CSCE 222: Discrete Structures for Computing
Section 503
Fall 2016

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Problem Set 1

Due: 4 September 2016 (Sunday) before 11:59 p.m. on eCampus (`ecampus.tamu.edu`).

Problem 1. (20 points)
You have n coins, exactly one of which is counterfeit. You know counterfeit coins weigh more than authentic coins. Devise an algorithm for finding the counterfeit coin using a balance scale[1]. Express your algorithm in pseudocode. For $n = 12$, how many weighings does your algorithm use?

Solution.

procedure $counterfeit\_coin(a_1 + a_2 + a_3 + ... + a_n$**: coin weights**)
    $n = 12$
    for $i = 1$ **to** n
        $a_{i-1} \rightarrow left\_scale$
        $a_i \rightarrow right\_scale$
        if $right\_scale < left\_scale$
            $counterfeit\_location = i$
        else if $left\_scale > right\_scale$
            $counterfeit\_location = i - 1$
    return $counterfeit\_location$

[1] A balance scale compares the weight of objects placed on it. The result of the comparison is either left side heavier, right side heavier, or both sides equal.

Problem 2. (20 points)
Devise an algorithm that takes as input a list of n integers in unsorted order, where the integers are not necessarily distinct, and outputs the location (index of first element) and length of the longest contiguous non-decreasing subsequence in the list. If there is a tie, it outputs the location of the first such subsequence. Express your algorithm in pseudocode. For the list $9, 7, 9, 4, 5, 8, 1, 0, 5, 9$, what is the algorithm's output? How many comparison operations between elements of the list are used?

Solution.

procedure *non-decreasing-location*$(a_1, a_2, a_3, ..., a_n$: integers)
location $:= 1$
for $i = 2$ **to** $n - 1$
    if $a_{i-1} < a_i$ **then**
    location $:= i$
else if $a_{i-1} = a_i$ **then**
return *non-decreasing-location*{*non-decreasing-location* is the location of the first element in the subsequence of non-decreacing elements}

Problem 3. (20 points)
Arrange the following functions in order such that each function is big-O of the next function: $2 \cdot 3^n$, $3n!$, $2019 \log n$, $\dfrac{n^3}{10^6}$, $n \log n$, $\sqrt{n}$, $3 \cdot 2^n$. Prove your answer is correct by giving the witnesses for each pair of consecutive functions.

Solution. The following functions are arranged such that that each function is a big-O of the next function.

1. $2019 \log n$

   $2019 \log n \leq C\sqrt{n}$

   $2019 \log n \leq 2019\sqrt{n} \qquad C = 2019$

   $\log n \leq \sqrt{n}$

   $0.0 \leq 1 \qquad k = 1$

   $2019 \log n$ is $O(\sqrt{n})$ by taking $C = 2019$ and $k = 1$ as witnesses.

2. $\sqrt{n}$

   $\sqrt{n} \leq C(n \log n)$

   $\frac{1}{\sqrt{n}} \leq \log n \qquad C = 1$

   $\frac{1}{2} \leq \log 4 \qquad k = 4$

   $0.5 \leq 1.386\,3$

   $\sqrt{n}$ is $O(n \log n)$ by taking $C = 1$ and $k = 4$ as witnesses.

3. $n \log n$

   $n \log n \leq C(\dfrac{n^3}{10^6})$

   $n \log n \leq \dfrac{n^3}{10^6} \qquad C = 1$

   $6044 \leq 6332 \qquad k = 1850$

   $n \log n$ is $O(\dfrac{n^3}{10^6})$ by taking $C = 1$ and $k = 1850$ as witnesses.

4. $\dfrac{n^3}{10^6}$

   $\dfrac{n^3}{10^6} \leq C(3 \cdot 2^n)$

   $\dfrac{n^3}{10^6} \leq 3 \cdot 2^n \qquad C = 1$

   $0 \leq 3 \qquad k = 0$

   $\dfrac{n^3}{10^6}$ is $O(3 \cdot 2^n)$ by taking $C = 1$ and $k = 0$ as witnesses.

5. $3 \cdot 2^n$

   $3 \cdot 2^n \leq C(2 \cdot 3^n)$

   $3 \cdot 2^n \leq \dfrac{3}{2}(2 \cdot 3^n) \qquad C = \dfrac{3}{2}$

   $3 \cdot 2^n \leq 3 \cdot 3^n$

   $2^n \leq 3^n$

   $2 \leq 3 \qquad k > 1$

   $3 \cdot 2^n$ is $O(2 \cdot 3^n)$ by taking $C = \dfrac{3}{2}$ and $k > 1$ as witnesses.

