Mastermind with n colors

Joseph MAROTTE, Lucas PESENTI

1 Introduction

1.1 Problem definition

In the Mastermind problem with n colors, there's a hidden secret $z \in \{1, \dots, n\}^n$ and we want to recover it by asking queries of the form:

- *input*: an element $x \in \{1, \dots, n\}^n$
- *output*: the number of positions on which x and z coincide

The goal is to find the secret while minimizing the number of queries. We analyze the blackbox complexity of this problem both from a theoretical and experimental perspective.

1.2 Algorithms

We implemented the following algorithms:

- Exhaustive Search
- Randomized Local Search
- $(\mu + \lambda)$-EA, (μ, λ)-EA[1]
- $(\mu + \lambda)$-GA, (μ, λ)-GA
- $1 + (\lambda, \lambda)$-GA
- An Erdős-Renyi-type approach

We provide pseudocodes for these algorithms at the end of this report. The implementation is available on `https://github.com/JosephMarotte/MPRI-OptimizationHeuristics-Project/`.

Remark. We initially planned to implement the algorithm with black-box complexity $O(n \log \log n)$ described in [5]. The main issue of this approach from a testing point of view is the time complexity. Apart from the fact that $f(m)$ is not explicit, there are two bottlenecks:

- checking whether the queries determine the 0-blocks at the end of the coin weighing part
- the final Erdős-Renyi part when k' is too small

Note that following [1] the coin weighing part can be done in polynomial time by constructing explicit detecting matrices[2] using Fourier analysis. In that case, the black-box complexity would be $O(n \log \log n)$ and the time complexity would be the same up to *polynomial* factors. Moreover,

[1] Note that for example we didn't analyze (μ, λ)-EA in practice, but it is straightforward to create new data from our implementation – we only need to change the selection method, which is a parameter of our generic evolutionary implementation.

[2] Matrices whose corresping linear transform is injective when we restrict it to the product set of the "interesting" integers what we need.

in the analysis of [5] we can see that we only need $k' = \omega(\log n)$ to run the main loop. So if we stop at $(\log n)^{1+\delta}$ instead of $\sqrt{n}$ for some $\delta > 0$, we would get a quasipolynomial time algorithm with $O(n \log\log n)$ black-box complexity for Mastermind with n colors. Notice that assuming SETH, this is faster than any approach based purely on Erdős-Renyi because the corresponding problem is NP-complete [6].

However, we couldn't understand the explicit construction of detecting matrices in [1][3]. The construction of search matrices is cleaner, so we unsuccessfully tried to bypass detecting matrices by reducing the coin weighing problem that we specifically need in Mastermind to search matrices instead (e.g. using trick mentioned in the section 2.2 of [2]). Note that [7] provides construction of detecting matrices using the probabilistic method, but as mentioned above, the "decoding" part cannot be done in polynomial time.

2 Theoretical results

Here we prove tight lower bounds and upper bounds for randomized local search (RLS), almost tight lower bounds and upper bounds for $(1+1)$-EA and an upper bound for the Erdős-Renyi approach. Pseudocodes of these algorithms are available in the appendix.

Notation.

- log denotes the logarithm in base e.
- $[n] \stackrel{\text{def}}{=} \{1, \dots, n\}$
- $H_n \stackrel{\text{def}}{=} \sum_{i=1}^{n} \frac{1}{i}$

Definition (MM_n)**.** As all our algorithms will be unbiased, we consider the objective function $f : [n]^n \to \mathbf{R}_+$ defined by:

$$f(x) \stackrel{\text{def}}{=} \sum_{i=1}^{n} \mathbf{1}_{x_i=0}$$

Remark. For RLS, we initialize $x \in_R [n]^n$ and at each step we choose an index $i \in_R [n]$ and a shift $s \in_R [n-1]$.

