Problem Set B

First Order Equations

The solution to Problem 5 appears in the *Sample Solutions*.

1. Consider the initial value problem

$$ty' + 3y = 5t^2, \qquad y(2) = 5.$$

   (a) Solve using **`dsolve`**. Define the solution function $y(t)$ in MATLAB, and then determine its behavior as t approaches 0 from the right and as t becomes large. This can be done by plotting the solution on intervals such as $0.5 \le t \le 5$ and $0.2 \le t \le 20$.

   (b) Change the initial condition to $y(2) = 3$. Determine the behavior of this solution, again by plotting on intervals such as those mentioned in part (a).

   (c) Find a general solution of the differential equation by solving

$$ty' + 3y = 5t^2, \qquad y(2) = c.$$

   Now find the solutions corresponding to the initial conditions

$$y_j(2) = j, \qquad j = 3, \ldots, 7.$$

   Plot the functions $y_j(x)$, $j = 3, \ldots, 7$, on the same graph. Describe the behavior of these solutions for small positive t and for large t. Find the solution that is not singular at 0. Identify its plot in the graph.

2. Consider the differential equation

$$ty' + 2y = e^t. \qquad \text{(B.1)}$$

   With initial condition $y(1) = 1$, this has the solution

$$y(t) = \frac{e^t(t-1)+1}{t^2}.$$

(a) Verify this using MATLAB, both by direct differentiation and by using **dsolve**.

(b) Graph $y(t)$ on the interval $0 < t < 2$. Describe the behavior of the solution near $t = 0$ and for large values of t.

(c) Plot the solutions $y_j(t)$ of (B.1) corresponding to the initial conditions

$$y_j(1) = j, \qquad j = -3, -2, \ldots, 2, 3,$$

all on the same graph. Use **axis** to adjust the picture.

(d) What do the solutions have in common near $t = 0$? for large values of t? Is there a solution to the differential equation that has no singularity at $t = 0$? If so, what is it?

3. Consider the initial value problem

$$ty' + y = 2t, \qquad y(1) = c.$$

(a) Solve it using MATLAB.

(b) Evaluate the solution with $c = 0.8$ at $t = 0.01, 0.1, 1, 10$. Do the same for the solutions with $c = 1$ and $c = 1.2$.

(c) Plot the solutions with $c = 0.8, 0.9, 1.0, 1.1, 1.2$ together on the interval $(0, 2.5)$.

(d) How do changes in the initial data affect the solution as $t \to \infty$? as $t \to 0^+$?

4. Solve the initial value problem

$$y' - 2y = \sin 2t, \qquad y(0) = c.$$

Use MATLAB to graph solutions for $c = -0.5, -0.45, \ldots, -0.05, 0$. Display all the solutions on the same interval between $t = 0$ and an appropriately chosen right endpoint. Explain what happens to the solution curves as t increases. You should identify three distinct types of behavior. Which values of c correspond to which behaviors? Now, based on this problem, and the material in Chapters 5 and 6, discuss what effect small changes in initial data can have on the global behavior of solution curves.

5. Consider the differential equation

$$\frac{dy}{dt} = \frac{t - e^{-t}}{y + e^{y}}$$

(*cf.* Problem 7, Section 2.2 in Boyce & DiPrima).

(a) Solve it using **dsolve**. Observe that the solution is given implicitly. Express it clearly in the form

$$f(t, y) = c.$$

(b) Use **contour** to see what the solution curves look like. For your t and y ranges, you might use **-1:0.05:3** and **-2:0.05:2**. Plot 30 contours.

(c) Plot the solution satisfying the initial condition $y(1.5) = 0.5$.

(d) To find a numerical value for the solution $y(t)$ from part (c) at a particular value of t, you can plug the value t into the equation

$$f(t, y) = f(1.5, 0.5)$$

and solve for y. Because there may be multiple solutions, you should look for one near $y = 0.5$. This can be done using **fzero**. (See Section 3.7.) Find $y(0), y(1), y(1.8), y(2.1)$. Mark these values on your plot.