6. $2 \cdot 3^n$

$$2 \cdot 3^n \leq C(3n!)$$

$$2 \cdot 3^n \leq \frac{2}{3}(3n!) \qquad C = \frac{2}{3}$$

$$2 \cdot 3^n \leq 2 \cdot n!$$

$$3^n \leq n!$$

$$2187 \leq 5040 \qquad\qquad k > 7$$

$2 \cdot 3^n$ is $O(3n!)$ by taking $C = \frac{2}{3}$ and $k > 7$ as witnesses

7. $3n!$

$3n!$ is $O(n!)$ with $C = 4$ and $k > 0$ as witnesses

Problem 4. (20 points)
For each of the following functions, give a big-O estimate, including witnesses, using a simple function $g(n)$ of the smallest order:

1. $(n^2+8)(n+1)$
2. $(n\log n+1)^2+(\log n+1)(n^2+1)$
3. $n^{2^n}+n^{n^2}$
4. $\dfrac{n^4+5\log n}{x^3+1}$
5. $2x^4+7x^3+5x+3$

Solution.

1. $(n^2+8)(n+1)$

   $(n^2+8)(n+1)\le C(n^3)$

   $n^3+n^2+8n+8\le C(n^3+n^3+n^3+n^3)$

   $n^3+n^2+8n+8\le 4(n^3)\qquad C=4$

   $68\le 108\qquad k>3$

   Thus n^3 is $O((n^2+8)(n+1))$ by taking $C=4$ and $k>3$ as witnesses.

2. $(n\log n+1)^2+(\log n+1)(n^2+1)$

   $(n\log n+1)^2+(\log n+1)(n^2+1)\le C(n^3)$

   $(n\log n+1)(\log n+1+n^2+1)\le C(n^3)$

   $(n\log n)^2+n\log n+n\log n+1+n^3\log n+n^2+n\log n+1\le C()$

   $n^3\log n+(n\log n)^2+5n\log n+n^3+2n^2+2\le C(n^3+n^3+n^3+n^3+n^3)$

   $n^3\log n+(n\log n)^2+5n\log n+n^3+2n^2+2\le 5n^3\qquad C=5$

   $8.06\le 40\qquad k>3$

   Thus $(n\log n+1)^2+(\log n+1)(n^2+1)$ is $O(n^3)$ with $C=5$ and $k>3$ as witnesses.

3. $n^{2^n}+n^{n^2}$

   $n^{2^n}+n^{n^2}\le C(n^{2^n})$

   $n^{2^n}+n^{n^2}\le 2n^{2^n}\qquad C=2$

   $2.33\cdot 10^{22}\le 4.66\cdot 10^{22}\qquad k>5$

   Thus $n^{2^n}+n^{n^2}$ is $O(n^{2^n})$ with $C=2$ and $k>5$ as witnesses.

4. $\dfrac{n^4+5\log n}{n^3+1}$

   $\dfrac{n^4+5\log n}{n^3+1}\le C(n)$

   $\dfrac{n^4}{n^3+1}+\dfrac{5\log n}{n^3+1}\le 2n\qquad C=2$

   $5\le 10\qquad k>5$

   Thus $\dfrac{n^4+5\log n}{n^3+1}$ is $O(x)$ with $C=2$ and $k>5$ as witnesses

5. $2x^4 + 7x^3 + 5x + 3$

$2x^4 + 7x^3 + 5x + 3 \leq 2x^4 + x^4 + x^4 + x^4$

$2x^4 + 7x^3 + 5x + 3 \leq 5x^4$ $\qquad C = 5$

$369 \leq 405$ $\qquad k > 3$

Thus $2x^4 + 7x^3 + 5x + 3$ is $O(x^4)$ with $C = 5$ and $k > 3$ as witnesses

Problem 5. (20 points)
For each of the following functions, determine whether that function is of the same order as n^2 either by finding witnesses or showing that sufficient witnesses do not exist:

1. $13n + 12$
2. $n^2 + 1000n \log n$
3. 3^n
4. $3n^2 + n - 5$
5. $\dfrac{n^3 + 2n^2 - n + 3}{4n}$

Solution.

Suppose that there are constants C and k for which $n^2 \leq C(13n+12)$ whenever $n > k$. We can divide both sides of $Cn^2 \leq 13n + 12$ by n and simplify to $n \leq 3C$ for values of $n > k$. However, no matter what C and k are, the inequality $n \leq 3C$ cannot hold for all n with $n > k$. In particular, once we set a value of k, we see that when n is larger than the maximum of k and C, it is not true that $n \leq C$ even though $n > k$. This contradiction shows that n^2 is not $O(13n + 12)$. Thus the function $13n + 12$ is not as the same order as n^2.

Suppose that there are particular witnesses C and k for which $n^2 + 1000n \log n \leq Cn^2$. After dividing both sides by n the resultant is $n + \log n^{1000} \leq Cn$.

3^n

$3n^2 + n - 5$

$\dfrac{n^3 + 2n^2 - n + 3}{4n}$

Aggie Honor Statement: On my honor as an Aggie, I have neither given nor received any unauthorized aid on any portion of the academic work included in this assignment.

Checklist:

1. Did you abide by the Aggie Honor Code?
2. Did you solve all problems and start a new page for each?
3. Did you submit the PDF to eCampus?