Theorem.

$$\mathbf{E}[\mathcal{T}(RLS, MM_n)] \underset{n\to\infty}{\sim} n^2 \log n$$

Proof. Let $x^0, \dots x^t$ be the successive queries of RLS. Let $T_i \stackrel{\text{def}}{=} \min\{t \mid f(x^t) \geq i\}$. Note that for all $i \geq i_0$, the variable $T_{i+1} - T_i$ conditioned on $f(x^0) = i_0$ follows a geometric distribution with parameter $\frac{n-i}{n}\frac{1}{n-1}$. So:

$$\mathbf{E}[T_{i+1} - T_i \mid f(x^0) = i_0] = \frac{n(n-1)}{n-i}$$

And it follows that:

$$\mathbf{E}[T_n \mid f(x^0) = i_0] = n(n-1) \sum_{i=i_0}^{n-1} \frac{1}{n-i} = n(n-1)H_{n-i_0}$$

[3]More precisely, in the proof of Theorem 4, why wouldn't the sum of all l_a be smaller or bigger than $n - @^{\nu}$? It doesn't seem like it's always the case, and this is confirmed by our implementation of what we thought was the algorithm described there.

If $c_n \stackrel{\text{def}}{=} \sqrt{n \log n}$, by additive Chernoff bounds:

$$\mathbf{P}[f(x^0) > 1 + c_n] \leq \exp(-2 \log n) = \frac{1}{n^2}$$

So:

$$\mathbf{E}[T_n] = n(n-1) \sum_{i_0 \leq 1 + c_n} H_{n-i_0} \mathbf{P}[f(x^0) = i_0] + o(1)$$

Finally:

$$\left(1 - \frac{1}{n^2}\right) \log(n - 1 - c_n) \leq \sum_{i_0 \leq 1 + c_n} H_{n-i_0} \mathbf{P}[f(x^0) = i_0] \leq 1 + \log n$$

Hence $\mathbf{E}[T_n] \underset{n \to \infty}{\sim} n^2 \log n$. □

Remark. For $(1+1)$-EA, we consider $p = \frac{1}{n}$.

Theorem.

$$\mathbf{E}[\mathcal{T}((1+1)\text{-}EA, MM_n)] \leq en^2(\log n + 1)$$

Proof. We apply the fitness local method. Let x be the current state of the variable and y the transformed one.

$$\mathbf{P}[f(y) > i \mid f(x) = i] \geq (n - i) \left(1 - \frac{1}{n}\right)^{n-1} \frac{1}{n^2} \geq \frac{n - i}{en^2}$$

So:

$$\mathbf{E}[\mathcal{T}((1+1)\text{-EA}, \text{MM}_n)] \leq en^2 H_n \leq en^2(\log n + 1)$$

□

Theorem.

$$\mathbf{E}[\mathcal{T}((1+1)\text{-}EA, MM_n)] \geq n^2 \log n + o(n^2 \log n)$$

Proof. Let $X_{j,t}$ be the indicator of the event "the j-th position is incorrect in x^0 and zero has never been drawn out for this position in the first t iterations". Let:

$$t_n \stackrel{\text{def}}{=} \left(1 - \frac{\log \log n}{\log n}\right) (n^2 - 1) \log n$$

Then at this time:

$$\mathbf{P}[X_{j,t_n} = 1] = \left(1 - \frac{1}{n}\right) \left(1 - \frac{1}{n^2}\right)^{\left(1 - \frac{\log \log n}{\log n}\right)(n^2 - 1) \log n} \geq \frac{\log n}{2n}$$

We deduce:

$$\mathbf{E}\left[\sum_{j=1}^{n} X_{j,t_n}\right] \geq \frac{\log n}{2}$$

And by multiplicative Chernoff bounds:

$$\mathbf{P}\left[\sum_{j=1}^{n} X_{j,t_n} \leq \frac{\log n}{4}\right] \leq \exp\left(-\frac{\log n}{16}\right) = o(1)$$

