for various n. This equation arises in quantum mechanics, but its solutions are not built into MATLAB. They do exist in Maple and so MATLAB can solve the equation—but see Chapter 5 for the limitations on dealing with Maple functions that emerge from MATLAB commands. This equation is considered in Problem 17 of Problem Set D.

Problem Set D

Second Order Equations

The solution to Problem 1 appears in the *Sample Solutions* in the back of the book.

1. Airy's equation is the linear second order homogeneous equation $y'' = ty$. Although it arises in a number of applications, including quantum mechanics, optics, and waves, it cannot be solved exactly by the standard symbolic methods. In order to analyze the solution curves, let us reason as follows.

   (a) For t close to zero, the equation resembles $y'' = 0$, which has general solution $y = c_1 t + c_2$. We refer to this as a "facsimile" solution. Graph a numerical solution to Airy's equation with initial conditions $y(0) = 0$, $y'(0) = 1$, and the facsimile solution (with the same initial data) on the interval $(-2, 2)$. How well do they match?

   (b) For $t \approx -K^2 \ll 0$, the equation resembles $y'' = -K^2 y$, and the corresponding facsimile solution is given by $y = c_1 \sin(Kt + c_2)$. Again using the initial conditions $y(0) = 0$, $y'(0) = 1$, plot a numerical solution of Airy's equation over the interval $(-18, -14)$. Using the value $K = 4$, try to find values of c_1 and c_2 so that the facsimile solution matches well with the actual solution. Why shouldn't we expect the initial conditions for Airy's equation to be the appropriate initial conditions for the facsimile solution?

   (c) For $t \approx K^2 \gg 0$, Airy's equation resembles $y'' = K^2 y$, which has solution $y = c_1 \sinh(Kt + c_2)$. (The hyperbolic sine function is called **sinh** in MATLAB.) Plot a numerical solution of Airy's equation together with a facsimile solution (with $K = 4$) on the interval $(14, 18)$. In analogy with part (b), you have to choose values for c_1 and c_2 in the facsimile solution.

   (d) Plot the numerical solution of Airy's equation on the interval $(-20, 2)$. What does the graph suggest about the frequency and amplitude of oscillations as $x \to -\infty$? Could any of that information have been predicted from the facsimile analysis?

2. Consider Bessel's equation of order zero

$$t^2y'' + ty' + t^2y = 0 \tag{D.1}$$

with initial data $y(0) = 1$, $y'(0) = 0$. The solution is the Bessel function of order zero of the first kind, $J_0(t)$. In this problem we solve equation (D.1) and two approximations of it with **dsolve** to learn about $J_0(t)$. Strictly speaking, this equation has a singularity at $t = 0$. However, this is one instance of a solution to a linear equation that exists outside the expected domain of definition. The singularity causes no difficulty in this problem.

   (a) For t close to 0, t^2 is very small compared to t. The equation (D.1) is therefore approximately $ty' = 0$. Solve this equation with the preceding initial data. What does this "facsimile" solution to the original problem suggest to you about the behavior of J_0 near near $t = 0$?

   (b) For t large and positive, t is small compared to t^2; and so we may approximate (D.1) by the equation $t^2(y'' + y) = 0$. Solve this equation with the same initial data. What does this suggest about the nature of the function J_0 for large t?

   (c) Still thinking of t as large and positive, rewrite (D.1) in the form

$$y'' + \frac{1}{t}y' + y = 0.$$

   If $t \approx K \gg 0$, we might approximate the equation by the constant coefficient equation

$$y'' + \frac{1}{K}y' + y = 0.$$

   Choose a specific value for K, say $K = 100$, and solve this equation and see what further information you obtain about $J_0(t)$ for large positive t.

   (d) Solve Bessel's equation with the given initial conditions, and plot the solution. Are the conclusions of your analysis confirmed?

