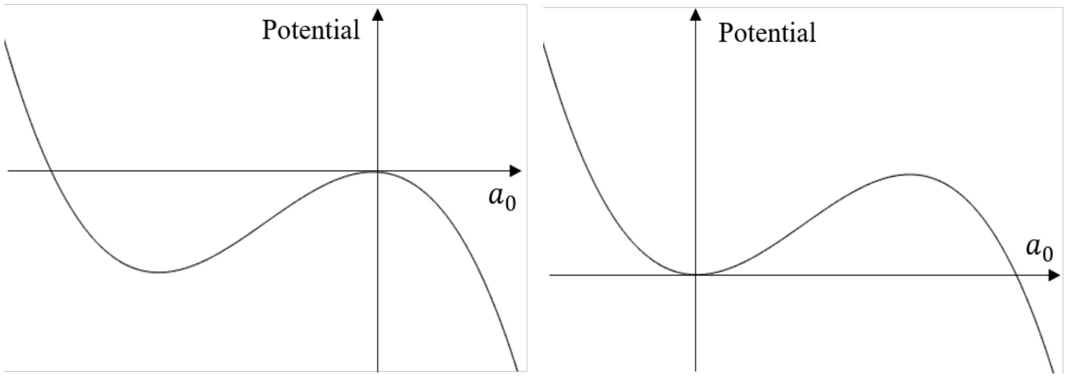

Figure 1: Potential U for two cases of sgn(CW). Left: sgn$=-1$. Right: sgn$=+1$.

However, the non-local term cannot be neglected and may give rise to peculiar particle behavior. This term has the greatest, cubic dependence on the field amplitude, and the particle's asymptotic position will strongly depend on it. Although the asymptotic behavior is not of an interest since before then the fluctuation assumption would break, the term still cannot be neglected. From the discussion of subsection 3.1, equation 99 is obviously independent of choice t_0. In fact, the t_0 dimensions are $[\zeta] = -1$, $[u_0] = 1$, $[W] = 1$, $[\bar{\delta}] = 0$, $[\bar{\alpha}] = -1$ that the equation has a consistent t_0 dimension of 2. Therefore the choice of t_0 does not modify any physics; rather, it only determines the time scale of $X(\zeta)$. Physical version equation is:

$$\begin{aligned} \frac{X''}{t_0^2} &= -\frac{\partial}{\partial X}\left[4\frac{|W|}{t_0}\mathrm{sgn}(CW)\left(\frac{\gamma}{\beta_2}\right)^{1/2} A_0 X^2 - 4\left(\frac{\gamma}{\beta_2}\right) A_0^2 X^3\right] \\ &- 4\left[2\mathrm{sgn}(C)\left(\frac{\gamma}{\beta_2}\right)^{1/2} A_0\frac{\eta\delta}{\beta_2} + \left(\frac{\gamma}{\beta_2}\right) A_0^2\alpha\right]\frac{X'}{t_0} + 16\mathrm{sgn}(C)\left(\frac{\gamma}{\beta_2}\right)^{3/2} A_0^3\frac{\eta\delta}{\beta_2}\int_{-\infty}^{\zeta} X^2 t_0 d\zeta' \end{aligned} \tag{101}$$

where $A_0 = |A(Z_0, T)|\exp(\eta\beta Z)$. Physical values are presented in equation 65, γ is nonlinear coefficient. Particle displays a variety of motion depending on the background amplitude A_0 and phase velocity $|W|$. We first investigate the friction term. Given $\mathrm{sgn}(C) = -1$, friction term's sign changes upon critical field amplitude $A_{0,critical1}$, equation 102. Below critical field amplitude, the coefficient is positive: particle's oscillation experiences growth rather than damping. Above it, the coefficient turns negative, thus particle's oscillation undergoes damping. The maximum positive coefficient is at $A_{0,critical1}/2$. The critical amplitude grows for larger $\eta\delta$ and smaller α i.e.

greater optical gain and small nonlinearity. For such oscillation to flourish, the critical amplitude should be higher: nonlinear effect should be smaller and optical gain be higher. However, small nonlinear effect and high optical gain generally incurs larger field amplitude. Therefore it would be difficult to reverse-engineer the critical nonlinearity strength and (unsaturated) optical gain for oscillatory wave behavior, while it is possible to explain the onset and cessation of oscillatory wave behavior at certain critical amplitude.

$$A_{0,critical1} = 2\left(\frac{1}{\gamma\beta_2}\right)^{1/2}\frac{\eta\delta}{\alpha} = (2.83\text{kW})^{1/2} \qquad (\text{parameters: } g_0 = 30\text{dB/m},\ n_{sat} = 0.4) \quad (102)$$