If T denotes the time at which we find the optimal solution:

$$\mathbf{P}[T \leq t_n] \leq \mathbf{P}\left[\sum_{j=1}^{n} X_{j,t_n} \leq \frac{\log n}{4}\right] = o(1)$$

In the end:

$$\mathbf{E}[T] \geq \mathbf{P}[T > t_n]t_n = (1 - o(1))\left(1 - \frac{\log\log n}{\log n}\right)(n^2 - 1)\log n \sim n^2 \log n$$

□

Remark. We are in a very different regime from what happens for $f : \{0,1\}^{n \log_2 n} \to \mathbb{R}_+$. Getting yes/no answers from groups of $\log_2 n$ pieces together somehow results in an additional n factor in the efficiency of $(1+1)$-EA.

We now give an upper bound for Erdős-Renyi method:

Theorem.

$$\mathbf{E}[\mathcal{T}(ER, MM_n)] \leq 2en(\log n + 1)$$

Proof. The probability that the color $c > 0$ at position $1 \leq p \leq n$ was not chosen in any sample x^i such that $f(x^i) = 0$ and $1 \leq i \leq t$ is upper bounded by:

$$\left(1 - \left(1 - \frac{1}{n}\right)^{n-1}\frac{1}{n}\right)^t \leq \exp\left(-t\left(1 - \frac{1}{n}\right)^{n-1}\frac{1}{n}\right) \leq \exp\left(-\frac{t}{en}\right)$$

Let T be the first point in time when all nonzero colors at every position have been chosen in a sample that evaluated to zero. Then by a union bound:

$$\mathbf{P}[T \geq t] \leq n^2 \exp\left(-\frac{t}{en}\right)$$

Let $t_n \stackrel{\text{def}}{=} 2en \log n$. Then:

$$\mathbf{E}[T] \leq \sum_{t=1}^{t_n} \mathbf{P}[T \geq t] + \sum_{t > t_n} \mathbf{P}[T \geq t] \leq t_n + n^2 \frac{e^{-\frac{t_n}{en}}}{1 - e^{-\frac{1}{en}}} = t_n + \frac{1}{1 - e^{\frac{1}{en}}} \leq 2en\log n + 2en$$

□

3 Experimental results

The implementation is in Python. We implemented the following abstractions in the `src/` directory:

- `blackbox.py`: act as a black-box: create an instance of Mastermind and handles the queries
- `erdos_renyi_like.py`: implementation of the Erdős-Renyi approach
- `evolutionary_algorithm_abstract.py`: generic abstraction for all evolutionary and genetic algorithms
- `log_log.py`: unsuccessful attempt to implement the construction of detecting matrices
- `mastermind_problem.py`: generic abstraction for all heuristics solving Mastermind
- `mu_plus_lambda_evolution_strategy.py` and others, `random_local_search.py`: implementation of the corresponding heuristics
- `selection_functions.py`: implementation of selection functions, two types corresponding either to (μ, λ) or $(\mu + \lambda)$
- `step_functions.py`: implementation of the selection of the next color at a given position
- `utils.py`: common utility functions for all heuristics

For each of the analyzed algorithms, we indicated the size of the instance, some quantiles, the mean, the standard deviation over all experiments and the number of experiments.

3.1 Exhaustive search

size	per 0.02	per 0.25	per 0.5	per 0.75	per 0.98	mean	std dev	count
2	1.0	1	2	3	4.0	2.378	1.134	45
4	11.52	76	164	199	241.24	141.6	68.785	45
8	1255927.0	6163844	7771397	13098828	14896442.88	8632855.889	4858003.181	9

For the corresponding sizes, the black-box complexity is roughly what we expect, namely $n^n/2$.