3. This and some of the following problems concern models for the motion of a pendulum, which consists of a weight attached to a rigid arm of length L that is free to pivot in a complete circle. Neglecting friction and air resistance, the angle $\theta(t)$ that the arm makes with the vertical direction satisfies the differential equation

$$\theta''(t) + \frac{g}{L}\sin(\theta(t)) = 0, \tag{D.2}$$

where $g = 32.2$ ft/sec^2 is the gravitational acceleration constant. We will assume the arm has length 32.2 ft and so replace (D.2) by the simpler form

$$\theta'' + \sin\theta = 0. \tag{D.3}$$

(Alternatively, one can rescale time, replacing t by $\sqrt{g/L}\,t$, to convert (D.2) to (D.3).) For motions with small displacements (θ small), $\sin\theta \approx \theta$, and (D.3) can be approximated by the linear equation

$$\theta'' + \theta = 0. \tag{D.4}$$

This equation has general solution $\theta(t) = A\cos(t - \delta)$, with *amplitude* A and *phase shift* δ. Hence all the solutions to the linear approximation (D.4) have period 2π, independent of the amplitude A. In this problem we consider solutions of equation (D.3) satisfying the initial conditions $\theta(0) = A$, $\theta'(0) = 0$. If $|A| < \pi$, these solutions are periodic. However, in contract to the linear equation (D.4), their periods depend on the amplitude A. We do expect that, for small displacements A, the solutions to the pendulum (D.3) will have periods close to 2π.

   (a) Investigate how the period depends on the amplitude A by plotting a numerical solution of equation (D.3) using initial conditions $\theta(0) = A$, $\theta'(0) = 0$ on an appropriate interval for various A. Estimate the periods of the pendulum for the amplitudes $A = 0.1, 0.7, 1.5$, and 3.0. Confirm these results by displaying the displacements at a sequence of times, and finding the time at which the pendulum returns to its original position.

   (b) The period is given by the formula

$$T = 4\int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}},$$

   where $k = \sin(A/2)$. This formula may be derived in your text; it can be found in Section 9.3 of Boyce & DiPrima, Problem 27. The integral is called an *elliptic integral*. It cannot be evaluated by an elementary formula, but it can be evaluated numerically using **quad** or **quadl**. Calculate the period for the values of A we are considering. Do the values agree with those obtained in part (a)?

   (c) Redo the numerical calculations in part (a) with different tolerances, choosing the tolerances so that the values you get agree with those calculated in part (b).

   (d) How does the period depend on the amplitude of the initial displacement? For A small, is the period close to 2π? What is happening to the accuracy of the linear approximation as the initial displacement increases?

4. In this problem, we'll look at what the pendulum does for various initial velocities (*cf.* Problem 3).

   (a) Numerically solve the differential equation (D.3) using initial conditions $\theta(0) = 0$, $\theta'(0) = 1$. Solve equation (D.4) with the same initial conditions. Plot on the same graph the solutions to both the nonlinear equation (D.3) and the linear equation (D.4) on the interval from $t = 0$ to $t = 40$, and compare the two. Be clear about which curve is the nonlinear solution and which is the linear solution.

(b) Repeat part (a) and compare the linear and nonlinear solutions for each of the following values of the initial velocity v: 1, 1.99, 2, 2.01. For the (numerical) nonlinear solution, interpret what the graph indicates the pendulum is doing physically. What do you think the exact solution does in each case?

5. In this problem, we'll investigate the effect of damping on the pendulum, using the model

$$\theta'' + b\theta' + \sin\theta = 0.$$

Prepare Simulink models that will take the value of b from a "Constant" block and

- plot the numerical solution of this differential equation with initial conditions $\theta(0) = 0$, $\theta'(0) = 4$, from $t = 0$ to $t = 20$;
- do the same for the linear approximation

$$\theta'' + b\theta' + \theta = 0.$$

(You do not need to send the plot to the printer if it displays in a Scope.) Compare the linear and nonlinear behavior for the values $b = 1$, 1.5, and 2. Interpret what is happening *physically* in each case; *i.e.*, describe explicitly what the graph says the pendulum is doing.

6. In this problem, we'll look at the effect of a periodic external force on the pendulum, using the model

$$\theta'' + 0.05\theta' + \sin\theta = 0.3\cos\omega t \tag{D.5}$$

(*cf.* Problems 3–5). We have chosen a value for the damping coefficient that is more typical of air resistance than the values in the previous problem. Prepare Simulink models for (D.5) and its linear approximation

$$\theta'' + 0.05\theta' + \theta = 0.3\cos\omega t.$$

The right-hand side can be produced by a Sine Wave block (which is in the Sources Library). When you install this block and left-click on it to bring up the **Block Parameters** menu, you will see an Amplitude box in which to insert the parameter 0.3 and a Frequency box in which to insert the parameter ω. Note that since we have a cosine, not a sine, you also have to adjust the Phase (to $\frac{\pi}{2}$, since $\cos\omega t = \sin(\omega t + \frac{\pi}{2})$).