6. Consider the differential equation

$$e^y + (te^y - \sin y)\frac{dy}{dt} = 0.$$

(a) Solve using **dsolve**. Observe that the solution is given implicitly in the form

$$f(t, y) = c.$$

(b) Use **contour** (see Section 3.8.6) to see what the solution curves look like. For your t and y ranges, you might use **-1:0.1:4** and **0:0.1:3**. Plot 30 contours.

(c) Plot the solution satisfying the initial condition $y(2) = 1.5$.

(d) Find $y(1), y(1.5), y(3)$. Mark these values on your plot. (See Problem 5, part (d) for suggestions.)

7. Consider the differential equation

$$y' = \frac{t^2}{1 + y^2}$$

(*cf.* Problem 8, Section 2.2 in Boyce & DiPrima).

(a) Solve it using **dsolve**. Observe that in some sense MATLAB is "too good" in that it finds three rather complicated explicit solutions. Note that two of them are complex-valued. Select the real one. In fact, for this solution it is easier to work with an implicit form; so here are instructions for converting your solution. Substitute $u = t^3$ (or $t = u^{(1/3)}$) to generate an expression in u. Set it equal to y and **solve** for u. Replace u by t^3 to get an implicit solution of the form

$$f(t, y) = c.$$

(b) Use **contour** to see what the solution curves look like. For both your t and y ranges, you might use **-1.5:0.05:1.5**. Plot 30 contours.

(c) Plot the solution satisfying the initial condition $y(0.5) = 1$.

(d) To find a numerical value for the solution $y(t)$ from part (c) at a particular value of t, you can plug the value t into the equation

$$f(t, y) = f(0.5, 1)$$

and solve for y. This can be done using **fzero**. (See Section 3.7.) Find $y(-1), y(0), y(1)$. Mark these values on your plot.

8. In this problem, we study continuous dependence of solutions on initial data.

(a) Solve the initial value problem

$$y' = y/(1 + t^2), \qquad y(0) = c.$$

(b) Let y_c denote the solution in part (a). Use MATLAB to plot the solutions y_c for $c = -10, -9, \ldots, -1, 0, 1, \ldots, 10$ on one graph. Display all the solutions on the interval $-20 \leq t \leq 20$.

(c) Compute $\lim_{t\to\pm\infty} y_1(t)$.

(d) Now find a single constant M such that for all real t, we have

$$|y_a(t) - y_b(t)| \leq M|a - b|,$$

for any pair of numbers a and b. Show from the solution formula that $M = 1$ will work if we consider only negative values of t.

(e) Relate the fact in (d) to Theorem 5.2.

9. Use **dsolve** to solve the following differential equations or initial value problems from Boyce & DiPrima. In some cases, MATLAB will not be able to solve the equation. (Before moving on to the next equation, make sure you haven't mistyped something.) In other cases, MATLAB may give extraneous solutions. (Sometimes these correspond to non-real roots of the equation it solves to get y in terms of t.) If so, you should indicate which solution or solutions are valid. You also might try entering alternative forms of an equation, for example, $M + Ny' = 0$ instead of $y' = -M/N$, or vice versa.

(a) $y' = ry - ky^2$ (Sect. 2.4, Prob. 29),

(b) $y' = t(t^2 + 1)/(4y^3)$, $y(0) = -1/\sqrt{2}$ (Sect. 2.2, Prob. 16),

(c) $(e^t \sin y + 3y)\, dt - (3t - e^t \sin y)\, dy = 0$ (Sect. 2.6, Prob. 8),

(d) $\dfrac{dy}{dt} = (2y - t)/(2t - y)$ (Sect. 2.2, Prob. 30),

(e) $\dfrac{dy}{dt} = (2t + y)/(3 + 3y^2 - t)$, $y(0) = 0$ (Ch. 2, Miscellaneous Problems, Prob. 3).

10. Use **dsolve** to solve the following differential equations or initial value problems from Boyce & DiPrima. See Problem 9 for additional instructions.