3.2 Erdős-Renyi

size	per 0.02	per 0.25	per 0.5	per 0.75	per 0.98	mean	std dev	count
2	1.0	1	2	3	5	2.3	1.317	30
4	3.58	5	6	7	9	6.033	1.45	30
8	17.0	17	17	17	17	17.0		1

As we mentioned at the beginning, it is difficult to analyze experimentally Erdős-Renyi-based approaches, because even though the black-box complexity is small ($\tilde{O}(n)$ according to our theoretical results), the time complexity is exponential, so we can't compare it to the heuristics for n sufficiently large.

3.3 Randomized local search

Remark. There are have several ways to extend randomized local search (and other evolution-based heuristics), depending on how we choose to change the color at each position. This corresponds to the notion of *step function* introduced in [4]. In the setting of the article, it makes sense not to take necessarily the new color uniformly at random because the objective function considered depends on the distance between the colors at each position. For Mastermind with n colors,

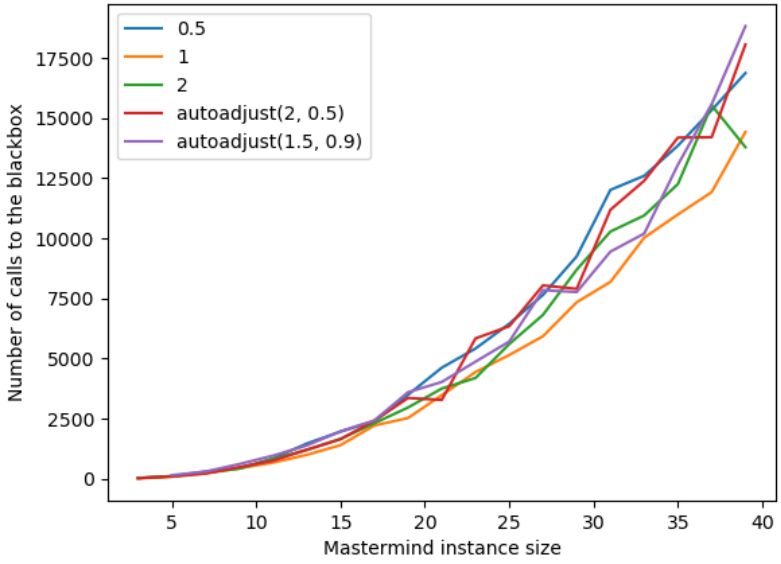
it doesn't make any difference (except for the unit mutation strategy, which actually performs worse), so we pick the new colour uniformly at random in the following experiments.

size	per 0.02	per 0.25	per 0.5	per 0.75	per 0.98	mean	std dev	count
2	1.0	2.0	3	4.0	5.0	2.8	1.373	15
4	5.68	10.5	13	24.5	41.72	18.4	11.667	15
8	67.12	109.5	159	215.0	285.8	161.733	70.673	15
16	501.0	714.5	848	981.5	1524.88	883.0	288.114	15
32	2078.4	4093.0	4276	4723.5	6293.48	4250.667	1129.859	15
64	13247.56	14633.5	20937	24208.0	27993.96	19967.133	5379.733	15
128	66483.76	75320.5	83684	87236.5	117226.24	84380.933	14373.753	15
256	313078.72	358965.0	382312	417418.0	494008.96	394791.8	56323.934	15

For large sizes, it seems to confirm the theoretical results of $n^2 \log n$ (recall that we talk about natural logarithm here).

3.4 The $(1+1)$-Evolutionary Algorithm

First, we tried to experimentally find the best choice of mutation rate for the $(1+1)$-EA. In the following figure, each curve corresponds to the black-box complexity of some strategy for the choice of mutation rate. We consider 3 different static mutation rates ($0.5/n$, $1/n$, $2/n$)) and two different autoadjusting mutation rates: in the strategy autoadjust(A, b), if the fitness of the offspring is better than the fitness of the original population, we multiply the mutation rate by A, otherwise we multiply it by b.