- Plot the numerical solution of this differential equation with initial conditions $\theta(0) = 0$, $\theta'(0) = 0$, from $t = 0$ to $t = 60$.
- Do the same for the linear approximation.

(You can send the plots to the printer from a Scope display using the "printer" icon in the upper-left corner.)

Compare the nonlinear and linear models for the following values of the frequency ω: $0.6, 0.8, 1, 1.2$. Which frequency moves the pendulum farthest away from its equilibrium position? For which frequencies do the linear and nonlinear equations have widely different behaviors? Which forcing frequency seems to induce resonance-type behavior in the pendulum? Graph that solution on a longer interval and decide whether the amplitude goes to infinity.

7. In this problem, we study the effects of air resistance.

(a) A paratrooper steps out of an airplane at a height of 1000 ft and after 5 seconds opens her parachute. Her weight, with equipment, is 195 lbs. Let $y(t)$ denote her height above the ground after t seconds. Assume that the force due to air resistance is $0.005y'(t)^2$ lbs in free fall and $0.6y'(t)^2$ lbs with the chute open. At what height does the chute open? How long does it take to reach the ground? At what velocity does she hit the ground? (This model assumes that air resistance is proportional to the *square* of the velocity and that the parachute opens instantaneously.)

(*Hint*: This problem can be solved most efficiently by using an ODE file to detect significant events. Pay attention to units. Recall that the mass of the paratrooper is $195/32$, measured in lb sec^2/ft. Here, 32 is the acceleration due to gravity, measured in ft/sec^2.)

(b) Let $v = y'$ be the velocity during the second phase of the fall (while the chute is open). One can view the equation of motion as an autonomous first order ODE in the velocity:

$$v' = -32 + \frac{192}{1950}v^2.$$

Make a qualitative analysis of this first order equation, finding in particular the critical or equilibrium velocity. This velocity is called the terminal velocity. How does the terminal velocity compare with the velocity at the time the chute opens and with the velocity at impact?

(c) Assume the paratrooper is safe if she strikes the ground at a velocity within 5% of the terminal velocity in (b). Except for the initial height, use the parameters in (a). What is the lowest height from which she may parachute safely? (*Please do not try this at home!*)

8. In many applications, second order equations come with *boundary conditions* rather than initial conditions. For example, consider a cable that is attached at each end to a post, with both ends at the same height (let us call this height $y = 0$). If the posts are located at $x = 0$ and $x = 1$, the height y of the cable as a function of x satisfies the differential equation

$$y'' = c\sqrt{1 + (y')^2},$$

with boundary conditions $y(0) = 0$, $y(1) = 0$. The constant c depends on the length of the cable; for this problem we'll use $c = 1$.

(a) Solve the boundary value problem with **dsolve**.

(b) Solve this problem numerically with the shooting method. Plot the solution on $[0, 1]$, and determine the maximum dip in the cable.

(c) Solve the auxiliary initial value problem using **dsolve**, and then find the critical value for the parameter s using **fzero**.

(d) Solve this boundary value problem using **bvp4c**.

Show that the solutions obtained in parts (a), (b), (c), and (d) agree with each other.

9. As explained in the last problem, sometimes one is faced with a second order differential equation with a boundary condition. Existence and uniqueness of solutions for such problems is more complicated than for initial value problems, as you will see in this problem. Consider the simple boundary value problem (BVP)

$$y'' + \alpha^2 y = 0, \qquad y(0) = 0, \quad y(1) = 1, \tag{D.6}$$

where $\alpha > 0$ is a parameter.

(a) Solve the problem explicitly with **dsolve** using only the left-hand condition $y(0) = 0$. Since this is a second order equation and you have not specified $y'(0)$, the answer should involve an undetermined constant $C1$. Solve for this constant (in terms of α) using **solve** in order to satisfy the right-hand condition $y(1) = 1$. For what values of α does a solution exist?

