

CSCE 222 [Sections 502, 503] Discrete Structures for Computing  
Spring 2017 – Hyunyoung Lee

**Problem Set 4**

**Due dates:** Electronic submission of *yourLastName-yourFirstName-hw4.tex* and *yourLastName-yourFirstName-hw4.pdf* files of this homework is due on **Friday, 2/24/2017 before 11:00 a.m.** on <http://ecampus.tamu.edu>. You will see two separate links to turn in the .tex file and the .pdf file separately. Please do not archive or compress the files. A signed paper copy of the pdf file is due on **Friday, 2/24/2017 at the beginning of class.** **If any of the three submissions are missing, your work will not be graded.**

**Name:** Joseph Martinsen

**Section:** 503

**Resources.** (All people, books, articles, web pages, etc. that have been consulted when producing your answers to this homework.)

On my honor, as an Aggie, I have neither given nor received any unauthorized aid on any portion of the academic work included in this assignment. Furthermore, I have disclosed all resources (people, books, web sites, etc.) that have been used to prepare this homework.

**Signature:** \_\_\_\_\_

**Problem 1.** (10 points) Let  $f_1, f_2, f_3, f_4$  be functions from the set  $\mathbf{N}$  of natural numbers to the set  $\mathbf{R}$  of real numbers. Suppose that  $f_1 = O(f_2)$  and  $f_3 = O(f_4)$ . Use the *definition* of Big Oh *given in class* to prove that

$$f_1(n) + f_3(n) = O(\max(|f_2(n)|, |f_4(n)|)).$$

**Solution.** By definition of Big Oh,

If  $f_1 \in O(f_2)$ , then there exist a  $k_1 \in \mathbb{R}$  and  $n_1 \in \mathbb{N}$  such that  $f_1(n) \leq k_1 f_2(n)$  for all  $n > n_1$

In the same manner, if  $f_3 \in O(f_4)$ , then there exist a  $k_2 \in \mathbb{R}$  and  $n_2 \in \mathbb{N}$  such that  $f_3(n) \leq k_2 f_4(n)$  for all  $n > n_2$

Let  $N = \max(n_1, n_2)$ . For all  $n > N$  it is true that

$$f_1(n) + f_3(n) \leq k_1 f_2(n) + k_2 f_4(n)$$

Let  $K = \max(k_1, k_2)$  and  $f_5(n) = \max(f_2(n), f_4(n))$

$$\begin{aligned} f_1(n) + f_3(n) &\leq K[f_2(n) + f_4(n)] \\ &\leq K f_5(n) = K \max(f_2(n), f_4(n)) \\ f_1(n) + f_3(n) &\leq K \max(f_2(n), f_4(n)) \end{aligned}$$

By definition of Big Oh, for  $n > N$

$$f_1(n) + f_3(n) \in O(\max(f_2(n), f_4(n))) \quad \square$$

**Problem 2.** (10 points) Let  $f_1, f_2, f_3$  be functions from the set  $\mathbf{N}$  of natural numbers to the set  $\mathbf{R}$  of real numbers. Suppose that  $f_1 = O(f_2)$  and  $f_2 = O(f_3)$ . Is it possible that

$$f_1(n) > f_3(n)$$

holds for all natural numbers  $n$ ? Give an example or given an argument that this is impossible.

**Solution.** Allow  $f_3 = f_2 + 1$  and  $f_2 = f_1 + 1$ , where  $f_1 = O(f_2)$  and  $f_2 = O(f_3)$  is still true.

$$\begin{aligned} f_3 &= f_2 + 1 \\ f_3 &= f_1 + 1 + 1 \\ f_3 &= f_1 + 2 \end{aligned}$$

For all  $n$ ,  $f_3$  is 2 bigger than  $f_1$ . It then follows that  $f_1 < f_3 \forall n$   $\square$

**Problem 3.** (5 pts  $\times$  4 = 20 points) Determine whether each of the following statements is true or false. In each case, answer true or false, and justify your answer.

- a)  $3n^2 - 42 = O(n^2)$
- b)  $n^2 = O(n \log n)$
- c)  $1/n = O(1)$
- d)  $n^n = \Omega(2^n)$

**Solution.**

**Problem 4.** (15 points) Does  $\Theta(n^3 + 2n + 1) = \Theta(n^3)$  hold? Justify your answer.

**Solution.**

A function  $f(n)$  is  $\Theta(g(n))$  iff

$f(n) \leq Ug(n)$  for all  $n > n_0$  *O definition*

and

$f(n) \geq Lg(n)$  for all  $n > n_0$   *$\Omega$  definition*

First, check if  $O(n^3 + 2n + 1) = O(n^3)$  holds

$$\begin{aligned} n^3 + 2n + 1 &\leq Un^3 \\ &\leq n^3 + 2n^3 + n^3 \\ &\leq 4n^3 \end{aligned}$$

$\therefore O(n^3 + 2n + 1) = O(n^3)$  with  $U = 4$  and  $n \geq 1$  as witnesses  
Next, check  $\Omega(n^3 + 2n + 1) = \Omega(n^3)$

$$\begin{aligned} n^3 + 2n + 1 &\geq Ln^3 \\ n^3 + 2n + 1 &\geq n^3 && \text{for } n \geq 1 \\ 1 + \frac{2}{n^2} + \frac{1}{n^3} &\geq 1 \\ 1 + \frac{2}{1} + \frac{1}{1} &\geq 1 && \text{for } n \geq 1 \end{aligned}$$

$\therefore O(n^3 + 2n + 1) = O(n^3)$   
 $\therefore \Omega(n^3 + 2n + 1) = \Omega(n^3)$   
 $\therefore \Theta(n^3 + 2n + 1) = \Theta(n^3) \quad \square$

**Problem 5.** (10 points) Let  $k$  be a fixed positive integer. Show that

$$1^k + 2^k + \cdots + n^k = O(n^{k+1})$$

holds.

