CSCE 222 [Sections 502, 503] Discrete Structures for Computing
Spring 2017 – Hyunyoung Lee

Problem Set 6

Due dates: Electronic submission of *yourLastName-yourFirstName-hw6.tex* and *yourLastName-yourFirstName-hw6.pdf* files of this homework is due on **Monday, 3/20/2017 before 11:00 a.m.** on `http://ecampus.tamu.edu`. You will see two separate links to turn in the .tex file and the .pdf file separately. Please do not archive or compress the files. A signed paper copy of the pdf file is due on **Monday, 3/20/2017** *at the beginning of class*. **If any of the three submissions are missing, your work will not be graded.**

Name: Joseph Martinsen **Section: 503**

Resources. (All people, books, articles, web pages, etc. that have been consulted when producing your answers to this homework.)

On my honor, as an Aggie, I have neither given nor received any unauthorized aid on any portion of the academic work included in this assignment. Furthermore, I have disclosed all resources (people, books, web sites, etc.) that have been used to prepare this homework.

Signature: ____

Problem 1. (5 points) Section 5.1, Exercise 50, page 331

Solution.
In the basis step of this proof, there is an error.

$$\frac{(1+\frac{1}{2})^2}{2} = \left(\frac{3}{2}\right)^2 \frac{1}{2} = \frac{9}{8}$$

and

$$\sum_{i=1}^{1} i = 1$$

are not true or equal as stated.

Problem 2. (15 points) Section 5.2, Exercise 4, pages 341–342

Solution.

a)

<u>**Basis Step**</u>
$P(18) : 1 \cdot 4$ cent stamps and $2 \cdot 7$ cent stamps
$P(19) : 3 \cdot 4$ cent stamps and $1 \cdot 7$ cent stamps
$P(20) : 5 \cdot 4$ cent stamps and $0 \cdot 7$ cent stamps
$P(21) : 0 \cdot 4$ cent stamps and $3 \cdot 7$ cent stamps

b) The inductive hypothesis of this proof is as follows. Any j¢ postage stamp can be made from a combination of 4¢ and 7¢ stamps for all j stamps with $18 \leq k$. where $k \geq 21$.

c) In order to prove the inductive step, show that $k + 1$¢ stamps can be made from 4¢ and 7¢ stamps.

d) With $k \geq 21$, $P(k-3)$ has been shown to be true by the inductive hypothesis. Adding another 4¢ stamp, it must follow that $P(k + 1)$ is also true.

e) With the basis step and the inductive step shown to be valid, P has been proved to hold for $n \geq 21$ by strong induction.

Problem 3. (10 points) Section 5.2, Exercise 12, page 342

Solution.
Basis Step: Show $P(0), P(1), P(2), P(3)$

$$2^0 = 1$$
$$2^1 = 2$$
$$2^2 = 4$$
$$2^3 = 8$$

$\therefore P(0), P(1), P(2), P(3)$ holds
Inductive Step Let $k \geq 1$, assume the claim holds for all n where $1 \leq n \leq k$

Case 1: Assume $k + 1$ is even. Then $\dfrac{k+1}{2}$ is an integer between 1 and k. The claim holds for $\dfrac{k+1}{2}$ by the strong induction hypothesis.

$$\frac{k+1}{2} = 2m_1 + 2m_2 + \cdots + 2m_{\mathbb{L}} \qquad \textbf{By Strong I.H.}$$
$$\frac{k+1}{2} = 2(m_1 + 1) + 2(m_2 + 1) + \cdots + 2(m_{\mathbb{L}} + 1)$$

$\therefore$ **the statement holds for when $k + 1$ is even**

Case 2: Assume $k + 1$ is odd. By Strong Induction Hypothesis, the claim holds for k.

$$k = 2m_1 + 2m_2 + \cdots + 2m_{\mathbb{L}} \qquad \textbf{By Strong I.H.}$$
$$k + 1 = 1 + 2m_1 + 2m_2 + \cdots + 2m_{\mathbb{L}}$$
$$k + 1 = 2^0 + 2m_1 + 2m_2 + \cdots + 2m_{\mathbb{L}}$$

$\therefore$ **the statement holds for when $k + 1$ is odd**

$\therefore$ **the statement holds by proof by Strong Induction**

Problem 4. (10 points) Section 5.2, Exercise 30, page 344

Solution.
The basis step shows that $a^0 = 1$. The correct induction hypothesis would be $a^0 = 1$ not $a^j = 1$. The incorrect inductive hypothesis is used in this proof.