(b) Redo part (a), but now modifying the right-hand condition in (D.6) to $y(1) = 0$. This time there is always a solution (namely $y = 0$), since the equation is now linear and homogeneous, but sometimes it is non-unique. For what values of α do you get more than one solution of the BVP? How much non-uniqueness is there in the solution?

(c) What happens to the solution of (D.6) (as computed in part (a)) when $\alpha \to \pi$? What do you observe about the solution as computed with **bvp4c** for α close to π, say 3.1415926? What about $\alpha = 3.1415926535898$? How do the plots of the solutions in the two cases differ, and how can you explain this?

10. The problem of finding the function $u(x)$ satisfying

$$\begin{cases} a(x)u''(x) + a'(x)u'(x) = f(x), & 0 \le x \le 1 \\ u(0) = u(1) = 0. \end{cases} \tag{D.7}$$

arises in studying the longitudinal displacements in a longitudinally loaded elastic bar. The bar is of length 1, its left end is at $x = 0$, and its right end at $x = 1$. In the differential equation above, $f(x)$ represents the external force on the bar (which is assumed to be longitudinal, *i.e.*, directed along the bar), $a(x)$ represents both the elastic properties and the cross-sectional area of the bar, and $u(x)$ is the longitudinal displacement of the bar at the point x. This problem is an example of a *boundary value problem*. The function $a(x)$ may be constant; this is the case if neither the elastic properties nor the cross-sectional area depends on the position x in the rod, *i.e.*, if the rod is uniform. But if the rod is not uniform, then $a(x)$ is not constant, and the equation has variable coefficients. Similarly, $f(x)$ will be constant only if the external force is applied uniformly along the rod.

(a) Take $a(x) = 1 + x$ and $f(x) = 5\sin^2(2\pi x)$. Here the force is applied symmetrically and is strongest at $x = 0.25$ and $x = 0.75$, as if someone is hanging by both hands from a relatively short rod. Use **dsolve** to find a solution $u(x)$ to the boundary problem (D.7), and plot it on the interval $[0, 1]$. Based on the result, do you think the rod is more flexible where $a(x)$ is larger or where it is smaller?

(b) Now suppose $a(x) = 1 + \exp(x)$ and $f(x)$ is the same as in part (a). Find and plot the corresponding solution $u(x)$ to (D.7). Since MATLAB cannot solve this equation symbolically, we will use a numerical method. We cannot apply **ode45** directly, since it requires initial conditions. Nonetheless, we can use the shooting method to find the value of $u'(0)$ that leads to a solution satisfying the condition $u(1) = 0$. By trial and error, find $u'(0)$ to at least two decimal places, and graph the resulting solution on $[0, 1]$. What is the maximum displacement, and where does it occur?

(c) Repeat part (b) using **bvp4c** instead of the shooting method.

11. This problem is based on Problem 32 in Boyce & DiPrima, Section 3.8. Consider a frictionless mass-spring system as in Figure 3.8.10 in Boyce & DiPrima (standard mass attached to a spring on a frictionless table). Suppose the restoring force of the spring is not given by Hooke's Law, but instead is of the form

$$F = -(ky + \epsilon y^3).$$

If $\epsilon = 0$, the assumption amounts to Hooke's Law, but in this problem we shall focus on $\epsilon \neq 0$ (either positive or negative). If we have air resistance present with damping coefficient γ, then the equation of motion (assuming the displacement is indicated by the variable y) becomes

$$my'' + \gamma y' + ky + \epsilon y^3 = 0$$

(see Boyce & DiPrima, 8th edition, p. 206). Henceforth we shall normalize by assuming that $m = k = 1$ and $\gamma = 0$, and then take the initial conditions

$$y(0) = 0, \quad y'(0) = 0.$$

(a) Plot the solution when $\epsilon = 0$. What is the amplitude and period of the solution?

(b) Let $\epsilon = 0.1$. Plot a numerical solution. Is the motion periodic? Estimate the amplitude and period.

(c) Repeat part (b) for $\epsilon = 0.2$, and then for $\epsilon = 0.3$.

(d) Plot your estimated values of the amplitude A and period T as functions of ϵ. How do A and T depend on ϵ?

(e) Repeat parts (b)–(d) for negative values of ϵ.