   (a) $t^3y' + 4t^2y = e^{-t}$, $y(-1) = 0$ (Sect. 2.1, Prob. 19),
   (b) $y' + (1/t)y = 3\cos 2t$, $t > 0$ (Sect. 2.1, Prob. 40),
   (c) $y' = ty(4 - y)/3$, $y(0) = y_0$, $t > 0$ (Sect. 2.2, Prob. 27),
   (d) $(\frac{y}{t} + 6t)\,dt + (\ln t - 2)\,dy = 0$, $t > 0$ (Sect. 2.6, Prob. 10),
   (e) $y' = \dfrac{t^2 + ty + y^2}{t^2}$ (Sect. 2.2, Prob. 31),
   (f) $ty' + ty = 1 - y$, $y(1) = 0$ (Ch. 2, Miscellaneous Problems, Prob. 6).

11. Chapter 6 describes how to plot the direction field for a first order differential equation. For each equation below, plot the direction field on a rectangle large enough (but not too large) to show clearly all of its equilibrium points. Find the equilibria and state whether each is stable or unstable. If you cannot determine the precise value of an equilibrium point from the equation or the direction field, use **fzero** or **solve** as appropriate.

   (a) $y' = -y(y - 2)(y - 4)/10$,
   (b) $y' = y^2 - 3y + 1$,
   (c) $y' = 0.1y - \sin y$.

12. In this problem, we use the direction field capabilities of MATLAB to study two nonlinear equations, one autonomous and one non-autonomous.

   (a) Plot the direction field for the equation

   $$\frac{dy}{dt} = 3\sin y + y - 2$$

   on a rectangle large enough (but not too large) to show all possible limiting behaviors of solutions as $t \to \infty$. Find approximate values for all the equilibria of the system (you should be able to do this with **fzero** using guesses based on the direction field picture), and state whether each is stable or unstable.

   (b) Plot the direction field for the equation

   $$\frac{dy}{dt} = y^2 - ty,$$

   again using a rectangle large enough to show the possible limiting behaviors. Identify the unique constant solution. Why is this solution evident from the differential equation? If a solution curve is ever below the constant solution, what must its limiting behavior be as t increases? For solutions lying above the constant solution, describe two possible limiting behaviors as t increases. There is a solution curve that lies along the boundary of the two limiting behaviors. What does it do as t increases? Explain (from the differential equation) why no other limiting behavior is possible.

(c) Confirm your analysis by using **dsolve** on the system $y' = y^2 - ty$, $y(0) = c$, and then examining different values of c.

13. The solution of the differential equation

$$y' = \frac{2y - t}{2t - y}$$

is given implicitly by $|t - y| = c|t + y|^3$. (This is not what **dsolve** produces, which is a more complicated explicit solution for y in terms of t, but it's what you get by making the substitution $v = y/t$, $y = tv$, $y' = tv' + v$ and separating variables. You do not need to check this answer.) However, it is difficult to understand the solutions directly from this algebraic information.

(a) Use **quiver** and **meshgrid** to plot the direction field of the differential equation.

(b) Use **contour** to plot the solutions with initial conditions $y(2) = 1$ and $y(0) = -3$. (Note that **abs** is the absolute value function in MATLAB.) Use **hold on** to put these plots and the vector field plot together on the same graph.

(c) For the two different initial conditions in part (b), use your pictures to estimate the largest interval on which the unique solution function is defined.

14. Consider the differential equation

$$y' = -ty^3.$$

(a) Use MATLAB to plot the direction field of the differential equation. Is there a constant solution? If $y(0) > 0$, what happens as t increases? If $y(0) < 0$, what happens as t increases?

(b) The direction field suggests that the solution curves are symmetric with respect to the y-axis, that is, if $y_1(t)$ is a solution to the differential equation, then so is $-y_1(t)$. Verify that this is so directly from the differential equation.

(c) Use **dsolve** to solve the differential equation, thereby obtaining a solution of the form

$$y(t) = \frac{\pm 1}{\sqrt{t^2 + c}}.$$

Note that $y(0) = \pm 1/\sqrt{c}$.

Let's consider the curves above the y-axis. The ones below it are their reflections. (What about the y-axis itself?) In graphing $y(t)$, it helps to consider three separate cases: (i) $c > 0$, (ii) $c = 0$, and (iii) $c < 0$.

