CSCE 222: Discrete Structures for Computing
Section 503
Fall 2016

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Problem Set 6

Due: 9 October 2016 (Sunday) before 11:59 p.m. on eCampus (`ecampus.tamu.edu`).
You must show your work in order to recieve credit.

Problem 1. (25 points)

1. Show that $\sum_{j=1}^{n}(a_j - a_{j-1}) = a_n - a_0$, where $a_0, a_1, \ldots, a_n$ is a sequence of real numbers.
   *This type of sum is called **telescoping**.*

2. Sum both sides of the identity $k^2 - (k-1)^2 = 2k - 1$ from $k = 1$ to $k = n$ and use the previous step to find:

   a. a formula for $\sum_{k=1}^{n}(2k-1)$.
   b. a formula for $\sum_{k=1}^{n} k$.

3. Use the technique given in step 1, together with the results of step 2, to derive the formula for $\displaystyle\sum_{k=1}^{n} k^2$.

   Hint: take $a_k = k^3$ in the telescoping sum in step 1.

Solution.

1.

$$\begin{aligned}\sum_{j=1}^{n}(a_j - a_{j-1}) &= a_1 - a_0 + a_2 - a_1 + a_3 - a_2 + a_4 - a_3 \ldots + a_n - a_{n-1} + a_{n-1} - a_{n-2} + a_{n-2} - a_{n-3} \\ &= \cancel{a_1} - a_0 + \cancel{a_2} - \cancel{a_1} + \cancel{a_3} - \cancel{a_2} + \cancel{a_4} - \cancel{a_3} \ldots + a_n - \cancel{a_{n-1}} + \cancel{a_{n-1}} - \cancel{a_{n-2}} + \cancel{a_{n-2}} - \cancel{a_{n-3}} \\ &= a_n - a_0\end{aligned}$$

2.

$$
\begin{aligned}
\sum_{k=1}^{n}(2k-1) &= \sum_{k=1}^{n} k^2-(k-1)^2 \\
\text{Let } a_k = k^2 & \\
&= \sum_{k=1}^{n} a_k - (a_k - 1)^2 \\
&= n^2 \\
\sum_{k=1}^{n}(2k-1) &= n^2 \\
\sum_{k=1}^{n} 2k - \sum_{k=1}^{n} 1 &= n^2 \\
2\sum_{k=1}^{n} k - n &= n^2 \\
2\sum_{k=1}^{n} k &= n^2 + n \\
\sum_{k=1}^{n} k &= \frac{n^2+n}{2} \\
\sum_{k=1}^{n}(2k-1) &= \frac{n(n+1)}{2}
\end{aligned}
$$

3.

$$
\begin{aligned}
\text{Let} a^k &= k^3 \text{ so,} \\
a_k - a_{k-1} &= k^3 - (k-1)^3 \\
k^3-(k-1)^3 &= k^3 - (k^3 - 1 - 3k^2 + 3k) \\
&= 1 + 3k^2 - 3k \\
k^2 &= \frac{k^3-(k-1)^3+3k-1}{3} \\
\sum_{k=1}^{n} k^2 &= \frac{1}{3}\sum_{k=1}^{n} k^3 - (k-1)^3 + 3k - 1 \\
&= \frac{1}{3}(\sum_{k=1}^{n} k^3 - (k-1)^3 + \sum_{k=1}^{n} 3k - \sum_{k=1}^{n} 1) \\
\sum_{k=1}^{n} k^2 &= n^3 - 0 + 3\frac{n(n+1)}{2} - n \\
\sum_{k=1}^{n} k^2 &= \frac{n(n+1)(2n+1)}{6}
\end{aligned}
$$

Problem 2. (25 points)
Solve the recurrence relation:

1. $A_n = 2 \cdot A_{n-1} + 3$, where $A_0 = 1$
2. $A_n = A_{n-1} + 4n - 2$, where $A_0 = 1$

Solution.

1.
$$\begin{aligned}
A_n &= 2 \cdot A_{n-1} + 3, \text{ where } A_0 = 1 \\
A_n &= 2A_{n-1} + 3 \\
A_{n-1} &= 2A_{n-2} + 3 \\
A_{n-2} &= 2A_{n-3} + 3 \\
A_n &= 2(2(2A_{n-3} + 3) + 3) + 3 \\
A_n &= 2^3 A_{n-3} + 3(3) \\
A_n &= 2^n A_0 + 3n \text{ Since } A_0 = 1 \\
A_n &= 2^n + 3n
\end{aligned}$$

2.
$$A_n = A_{n-1} + 4n - 2, \text{ where } A_0 = 1$$

Problem 3. (25 points)
Let $V = \{S, A, B, a, b\}$ and $T = \{a, b\}$. Find the language generated by the grammar (V, T, S, P) when the set P of production rules consists of

1. $S \to AB$, $A \to ab$, $B \to bb$.
2. $S \to AB$, $S \to aA$, $A \to a$, $B \to ba$.
3. $S \to AB$, $S \to AA$, $A \to aB$, $A \to ab$, $B \to b$.
4. $S \to AA$, $S \to B$, $A \to aaA$, $A \to aa$, $B \to bB$, $B \to b$.
5. $S \to AB$, $A \to aAb$, $B \to bBa$, $A \to \lambda$, $B \to \lambda$.