12. In this problem we study how solutions of the initial value problem

$$\frac{d^2y}{dt^2} + 0.15\frac{dy}{dt} - y + y^3 = 0, \quad y(0) = c, \quad y'(0) = 0$$

depend on the initial value c.

(a) Plot a numerical solution of this equation from $t = 0$ to $t = 40$ for each of the initial values $c = 0.5, 1, 1.5, 2, 2.5$. Describe how increasing the initial value of y affects the solutions, both in terms of their limiting behavior and their general appearance.

(b) Plot all five solutions on one graph. Would such a picture be possible for solutions of a first order differential equation? Why or why not?

13. In this problem, we consider the long-term behavior of solutions of the initial value problem

$$\frac{d^2y}{dt^2} + 0.2\frac{dy}{dt} - y + y^3 = 0.3\cos(\omega t), \quad y(0) = 0, \quad y'(0) = 0 \tag{D.8}$$

for various frequencies ω in the forcing term.

Plot a numerical solution of (D.8) from $t = 0$ to $t = 100$ for each of the eight frequencies $\omega = 0.8, 0.9, \ldots, 1.4, 1.5$.

Describe and compare the different long-term behaviors you see. Due to the forcing term, all solutions will oscillate, but pay particular attention to the magnitude of the oscillations and to whether or not there is a periodic pattern to them. Are there any similarities between your results for this nonlinear system and the phenomenon of resonance for linear systems with periodic forcing?

14. In this problem, we study the zeros of solutions of the second order differential equation

$$y'' + (3 - \cos t)y = 0. \tag{D.9}$$

(a) Compute and plot several solutions of this equation with different initial conditions: $y(0) = c$, $y'(0) = d$. To be specific, choose three different values for the pair c, d and plot the corresponding solutions over $[0, 20]$. By inspecting your plots, find a number L that is an upper bound for the distance between successive zeros of the solutions. Then find a number l that is a lower bound for the distance between successive zeros.

(b) Information on the zeros of solutions of linear second order ODEs can be obtained from the Sturm Comparison Theorem (see Theorem 10.1 in Chapter 10). By comparing (D.9) with the equation $y'' + 2y = 0$, you should be able to get a value for L, and by comparing (D.9) with $y'' + 4y = 0$, you should be able to find l. How do these values compare with the values obtained in part (a)? (You will find it useful to note that the general solution of $y'' + ky = 0$, where k is a positive constant, can be written as $y = R\cos(\sqrt{k}x - \delta)$, with $R \geq 0$, δ arbitrary. R is called the *amplitude* and δ the *phase shift*.)

(c) Plot a solution of (D.9) and a solution of $y'' + 2y = 0$ on the same graph, and verify that between any two zeros of the latter solution, there is at least one zero of the solution of (D.9).

15. This problem is based on Problems 18 and 19 in Boyce & DiPrima, Section 3.9. Consider the initial value problem

$$u'' + u = 3\cos(\omega t), \quad u(0) = 0, \quad u'(0) = 0.$$

(a) Find the solution (using **dsolve**). For $\omega = 0.5, 0.6, 0.7, 0.8, 0.9$, plot the solution curves on the interval $0 \leq t \leq 15$. Note that $\omega_0 = 1$ is the natural frequency of the homogeneous equation. Describe how the solution curves change as ω gets closer to 1.

(b) Note that the formula you found in part (a) is invalid when $\omega = 1$. Find and plot the solution curve for $\omega = 1$ on the interval $0 \leq t \leq 15$. Based on the discussion of forced vibrations in your text, what phenomenon should be exhibited for this value of ω? Corroborate your answer by plotting on a longer interval.

(c) Plot the solution for $\omega = 0.9$ on a longer interval and compare it with the solution from part (b). What phenomenon is exhibited by the curve for $\omega = 0.9$?

16. A solution of a second order linear differential equation is called *oscillatory* if it changes sign infinitely many times and *nonoscillatory* if it changes sign only finitely many times. In this problem, we will be interested in determining the oscillatory nature of nonzero solutions to some second order linear ODEs. First consider the equation $y'' + ky = 0$, where k is a constant. If $k > 0$, it has the general solution $y = R\cos(\sqrt{k}x - \delta)$, with $R \geq 0, \delta$ arbitrary. The constant R is called the *amplitude*, and δ is called the *phase shift*. From this formula we see that every solution $y(x)$ has infinitely many zeros, and hence changes sign infinitely many times, *i.e.*, is oscillatory. If $k < 0$, the general solution is $y(x) = c_1e^{\sqrt{-k}x} + c_2e^{-\sqrt{-k}x}$, while if $k = 0$, the general solution is $y(x) = c_1x + c_2$. These solutions change sign at most once, and so are nonoscillatory.