Overall it seems that the best strategy among those that we tested is to pick the mutation rate $p = 1/n$, so that's what we will do in all the following experiments.

size	mu	lambda	mutation rate	per 0.02	per 0.25	per 0.5	per 0.75	per 0.98	mean	std dev	count
2	1	1	0.5	1.0	1.0	3	3.5	15.32	4.067	4.667	15
4	1	1	0.25	9.68	19.5	34	40.5	74.76	33.333	19.059	15
8	1	1	0.125	122.24	186.5	234	318.5	571.32	288.333	155.646	15
16	1	1	0.062	1307.76	1663.0	2269	3031.0	4954.72	2499.933	1102.725	15
32	1	1	0.031	5534.24	7497.0	8427	11231.5	16616.04	9440.4	3299.962	15
64	1	1	0.016	28051.6	37844.5	41506	46798.5	70748.48	44194.533	12229.526	15
128	1	1	0.008	143334.24	160700.5	211083	225974.0	253009.92	197100.333	39450.459	15
256	1	1	0.004	867953.0	867953.0	867953	867953.0	867953.0	867953.0		1

It seems that the black-box complexity is a little bit better than our theoretical upper bound of $2en^2 \log n$, so the constant that we found is probably not tight (but it looks like it is on the right scale).

3.5 $(\mu + \lambda)$-Genetic Algorithm

size	mu	lambda	mutation rate	per 0.02	per 0.25	per 0.5	per 0.75	per 0.98	mean	std dev	count
2	2	1	0.5	2.0	2.5	3	4.5	15.48	4.667	4.135	15
2	10	6	0.5	10.0	13.0	16	16.0	16.0	14.4	2.746	15
4	2	1	0.25	18.56	23.0	27	45.5	72.56	35.267	17.392	15
4	10	6	0.25	17.68	40.0	46	61.0	78.64	48.4	17.683	15
8	2	1	0.125	122.56	134.0	177	326.0	481.64	242.933	133.195	15
8	10	6	0.125	139.36	172.0	184	241.0	403.12	217.6	79.074	15
16	2	1	0.062	839.8	963.0	1342	2351.5	2625.92	1552.2	729.231	15
16	10	6	0.062	526.72	793.0	1018	1216.0	1624.48	1033.6	334.557	15
32	2	1	0.031	3719.64	4577.5	5345	7144.0	13422.76	6537.533	3094.985	15
32	10	6	0.031	3300.64	4024.0	5026	6439.0	9895.12	5548.4	2015.521	15
64	2	1	0.016	20285.48	28036.5	30694	31764.0	42031.84	30862.267	6039.868	15
64	10	6	0.016	20412.64	22060.0	25858	30442.0	59236.72	30342.4	12771.794	15
128	2	1	0.008	168539.0	168539.0	168539	168539.0	168539.0	168539.0		1
128	10	6	0.008	125638.0	125638.0	125638	125638.0	125638.0	125638.0		1

Having a bigger value for μ and λ doesn't seem to yield significantly better results.

Values for μ and λ are not easy to choose. In some cases, I believe having a smaller value for μ speeds up the algorithm but may induce good assignments of a cell being forgotten (for example, 1 good cell change to be wrong, but 2 wrong changes to be good). Having a larger value for μ means older value are not immediately deleted, so in the same case, the information of the good cell which was changed may be retrieved, but it means in some cases only not so good will be used in the majority vote of A and the generated offspring may not be the best because of that.

I believe the $1 + (\lambda, \lambda)GA$ we present now is a nice compromise.

3.6 $1 + (\lambda, \lambda)$-Genetic Algorithm

Finally we tried to adapt heuristically the $1 + (\lambda, \lambda)$-GA of [3]. In the pseudocode from the appendix, F is the update strength, c is the crossover rate and is equal to $1/\lambda$, and the mutation rate is equal to λ/n. The function $\text{cross}_c(x, y)$ outputs an array z such that $z_i = x_i$ with probability c and $z_i = y_i$ otherwise. In the adaptive version, c and the mutate rate change with the value of λ. In the non-adaptive version, they remain the same.