Solution.

1. $S \to AB$, $A \to ab$, $B \to bb$.
   From S we get AB
   From A we now have abB
   From B we now have $aabb$
   $\therefore$ the language is $\{aabb\}$
2. $S \to AB$, $S \to aA$, $A \to a$, $B \to ba$.
   From S we get AB
   From A we now have aB
   From B we now have $\{aba\}$
   Also, from S we get aA
   From A we now have $\{aa\}$
   $\therefore$ the language is $\{aba, aa\}$
3. $S \to AB$, $S \to AA$, $A \to aB$, $A \to ab$, $B \to b$. From S we get AB
   From A we now have aBB
   Using B twice we get abb
   From S we get AA
   From A we now have $aBaB$
   Using B twice we get $abab$
   $\therefore$ the language is $\{abb, abab\}$
4. $S \to AA$, $S \to B$, $A \to aaA$, $A \to aa$, $B \to bB$, $B \to b$.
   From S we get AA
   From both A's we forms of A such that $aaaa$ or $aaaaa$ or $aaaaaa$ or $aaaaaaaa$, strings of even a's with minimum size of 4
   Using $S \to B$ to dervive B we get forms of b, bb, bbb, such that it results in strings of b greater than 1
   $\therefore$ the language is $\{a \cdot 2^n \mid n \geq 2\} \cup \{b^2 \mid n \geq 1\}$
5. $S \to AB$, $A \to aAb$, $B \to bBa$, $A \to \lambda$, $B \to \lambda$ From S we get AB
   From A we get aBa all the way to $ababababab \ldots AB$ until λ depending on x number of repetitions
   From B we get $abbaAB$ all the way to $ababababab \ldots AB$ until λ depending on y number of repetitions
   $\therefore$ the language is of the form $\{a^{x+y} b^{x+y} \mid x \geq 0, y \geq 0\}$

Problem 4. (25 points)
Find a phrase-structure grammar for each of these languages.

1. the set consisting of the bitstrings 00, 11, and 010.
2. the set of bitstrings that start with 10 and end with one or more 1s.
3. the set of bitstrings consisting of an odd number of 0s followed by a final 1.
4. the set of bitstrings that have neither two consecutive 0s nor two consecutive 1s.

Solution.
Let $V = \{S, A, B, a, b\}$ and $T = \{a, b\}$. Find the language generated by the grammar (V, T, S, P) when the set P of production rules consists of

1. the set consisting of the bitstrings 00, 11, and 010. With $L = \{00, 11, 010\}$
   The phrase structure gammer of this language is $G = \{V, T, S, P\}$
   The Vocabulary $(V) = \{0, 1, S\}$ and the terminal symbols are $T = \{0, 1\}$
   The productions are $S \to 00, S \to 11$, and $S \to 010$
2. the set of bitstrings that start with 10 and end with one or more 1s.
   The language $L = \{a : a$ is a bit string that starts with 10 and end with one or more 1s$\}$
   The phrase structure gammer of this language is $G = \{V, T, S, P\}$
   The Vocabulary $(V) = \{0, 1, S, A, B\}$ and the terminal symbols are $T = \{0, 1\}$
   The productions are $S \to 10AB, A \to AA, A \to 1, A \to 0, B \to BB, B \to 1$
3. the set of bitstrings consisting of an odd number of 0s followed by a final 1.
   The language $L = \{a : a$ is a bit string of odd number of 0s followed by a final 1$\}$
   The phrase structure gammer of this language is $G = \{V, T, S, P\}$
   The Vocabulary $(V) = \{0, 1, S, A, B, C\}$ and the terminal symbols are $T = \{0, 1\}$
   The productions are $S \to A1, A \to \lambda, A \to BBC, A \to BCB, A \to CBB, B \to CB, B \to BC, B \to 1, C \to 0$
4. the set of bitstrings that have neither two consecutive 0s nor two consecutive 1s.
   The language $L = \{a : a$ is a bit string that has neither two consecutive 0s nore two consecutive 1s$\}$
   The phrase structure gammer of this language is $G = \{V, T, S, P\}$
   The Vocabulary $(V) = \{0, 1, S, A, B\}$ and the terminal symbols are $T = \{0, 1\}$
   The productions are $S \to A, A \to AA, A \to 01, A \to \lambda, S \to B, B \to BB, B \to 10, B \to \lambda$

Aggie Honor Statement: On my honor as an Aggie, I have neither given nor received any unauthorized aid on any portion of the academic work included in this assignment.

Checklist: Did you...
1. abide by the Aggie Honor Code?
2. solve all problems?
3. start a new page for each problem?
4. show your work clearly?
5. type your solution?
6. submit a PDF to eCampus?