Next, consider Airy's equation

$$y'' = xy, \tag{D.10}$$

which arises in various applications. Since (D.10) has a variable coefficient, we cannot study it by elementary methods; in particular, we cannot find the general solution in terms of elementary functions as we did for the constant coefficient equation. We will instead first make a graphical study of the solutions of (D.10) and then study the solutions using the Sturm Comparison Theorem (see Chapter 10).

(a) Compute and plot several solutions of (D.10) with different initial conditions $y(0) = c$, $y'(0) = d$ over the interval $[-10, 5]$. Use different combinations of positive, negative, and zero values for c and d. What can you say in general about the zeros of the solution? Where do they occur on the x axis, and how far apart are the successive zeros of a solution? Are the solutions oscillatory as $x \to \infty$? As $x \to -\infty$?

(b) Information on the zeros of solutions of linear second order ODEs, and hence information on their oscillatory nature, can also be obtained from the Sturm Comparison Theorem. By comparing (D.10) with the equation $y'' + by = 0$ for $x \leq -b < 0$, what do you learn about zeros on the negative x axis and their spacing, and hence about the oscillatory nature of solutions of (D.10) for $x \leq 0$? By comparing (D.10) with $y'' = 0$ for $0 < x$, what do you learn about the oscillatory nature of solutions for $x > 0$? How many zeros could a solution have on the positive x-axis? Do the graphs you plotted in part (a) agree with these results?

17. In this problem, we study solutions of the parabolic cylinder equation

$$y'' + \left(n + \frac{1}{2} - \frac{x^2}{4}\right) y = 0, \tag{D.11}$$

which arises in the study of quantum-mechanical vibrations. Since the equation (D.11) is unchanged if x is replaced by $-x$, any solution function will be symmetric with respect to the y-axis. Therefore, we focus our attention on $x \geq 0$.

(a) Find the corresponding first order equation for $z = \arctan(y/y')$, as described in Section 10.4.

(b) For $n = 1$, plot the direction field for the z equation from $x = 0$ to $x = 10$. (Remember to use $-\pi/2 \leq z \leq \pi/2$.) Based on the plot, predict what the solutions y to the parabolic cylinder equation look like near $x = 0$ and for larger x. Is there a value of x around which you expect their behavior to change?

(c) Now solve the parabolic cylinder equation numerically for $n = 1$ with the two sets of initial conditions $y(0) = 1$, $y'(0) = 0$ and $y(0) = 0$, $y'(0) = 1$. Plot the two solutions on the same graph. (It will probably help to change the range on the plot.) Do the solutions behave as you expected?

(d) Repeat parts (b) and (c) for $n = 5$. Point out any differences from the case $n = 1$.

(e) Repeat parts (b) and (c) for $n = 15$. Discuss how the solutions are changing as n increases.

(f) Now consider the three solutions (for $n = 1$, 5, and 15) with $y(0) = 0$. Point out similarities and differences. By drawing an analogy with Airy's equation, argue from the direction fields that, for any n, exactly one solution function decays for large x while all others grow. Do you have enough graphical evidence from the numerical plots to conclude that the solution corresponding to the initial data $y(0) = 0$, $y'(0) = 1$ is that function? Why or why not? (*Hint*: Look at previous discussions of stability of solutions as a guide.)

18. Consider Bessel's equation of order zero

$$x^2y'' + xy' + x^2y = 0.$$

(a) Compute and plot several solutions of this equation with different initial conditions: $y(0.1) = c, y'(0.1) = d$. To be specific, choose three different values for the pair c, d and plot the corresponding solutions over $[0.1, 20]$. (Why isn't it a good idea to use $x = 0$ in the initial conditions?) By inspecting your plots, find a number L that is an upper bound for the distance between successive zeros of the solutions. Then find a number l that is a lower bound for the distance between successive zeros.