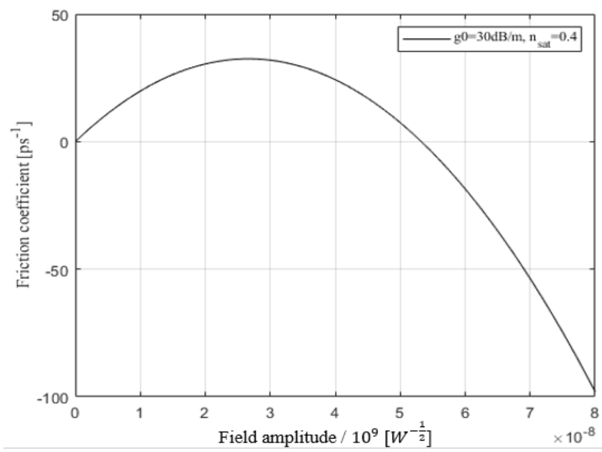

Figure 2: Friction coefficient as a function of background field amplitude.

To accurately simulate evolution of the fractional amplitude X, we should determine phase velocity W. W governs 'potential' shape, and first entered into formulas as a variable in equation 98. One way of estimating W comes from numerical simulation results. Equation 71 presents background wave's inverse phase velocity as $dt/dx = |u_0| = |A|t_0(\gamma/\beta_2)^{1/2}$ (dimensionless). According to transformation rules, $dT/dZ = (\beta_2/t_0)(dt/dx) = |A|(\beta_2\gamma)^{1/2}$ (physical). Therefore, fluctuation (situated in the middle of 99) moves -0.7ps/m for background amplitude $(2.5\text{kW})^{1/2}$. However, oscillating waves moved $0.3 \sim 0.4$ps/m in $t > 0$ region, $-0.1 \sim -2.0$ps/m in $t < 0$ region in numerical simulations. This implies the $|W|$ lies between scale of ± 1ps/m (physical), ± 200 (dimensionless, choosing $t_0 = 1$ps).

3.5 Perturbation to all orders

In this section, perturbation is expanded to all orders and possible modification to the governing equation is investigated. First, we look at higher-than-second order dispersions.

$$\text{(higher order dispersions)} = \frac{\beta_3}{6}\frac{\partial^3 A}{\partial T^3} + \frac{i\beta_4}{24}\frac{\partial^4 A}{\partial T^4} + ... \xrightarrow[dimensionless]{rescaling} \frac{i\beta_3}{6\beta_2 t_0}\frac{\partial^3 u}{\partial t^3} - \frac{\beta_4}{24\beta_2 t_0^2}\frac{\partial^4 u}{\partial t^4} + ...$$
$$\frac{\partial^n u}{\partial t^n} = \exp\left[\bar{\beta}x + i\int^x u_0^2(\exp(2\bar{\beta}x') - 1)dx'\right]\frac{\partial^n A}{\partial t^n} \xrightarrow[eliminate]{exp} \frac{\partial^n A}{\partial t^n} \qquad (n = 3, 4) \tag{103}$$

Taylor expansion for the third order dispersion is as follows.

$$i\frac{\partial^3}{\partial t^3}(u_0 + a)\exp(i(u_0^2 x + \phi)) = i\left[\frac{\partial^3 a}{\partial t^3} - 3\frac{\partial a}{\partial t}\left(\frac{\partial \phi}{\partial t}\right)^2 - 3(u_0 + a)\frac{\partial \phi}{\partial t}\frac{\partial^2 \phi}{\partial t^2}\right.$$
$$\left. + 3i\frac{\partial^2 a}{\partial t^2}\frac{\partial \phi}{\partial t} + 3i\frac{\partial a}{\partial t}\frac{\partial^2 \phi}{\partial t^2} - i(u_0 + a)\left(\frac{\partial \phi}{\partial t}\right)^3 + i(u_0 + a)\frac{\partial^3 \phi}{\partial t^3}\right]\exp(i(u_0^2 x + \phi)) \tag{104}$$

Real and imaginary parts are series expanded as in equation 76, and all orders higher than 5(for imaginary) and 4(for real) are truncated since we only manipulate two lowest order equations to derive the governing equation for leading amplitude fluctuation.

$$\text{(Imaginary)} = \epsilon^5\left[\frac{\partial^3 a_0}{\partial \tau^3} - 3u_0\frac{\partial \phi_0}{\partial \tau}\frac{\partial^2 \phi_0}{\partial \tau^2}\right] + \text{H.O.T.} \qquad \text{(Real)} = \epsilon^4\left[-u_0\frac{\partial^3 \phi_0}{\partial \tau^3}\right] + \text{H.O.T.} \tag{105}$$