In the non-adaptive version, having a large λ doesn't seem to yield significantly better results than having a smaller one (keep in mind the number of experiment is equal to 15).

The adaptive case has better results. I believe the reason is that at all stages of the algorithm, the value of λ is changed to have steps with better performances: a small λ at the early stage of the algorithm because when most of the cells are different, changing only a few cells mean one of them is probably wrong and hence we have good chance of finding the real value of a cell. In the late stage of the algorithm, I believe the value of λ increases, hence we have more mutation and more offspring. Among these offsprings, the best one X is probably a poor choice in itself (because a lot of correct values of S were mutated) and the cross function will have a small value for c so we are almost ensured to keep the good values of the initial S and only change the new one found by X.

size	lambda	mutation rate	crossover probability	per 0.02	per 0.25	per 0.5	per 0.75	per 0.98	mean	std dev	count
2	4	1.0	0.25	1.0	5	9	17	36.52	12.2	10.818	15
2	8	1.0	0.125	1.0	1	17	41	95.08	27.667	30.035	15
2	12	1.0	0.083	1.0	25	25	61	97.0	42.6	33.288	15
4	4	1.0	0.25	17.0	25	41	49	99.56	43.133	24.465	15
4	8	1.0	0.125	33.0	49	65	81	124.52	69.267	29.913	15
4	12	1.0	0.083	49.0	61	73	73	155.56	76.2	31.248	15
8	4	0.5	0.25	139.24	193	273	329	638.12	292.2	148.279	15
8	8	1.0	0.125	117.48	153	209	249	323.56	209.0	64.569	15
8	12	1.0	0.083	127.72	217	241	337	419.56	266.6	90.013	15
16	4	0.25	0.25	781.8	1429	1881	2201	3464.68	1943.933	794.782	15
16	8	0.5	0.125	1173.48	1393	1793	2433	4527.72	2122.6	1029.748	15
16	12	0.75	0.083	933.16	1105	1705	2233	3712.36	1852.2	861.323	15
32	4	0.125	0.25	5438.12	7933	10001	13081	14486.12	10324.733	2968.613	15
32	8	0.25	0.125	6578.92	7745	9057	9761	13406.44	9096.467	1974.149	15
32	12	0.375	0.083	5886.76	7537	9049	12325	17554.6	10114.6	3637.72	15
64	4	0.062	0.25	32453.8	39401	41993	56233	63748.52	46511.933	10436.636	15
64	8	0.125	0.125	35830.12	39417	40721	53049	60246.76	45486.867	8754.413	15
64	12	0.188	0.083	27965.8	40561	46993	52933	89598.76	49061.8	17192.228	15
128	4	0.031	0.25	162769.0	185665	201921	241385	362813.48	223105.0	60037.529	15
128	8	0.062	0.125	145112.68	190601	211649	248521	304951.4	217258.6	48584.418	15
128	12	0.094	0.083	147265.96	166501	221041	251521	313345.96	215831.4	53732.739	15
256	4	0.016	0.25	682281.0	682281	682281	682281	682281.0	682281.0		1
256	8	0.031	0.125	1205761.0	1205761	1205761	1205761	1205761.0	1205761.0		1
256	12	0.047	0.083	897889.0	897889	897889	897889	897889.0	897889.0		1

Figure 1: Non-adaptative version

size	update strength	per 0.02	per 0.25	per 0.5	per 0.75	per 0.98	mean	std dev	count
2	1.1	1.0	1	5	13	25.0	7.667	8.902	15
4	1.1	11.24	25	41	69	124.52	50.6	34.502	15
8	1.1	101.48	161	195	249	348.52	210.2	71.896	15
16	1.1	786.52	905	997	1257	1390.6	1066.067	215.396	15
32	1.1	4150.68	4546	4927	5245	6254.12	4921.933	629.403	15
64	1.1	17671.64	19474	21111	22398	22991.72	20761.4	1822.801	15
128	1.1	89014.68	90249	91869	93332	98975.72	92410.867	2985.324	15
256	1.1	373531.0	373531	373531	373531	373531.0	373531.0		1