(b) Now confirm your findings with the Sturm Comparison Theorem. It doesn't apply directly, but if we introduce the new function $z(x) = x^{1/2}y(x)$, Bessel's equation becomes

$$z'' + (1 + 1/(4x^2))z = 0,$$

which has the form of the equation in the Comparison Theorem. Since y will have a zero wherever z has a zero, we can study the zeros of y by studying the zeros of z. By comparing the equation for z with the equation $z'' + z = 0$, determine an upper bound L on the distance between successive zeros of any solution of Bessel's equation.

(c) Let a be a positive number. For $x \in [a, \infty)$, the quantity $1+1/(4x^2)$ is less than or equal to the constant $1 + 1/(4a^2)$. By making an appropriate comparison, determine a lower bound l on the distance between successive zeros of any solution of Bessel's equation (for $x > a$). What is the limiting value of l as a goes to ∞? Approximately how far apart are the zeros when x is large? Did your graphical study lead to comparable values for l and L?

19. In this problem, we study solutions of Bessel's equation of order n,

$$y'' + \frac{1}{x}y' + \left(1 - \frac{n^2}{x^2}\right) y = 0, \tag{D.12}$$

for $n > 0$. Solutions of this equation, called *Bessel functions* of order n, are used in the study of vibrations and waves with circular symmetry. Since (D.12) is unchanged if x is replaced by $-x$, we focus our attention on $x \geq 0$.

(a) Find the corresponding first order equation for $z = \arctan(y/y')$, as described in Section 10.4 of Chapter 10.

(b) For $n = 1$, plot the direction field for the z equation from $x = 0$ to $x = 20$. (Remember to use $-\pi/2 \le z \le \pi/2$.) Based on the plot, predict what the Bessel functions of order 1 look like for small x and then for large x. Is there a value of x around which you expect their behavior to change?

(c) Now plot the Bessel functions **besselj(n, x)** and **bessely(n, x)** for $n = 1$ on the same graph. (It will probably help to change the range on the plot.) Do the solutions behave as you expected?

(d) Repeat parts (b) and (c) for $n = 5$. Point out any differences from the case $n = 1$.

(e) Repeat parts (b) and (c) for $n = 15$. Discuss how the solutions are changing as n increases.

20. Consider the mass-spring system with dry friction, depicted in Figure D.1. This

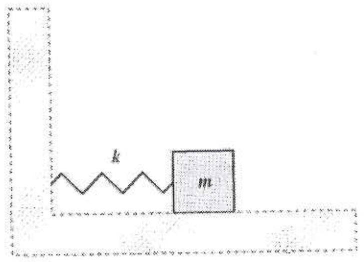

Figure D.1: Mass-Spring System with Friction

system is governed by the following equations:

$$\begin{cases} my'' = -ky - \mu mg\,\mathrm{sign}(y'), & \text{unless } y' = 0 \text{ and } |ky| < \mu mg, \\ y'(t) \equiv 0, \quad t \ge t_0 & \text{if } y'(t_0) = 0 \text{ and } |ky(t_0)| < \mu mg. \end{cases} \tag{D.13}$$

The first equation in (D.13) governs *kinetic friction*, and says that the object of mass m is subject to a restoring force opposite and proportional to the displacement y, with proportionality constant the spring constant k, as well as to a friction force opposite to the direction of motion, given by the friction coefficient μ times the force of gravity on the mass. The second equation in (D.13) governs *static friction*, and says that if the mass is instantaneously not moving at time t_0 and the force exerted on it by the spring is not big enough, then it will never move at all. In this model we are assuming for simplicity (even though this is usually not the case) that the coefficients of static and kinetic friction are equal. (In a more realistic model, the μ in the second equation would be larger than the μ in the first equation.) We are also

assuming that the force of friction is independent of velocity in the kinetic friction case, which is not 100% correct, but is a reasonable first approximation to the truth. Note that the equations (D.13) really only depend on two constants, $d = \mu g$, and $\omega = \sqrt{k/m}$. When $\mu = 0$ (case of no friction), they reduce to the usual spring equation $y'' + \omega^2 y = 0$ with angular frequency ω. On the other hand, in the limiting case where $\mu \to \infty$, the object comes to an immediate stop and then doesn't move at all.