Series expansion for the fourth order dispersion is a derivative of equation 104 after dropping out terms that give no contribution to equation 105. The omission does not affect consequences since only two lowest order equations are important. It can be readily found out that a single term comes out. There are no low order contributions from higher-than-fourth order dispersions.

$$\frac{\partial^4 u}{\partial t^4} \rightarrow \text{(Imaginary)} = \epsilon^5\left[u_0\frac{\partial^4 \phi_0}{\partial \tau^4}\right] + \text{H.O.T.} \qquad \text{(Real)} = 0 + \text{H.O.T.} \tag{106}$$

We may also investigate extra terms coming form second, third, ... Taylor expansions of Raman response (equation 48) and third, fourth, ... Taylor expansions of optical gain term (equation

56) by the same manner. For Raman response, (refer to equation 62)

$$
\begin{aligned}
&i\gamma A \int_{-\infty}^{+\infty} R(T')|A(T-T')|^2 dT' \xrightarrow[dimensionless]{rescaling} -u|u|^2 + \frac{f_R}{t_0}\int_{-\infty}^{+\infty} t_0 t' h_R(t_0 t') t_0 dt' \cdot u\frac{\partial}{\partial t}|u|^2 \\
&-\frac{f_R}{2t_0^2}\int_{-\infty}^{+\infty} (t_0 t')^2 h_R(t_0 t') t_0 dt' \cdot u\frac{\partial^2}{\partial t^2}|u|^2 + \frac{f_R}{6t_0^3}\int_{-\infty}^{+\infty} (t_0 t)'^3 h_R(t_0 t') t_0 dt' \cdot u\frac{\partial^3}{\partial t^3}|u|^2 + \text{H.O.T.} \\
&\equiv -u|u|^2 + \bar{\alpha}\cdot u\frac{\partial}{\partial t}|u|^2 + \bar{\alpha'}\cdot u\frac{\partial^2}{\partial t^2}|u|^2 + \bar{\alpha''}\cdot u\frac{\partial^3}{\partial t^3}|u|^2 + \text{H.O.T.} \qquad (107)
\end{aligned}
$$

Former two terms of right-hand side are already included in equation 64; latter three give higher order Taylor terms. However, it can be seen that $u(\partial^3/\partial t^3)|u|^2$ term's lowest order is ϵ^5, which is not of our interest since all $\epsilon^{n>4}$ terms are truncated for the real part. Only contribution:

$$
\begin{aligned}
&u\frac{\partial^2}{\partial t^2}|u|^2 = (u_0+a)\left[2\left(\frac{\partial a}{\partial t}\right)^2 + 2(u_0+a)\frac{\partial^2 a}{\partial t^2}\right]\exp(i(u_0^2 x+\phi)) \\
&\bar{\alpha'}\cdot u\frac{\partial^2}{\partial t^2}|u|^2 \rightarrow (\text{Imaginary}) = 0 \qquad (\text{Real}) = \epsilon^4\left[2\bar{\alpha'}u_0^2\frac{\partial^2 a_0}{\partial \tau^2}\right] + \text{H.O.T.} \qquad (108)
\end{aligned}
$$

Finally, for the optical gain term (refer to equation 56), higher order Taylor terms are time derivatives of amplitude, as are higher order dispersions. Taylor terms higher than the fourth order gives no contribution. Now equations 78 and 80 are modified using equation 103, 105, 106, 108; subsequent procedures yield a modified KdV equation for leading amplitude fluctuation a_0 with extra terms. The equation is, to the best of author's knowledge, the first fully analytic approach for the combination of optical gain and nonlinear response.

4 Conclusion

In this dissertation, the theoretical framework for analysis of gain-embedded highly nonlinear optical fiber was developed. Terms other than second order dispersion and instantaneous nonlinear response were treated as perturbations. These include optical gain, higher order dispersions, and Raman delayed response. Perturbation analysis is the starting point to correctly describe the developing dark solitons and dark soliton generation criterion.

Step-by-step approach was taken. In the second section, background theories, including MBE and periodic pulse modeling were introduced to obtain the analytic form for frequency dependent optical gain. In the third section, first, perturbation terms were rigorously included into the most simple form of NLSE via Taylor series. Then, several field redefinition and rescaling steps were taken to transform the equation into a mathematically tractable form. Perturbation expansion was used to derive the governing equation for leading amplitude of the fluctuation. Finally, perturbation terms coming from higher order Taylor terms were considered.

There are several possible developments stemming from this framework. Two most obvious researches are identification of fluctuation as a dark soliton, and a search for oscillating wave solutions supported by the equation of amplitude fluctuation. These future researches would provide a rich physics of the combination of optical and Raman gain. I hope the complete description for dark soliton generation comes from more elaborate perturbational analysis.

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