CSCE 222: Discrete Structures for Computing
Section 503
Fall 2016

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Problem Set 10

Due: 6 November 2016 (Sunday) before 11:59 p.m. on eCampus (`ecampus.tamu.edu`). You must show your work in order to recieve credit.

Problem 1. (25 points)

Use induction on n to prove that $\displaystyle\sum_{i=0}^{n-1} \frac{i}{2^i} = 2 - \frac{n+1}{2^{n-1}}$

Solution.

$$\sum_{i=0}^{n-1} \frac{i}{2^i} = 2 - \frac{n+1}{2^{n-1}} \text{ is true for all } n \geq 1$$

<u>Basis Step:</u> Show $P(1)$

$$\begin{aligned} P(1) &= 2 - \frac{1+1}{2^{1-1}} \\ &= 2 - \frac{2}{1} = 0 \\ P(1) &= \sum_{i=0}^{1-1} \frac{i}{2^i} \\ &= \frac{0}{2^0} = 0 \end{aligned}$$

$\therefore P(0)$ holds

<u>Inductive Step:</u> Show $P(k) \to P(k+1)$

$$\text{Assume } P(k) \text{for arbitrary } k > 1 : \sum_{i=0}^{k-1} \frac{i}{2^i} = 2 - \frac{k+1}{2^{k-1}}$$

$$\text{Show } P(k+1) : \sum_{i=0}^{k} \frac{i}{2^i} = 2 - \frac{k+2}{2^k}$$

$$\begin{aligned} \sum_{i=0}^{k} \frac{i}{2^i} &= \frac{k}{2^k} + \sum_{i=0}^{k-1} \frac{i}{2^i} \\ &= \frac{k}{2^k} + 2 - \frac{k+1}{2^{k-1}} && \text{By HI} \\ &= 2 - \frac{k+2}{2^k} \end{aligned}$$

$\therefore P(k) \to P(k+1)$ holds

$$\therefore \sum_{i=0}^{n-1} \frac{i}{2^i} = 2 - \frac{n+1}{2^{n-1}} \text{ holds for all } n \geq 1 \text{ by mathematical induction}$$

Problem 2. (25 points)
A guest at a party is a **celebrity** if this person is known by every other guest, but knows none of them. There is at most one celebrity at a party[1]. Your task is to find the celebrity, if one exists, at a party by asking only one type of question – asking a guest whether they know a second guest. Everyone must answer your questions truthfully. That is, if Alice and Bob are two people at the party, you can ask Alice whether she knows Bob; she must answer correctly. Use mathematical induction to show that if there are n people at the party, then you can find the celebrity, if there is one, with $3(n-1)$ questions. *Hint: First, ask a question to eliminate one person as a celebrity. Then use the inductive hypothesis to identify a potential celebrity. Finally, ask two more questions to determine whether that person is actually a celebrity.*

Solution.
Base Step: if two people A and B are at a paty, you ask if they know one another. If one of the two peole says Yes, and the other one says No, then the person who said No is a celebrity, else there is not a celebrity present
Inductive Step: Assume that the above statement is true for a party of k people and prove that it is also true for a party of $k+1$ people.
Let A and B be party members. Ask A if they know B. If they answer Yes, A is not a celebrity. Else, if they answer no then it follows that B is not a celebrity. One party member has now been ruled out as being a celebrity. Going on to the next n party members, use the inductive hypothesis to find the celebrity with $3(n-1)$ questions.
If there is a celebrity C, ask C if they know the last person. Also ask the last person if they know C. If C does not know the last party member and the last party member knows C, then C is a celebrity.

[1] If there were two, they would know each other. A particular party may have no celebrity

Problem 3. (25 points)
Determine which Fibonacci numbers are divisible by 3. Use strong induction on n to prove your conjecture. The Fibonacci sequence satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ where $f_0 = 0$ and $f_1 = 1$.

Problem 4. (25 points)
Restaurant 222 offers gift certificates in denominations of \$8 and \$15. Determine the possible total amounts you can form using these denominations of gift certificates. Prove your answer using strong induction.

Solution.

$$P(n) =$$

Aggie Honor Statement: On my honor as an Aggie, I have neither given nor received any unauthorized aid on any portion of the academic work included in this assignment.

Checklist: Did you...
1. abide by the Aggie Honor Code?
2. solve all problems?
3. start a new page for each problem?
4. show your work clearly?
5. type your solution?
6. submit a PDF to eCampus?