Problem 5. (30 points) Section 5.3, Exercise 6, page 357. To prove that your formula is valid, use mathematical induction. For the invalid recursive definitions, explain why they are invalid.

Solution.

a) $f(0) = 1 \quad f(1) = -1 \quad f(2) = 1$. It is apparent that the closed form solution is $f(n) = (-1)^n$
Basis Step: Show $P(1)$

$$P(1) = f(1) = -f(0) = -1$$
$$P(1) = f(1) = (-1)^1 = -1$$

$\therefore P(2)$ holds
Inductive Step: Show $P(k) \rightarrow P(k+1)$
Assume $P(k)$ **for arbitrary** $k \geq 1$: $f(k) = (-1)^k$
Show P(k+1):

$$\begin{aligned} P(k+1) &= -f(k+1-1) \\ &= (-1)f(k) \\ &= (-1)^1(-1)^k && \textbf{By Inductive Hypothesis} \\ &= (-1)^{k+1} \end{aligned}$$

$\therefore P(k) \rightarrow P(k+1)$ **holds**
$\therefore$ **the statement holds for all** $n \geq 1$ **by mathematical induction**

b) $f(0) = 1 \quad f(1) = 0 \quad f(2) = 2 \quad f(3) = 2 \quad f(4) = 0 \quad f(5) = 4$
2^n

c) It is not well defined because each $n \geq 2$ is dependent on the the value of $f(n+1)$ which is unknown.

d) It is not well defined because the base case $f(1)$ is given as $f(1) = 1$ but also for $n \geq 1$ it is also given that $f(1) = 2f(1-1) = 2f(0) = 0$. Because of 2 different definitions for the same value, this is not well defined.

e) $f(0) = 2 \quad f(1) = f(0) = 2 \quad f(2) = 2f(0) = 4 \quad f(3) = f(2) = 4 \quad f(4) = 2f(3) = 8 \quad f(5) = 8 \quad f(6) = 16 \quad f(7) = 16$. A closed form solution is given by $f(n) = 2^{\lfloor (n+1)/2 \rfloor}$ **Basis Step: Show** $P(1)$

$$P(1) = f(1) = f(0) = 2$$
$$P(1) = f(1) = 2^{\lfloor (1+1)/2 \rfloor} = 2^1 = 2$$

$\therefore P(1)$ holds
Inductive Step: Show $P(k) \to P(k+1)$
Assume $P(k)$ **for arbitrary** $k \geq 1$: $f(k) = 2^{\lfloor (k+1)/2 \rfloor}$
Show P(k+1):
Case 1: assume k+1 is even

$$\begin{aligned} P(k+1) &= 2f(k+1-2) \\ &= 2f(k-1) \\ &= 2 \cdot 2^{\lfloor (k-1+1)/2 \rfloor} && \textbf{By Inductive Hypothesis} \\ &= 2^{\lfloor 1+k/2 \rfloor} \\ &= 2^{\lfloor (k+2)/2 \rfloor} \\ &= 2^{\lfloor ((k+1)+1)/2 \rfloor} \end{aligned}$$

$\therefore P(k) \to P(k+1)$ **holds** when $k+1$ is even
Case 2: assume $k+1$ is odd

$$\begin{aligned} P(k+1) &= f(k+1-1) \\ &= f(k) && \textbf{This holds by the Inductive Hypothesis} \end{aligned}$$

$\therefore$ **the statement holds for all** $n \geq 1$ **by mathematical induction**

Problem 6. (15 points) Section 5.3, Exercise 16, page 358 (use mathematical induction)