(a) Write a function M-file `friction.m` starting with the line

```
function [T, Y] = friction(time, d, omega, y0, v0)
```

that takes as parameters the time interval **time**, d, ω, and initial values **y0** of position and **v0** of velocity, and outputs a table of values of y with the corresponding values of position and velocity. There are (at least) two ways to structure your M-file. The most natural method is to have lines that say

```
options = odeset('Events', static);
[T, Y, TE, YE, IE] = ...
   ode45(rhs, time, [y0 v0], options);
```

Ignoring the **options** for the moment, the second line calls **ode45** on an anonymous function **rhs** (whose definition will involve the parameters **d** and **omega**) that represents the right-hand side of the kinetic friction equation, expressed as a first order system. But **odeset** is used here to call the event detection feature of the ODE solver, to check for the condition

$$|y| < \frac{\mu mg}{k} = \frac{d}{\omega^2}$$

in the static friction equation. This condition should be encoded in an anonymous function **static**. Since **ode45** insists that an "events" function have only two inputs, and since the static friction condition also involves **d** and **omega**, the easiest thing is to define **static** by a line

```
static = @(t, y) static1(t, y, d, omega)
```

and to define an additional function M-file `static1.m` just as in Section 8.6.3, but with four inputs instead of two. The first output of `static1.m` should be **y(2)**, the velocity, but the second output, the "stop function," should be 1 if **abs(y(1))** is small enough (meaning one should stop in this case), 0 otherwise. Once the integration of the kinetic friction equation stops, the mass should remain constant thereafter. The alternative method is to call **ode45** on a "right-hand side" function that is discontinuous, and is given by the two different equations in (D.13), depending on conditions.

(b) Suppose we denote the initial data by $y_0 = y(0)$, $v_0 = y'(0)$. Use your function M-file `friction.m` to solve the IVP when $y_0 = 4$, $v_0 = 0$, $\omega = 1$, and $d = 0.75$. Draw two graphs: on one, graph the displacement y for $0 \le t \le 10$; on the second, draw a phase diagram for y, y' over the same time interval. Explain what the pictures mean. Where does the mass come to rest?

(c) Answer part (b) but with $y_0 = 3$, then with $y_0 = 2$.

(d) Now change the initial data to $y_0 = 0$, $v_0 = 5$ and leave $\omega = 1$, $d = 0.75$. What kind of motion ensues? (*Hint*: You may need to lengthen the time interval.)

(e) Find (to the nearest tenth) the value of v_0 (with $y_0 = 0$, $\omega = 1$, $d = 0.75$) such that initial velocities below or equal to that value do not propel the mass out of the "friction well" $-0.75 \leq y \leq 0.75$, but initial velocities above it propel it so that that it goes beyond the well once, but when it returns to the well, it does not escape it again.

Chapter 11

Series Solutions

A primary theme of this book is that numerical, geometric, and qualitative methods can be used to study solutions of differential equations, even when we cannot find an exact formula solution. Of course, formula solutions are extremely valuable, and there are many techniques for finding them. For instance, techniques for finding formula solutions of second order linear differential equations include:

- The exponential substitution, which leads to solutions of an arbitrary constant coefficient homogeneous equation.
- The method of *reduction of order*, which produces a second linearly independent solution of a homogeneous equation when one solution is already known.
- The method of *undetermined coefficients*, which solves special kinds of inhomogeneous equations with constant coefficients.
- The method of *variation of parameters*, which yields the solution of a general inhomogeneous equation, given a fundamental set of solutions to the homogeneous equation.

Each of these techniques reduces the search for a formula solution to a simpler problem in algebra or calculus: finding the root of a polynomial, computing an antiderivative, or solving a pair of simultaneous linear equations.

The process of finding formulas for exact solutions of equations, either by hand or by computer, is called *symbolic computation*. The **`dsolve`** command incorporates all of the techniques listed above. It enables us to find exact solutions more rapidly and more reliably than we could by hand.

There are many differential equations that do not yield to the techniques listed above. By using more advanced ideas from calculus, however, we can find exact or approximate solutions for a wider class of differential equations. In this chapter and the next, we discuss two such calculus-based techniques for finding solutions of differential equations: *series solutions* and *Laplace transforms*. The method of series solutions constructs power series solutions of linear differential equations with variable coefficients; in many examples, we