Figure 2: Adaptative version

4 Conclusion

In conclusion, it seems that all evolutionary-based algorithms that we presented are in average close to randomized local search, whose theoretical guarantee is asymptotically $n^2 \log n$. Only the adaptative version of $1 + (\lambda, \lambda)$-GA finds the target in a smaller expected number of queries than RLS (but we conjecture that the theoretical gain is at most a constant factor).

Therefore, the "algebraic" approaches (Erdős-Renyi-based and $O(n \log \log n)$) are probably also much better in practice, but the time complexity is a serious bottleneck in order to test these against the other heuristics. It would have been interesting to see how the quasi-polynomial time implementation of [5] performs in terms of black-box complexity in practice.

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Appendix: pseudocodes

while *F(S) != N* **do**
  $S \leftarrow$ next array;
  Evaluate $F(S)$;
end

Algorithm 1: Exhaustive Search

$S \leftarrow$ random array
while *F(S) != N* **do**
  $i \leftarrow$ randint(0, N-1)
  $S' \leftarrow S$
  $S'[i] \leftarrow step()$
  if *F(S') > F(S)* **then**
    $S \leftarrow S'$
  end
end

Algorithm 3: Randomized Local Search

```
begin
    Pick a Set S of random array in {0, ..., N − 1}^N to Evaluate.
    foreach A in S do
        compute F(A)
    end
    Create an empty set of possible solution S′ foreach Possible Array A′ do
        foreach A in S do
            if distance(A, A′) != F(A) then
                A′ can't be solution
                Continue to next iteration step
            end
        end
        Add A′ to S′
    end
end
```

Algorithm 2: Erdős-Renyi Algorithm

```
S ← random array
while F(S) != N do
    S′ ← S
    for i = 0; i < N; i++ do
        if random() <= mutation_rate then
            S′[i] ← step()
        end
    end
    if F(S′) > F(S) then
        S ← S′
    end
end
```

Algorithm 4: $(1 + 1)$-EA

```
S ← Set of μ random array
while max(F(S)) != N do
    S′ ← empty Set
    for i = 0; i < λ; i++ do
        A ← majority_vote(S, k)
        for j = 0; j < N; j++ do
            if random() <= mutation_rate then
                A[i] ← step()
            end
            Add A to S′
        end
        S ← Elitist_Selection(S, S′)
    end
end
```

Algorithm 5: $(\mu + \lambda)$-GA

$S \leftarrow$ random array
while *F(S) !=* N **do**
  Mutation phase
  begin
    for $i = 0;\ i < \lambda;\ i++$ **do**
      $X_i \leftarrow S$ **for** $j = 0;\ j < N;\ j++$ **do**
        if *random()* <= *mutation_rate* **then**
          $X_i[j] \leftarrow step()$
        end
      end
      Compute(F(X_i))
    end
    $X \leftarrow X_i$ such that $f(X)$ is maximized
  end
  Crossover phase
  begin
    for $i = 0;\ i < \lambda;\ i++$ **do**
      for $j = 0;\ j < N;\ j++$ **do**
        $Y_i[j] \leftarrow cross_c(S, X)$
      end
      Compute(F(Y_i))
    end
    $Y \leftarrow Y_i$ such that $f(Y_i)$ is maximized
  end
  Optional Auto Adjustment Rate begin
    if *F(Y) > F(S)* **then**
      $\lambda \leftarrow \max(\lambda/F, 1)$
    end
    else
      $\lambda \leftarrow \min(\lambda F^{1/4}, N)$
    end
  end
  if *F(Y) > F(S)* **then**
    $S \leftarrow Y$
  end
end

Algorithm 6: $1 + (\lambda, \lambda)$-GA