**Solution.**

**Problem 6.** (15 points) Suppose that you have two algorithms  $A$  and  $B$  that solve the same problem. Algorithm  $A$  has worst case running time  $T_A(n) = 2n^2 - 2n + 1$  and Algorithm  $B$  has worst case running time  $T_B(n) = n^2 + n - 1$ .

**Solution.**

a) Show that both  $T_A(n)$  and  $T_B(n)$  are in  $O(n^2)$ .

$$\begin{aligned} T_A(n) = 2n^2 - 2n + 1 &\leq Un^2 \\ &\leq 2n^2 + n^2 + n^2 \\ &\leq 4n^2 && \text{for } n > 1 \end{aligned}$$

$\therefore T_A(n)$  is  $O(n^2)$  with  $U = 4$  and  $n > 1$  as witnesses

$$\begin{aligned} T_B(n) = n^2 + n - 1 &\leq Un^2 \\ &\leq 2n^2 + n^2 + n^2 \\ &\leq 4n^2 && \text{for } n > 1 \end{aligned}$$

$\therefore T_B(n)$  is  $O(n^2)$  with  $U = 4$  and  $n > 1$  as witnesses

b) Show that  $T_A(n) = 2n^2 + O(n)$  and  $T_B(n) = n^2 + O(n)$ .

$$\begin{aligned} -2n + 1 &\leq Un \\ &\leq n + n \\ &\leq 2n \\ -2n + 1 &\in O(n) \end{aligned}$$

$$\therefore T_A(n) = 2n^2 - 2n + 1 = 2n^2 + O(n)$$

$$\begin{aligned} n - 1 &\leq Un \\ &\leq n + n \\ &\leq 2n \\ n - 1 &\in O(n) \end{aligned}$$

$$\therefore T_A(n) = n^2 + n - 1 = n^2 + O(n)$$

c) Explain which algorithm is preferable.

For large  $n$   $T_B(n)$  is preferable because due to the fact that the coefficient of  $n^2$  is 1 instead of 2, like in  $T_A(n)$

**Problem 7.** (10 points) Section 3.3, Exercise 14 on page 230.

**Solution.**

a)

```
n = 2 a2 = 3 a1 = 1 a0 = 1
Line 1: y := 3
For Loop:
i := 1 y := 3 * 2 + 1 => 7
i := 2 y := 7 * 2 + 1 => 15
return 15
```

b)

There is one multiplication and one addition within the for loop. The loop executes  $n$  times for  $x = n$ .

$\therefore$  there are  $n$  additions and  $n$  multiplications

**Problem 8.** (10 points) Section 3.3, Exercise 16 a), d), g) and h) on page 230.

**Solution.**

Seconds in a day =  $10^5 \text{ sec}$

# of Operations =  $10^{16}$

a)

$$\begin{aligned} f(n) &\leq 10^{16} \\ \log n &\leq 10^{16} \\ n &\leq 2^{10^{16}} \end{aligned}$$

Largest value of  $n$  is  $2^{10^{16}}$

d)

$$\begin{aligned}
 f(n) &\leq 10^{16} \\
 1000n^2n &\leq 10^{16} \\
 n^2 &\leq 10^{13} \\
 n &\leq 10^{13/2}
 \end{aligned}$$

Largest value of  $n$  is  $10^{13/2}$

g)

$$\begin{aligned}
 f(n) &\leq 10^{16} \\
 2^{2n} &\leq 10^{16} \\
 2n &\leq \log_2 10^{16} \\
 n &\leq \frac{1}{2} \log_2 10^{16}
 \end{aligned}$$

Largest value of  $n$  is  $\frac{1}{2} \log_2 10^{16}$

h)

$$\begin{aligned}
 f(n) &\leq 10^{16} \\
 2^{2^2n} &\leq 10^{16} \\
 2^n &\leq \log_2 10^{16} \\
 n &\leq \log_2(\log_2 10^{16})
 \end{aligned}$$

Largest value of  $n$  is  $\log_2(\log_2 10^{16})$

**Checklist:**

- ☐ Did you type in your name and section?
- ☐ Did you disclose all resources that you have used?  
(This includes all people, books, websites, etc. that you have consulted.)
- ☐ Did you sign that you followed the Aggie Honor Code?
- ☐ Did you solve all problems?
- ☐ Did you submit the .tex and .pdf files of your homework to the correct link on eCampus?
- ☐ Did you submit a signed hardcopy of the pdf file in class?