Solution.
f_n is the n^{th} Fibonacci number.
Assume $f_0 - f_1 + f_2 - \cdots - f_{2n-1} + f_{2n} = f_{2n-1} - 1$ for $n \geq 1$
Basis Step: Show $P(1)$

$$P(1) = f_0 - f_1 + f_2 = 0 - 1 + 1 = 0$$
$$P(1) = f_{2(1)-1} - 1 = f_1 - 1 = 1 - 1 = 0$$

$\therefore P(1)$ holds
Inductive Step: Show $P(k) \to P(k+1)$
Assume $P(k)$ **for arbitrary** $k \geq 1$
Show P(k+1):

$$\begin{aligned}
f_0 - f_1 + f_2 - \cdots - f_{2(k+1)-1} + f_{2(k+1)} &= f_0 - f_1 + f_2 - \\
&\cdots - f_{2k-1} + f_{2k} - f_{2k+1} + f_{2k+2} \\
&= f_{2k-1} - 1 - f_{2k+1} + f_{2k+2} && \textbf{By Inductive Hypothesis} \\
&= f_{2k-1} - 1 + f_{2k} \\
&= f_{2k+1} - 1 && \textbf{By Fib.} \\
&= f_{2(k+1)-1} - 1
\end{aligned}$$

$\therefore P(k) \to P(k+1)$ **holds**
$\therefore$ **the statement holds for all $n \geq 1$ by mathematical induction**

Problem 7. (15 points) Section 5.3, Exercise 44, page 359

Solution.
$l(T)$ is the number of leaves of a full binary tree T. $i(T)$ is the number of internal vertices of T
$P : l(T) = 1 + i(T)$
Basis Step
$P(0)$ is a tree with a single vertex. By the basis step of the given recursive definition, the single root is a leaf and there are no internal vertices.

$$\begin{aligned}
l(t) &= 1 \\
i(t) &= 0 \\
l(t) &= 1 + i(t) = 1
\end{aligned}$$

$\therefore P(1)$ holds
Let t be a tree smaller than T. Assume that the result is true for all t.
Recursive Step: Let T_1 be a left subtree and T_2 be a right subtree consisting of a root r. By the given definition $T = T_1 \cdot T_2$. It follows that

$$l(T) = l(T_1) + l(T_2) \tag{1}$$

The internal vertices of T are the root r of T and the union of the set of internal vertices of T_1 and the set of internal vertices of T_2.

$$i(T) = i(T_1) + i(T_2) + 1 \tag{2}$$

$$\begin{aligned}
l(T) &= l(T_1) + l(T_2) && \textbf{By (1)} \\
&= i(T_1) + 1 + i(T_2) + 1 && \textbf{By Assumtion, } T_1, T_2 \textbf{ are smaller than } T \\
&= i(T) + 1 \quad \square && \textbf{By (2)}
\end{aligned}$$

$\therefore$ the statment has been proven by structural induction.

Problem 8. (**Extra credit 10 points**) Section 5.3, Exercise 36, page 359

Solution.
Let w_1 = abc and w_2 = def.
Basis Step

$$
\begin{align}
w_1 &= cba \tag{3}\\
w_2 &= fed \tag{4}\\
w_1 w_2 &= abcdef \tag{5}\\
(w_1 w_2)^r &= fedcba \tag{6}
\end{align}
$$

Recursive Step

$$
\begin{aligned}
(w_2)^r (w_1)^r &= (fed)(cba) && \textbf{by (3)}\\
&= (w_1 w_2)^r && \textbf{by (6)}
\end{aligned}
$$

Checklist:
- □ Did you type in your name and section?
- □ Did you disclose all resources that you have used?
  (This includes all people, books, websites, etc. that you have consulted.)
- □ Did you sign that you followed the Aggie Honor Code?
- □ Did you solve all problems?
- □ Did you submit the .tex and .pdf files of your homework to the correct link on eCampus?
- □ Did you submit a signed hardcopy of the pdf file in class?