(d) In each case, graph the solution for several specific values of c, and identify important features of the curves. In particular, in case (iii), compute the location of the asymptotes in terms of c.

(e) Identify five different types of solution curves for the differential equation that lie above the y-axis. In each case, specify the t-interval of existence, whether the solution is increasing or decreasing, and the limiting (asymptotic) behavior at the ends of the interval.

(f) Combine the direction field plotted in (a) with the graphs plotted in (d).

15. Use **dsolve** to solve the differential equation

$$y' = ty^3 \tag{B.2}$$

with the initial condition $y(0) = c$.

(a) How many solutions does MATLAB give? Do they all satisfy the initial condition? Why do you think that MATLAB gives multiple answers?

(b) What happens if you substitute $c = 0$ in the solution to (B.2)? What happens if you instead take the limit of the solution as $c \to 0$? Do you indeed get the correct solution of the equation for this case?

(c) Plot the solutions to (B.2) for $c = -5, -4, \ldots, 4, 5$ on the same set of axes. Is the initial value problem stable?

16. Consider the critical threshold model for population growth

$$y' = -(2 - y)y.$$

(a) Find the equilibrium solutions of the differential equation. Now draw the direction field, and use it to decide which equilibrium solutions are stable and which are unstable. In particular, what is the limiting behavior of the solution if the initial population is between 0 and 2? greater than 2?

(b) Use **dsolve** to find the solutions with initial values 1.5, 0.3, and 2.1, and plot each of these solutions. Find the inflection point for the first of these solutions.

(c) Plot the three solutions together with the direction field on the same graph. Do the solutions follow the direction field as you expect it to?

17. Consider the following logistic-with-threshold model for population growth:

$$y' = y(1 - y)(y - 3).$$

(a) Find the equilibrium solutions of the differential equation. Now draw the direction field, and use it to decide which equilibrium solutions are stable and which are unstable. In particular, what is the limiting behavior of the solution if the initial population is between 1 and 3? greater than 3? exactly 1? between 0 and 1?

(b) Next replace the logistic law by the Gompertz model, but retain the threshold feature. The equation becomes

$$y' = y(1 - \ln y)(y - 3).$$

Once again, find the equilibrium solutions and draw the direction field. You will have difficulty "reading the field" between 2.5 and 3. There appears to be a continuum of equilibrium solutions.

(c) Plot the function $f(y) = y(1 - \ln y)(y - 3)$ on the interval $0 \le y \le 4$, and then use **`limit`** to evaluate $\lim_{y \to 0} f(y)$.

(d) Use these plots and the discussion in Chapter 6 to decide which equilibrium solutions are stable and which are unstable. Now use the last plot to explain why the direction field (for $2.5 \le y \le 3$) appears so inconclusive regarding the stability of the equilibrium solutions. (*Hint*: The maximum value of f is a relevant number.)

18. This problem is based on Example 1 in Section 2.3 of Boyce & DiPrima: "A tank contains Q_0 lb of salt dissolved in 100 gal of water. Water containing 1/4 lb of salt per gallon enters the tank at a rate of 3 gal/min, and the well-stirred solution leaves the tank at the same rate. Find an expression for the amount of salt $Q(t)$ in the tank at time t."

The differential equation

$$Q'(t) = 0.75 - 0.03Q(t)$$

models the problem (*cf.* equation (2) in Section 2.3 of Boyce & DiPrima).

(a) Plot the right-hand side of the differential equation as a function of Q, and identify the critical point.

(b) Analyze the long-term behavior of the solution curves by examining the sign of the right-hand side of the differential equation, in a similar fashion to the discussion in Section 6.3.

(c) Use MATLAB to plot the direction field of the differential equation. In choosing the rectangle for the direction field be sure to include the point $(0, 0)$ and the critical value of Q.

(d) Use the direction field to estimate the limiting amount of salt and to determine how the amount of salt approaches this limit.

(e) Use **`dsolve`** to find the solution $Q(t)$ and plot it for several specific values of Q_0. Do the solutions behave as indicated in parts (b) and (d)? You should combine the direction field plot from (c) with that of the solution curves.

19. A 10-gallon tank contains a mixture consisting of 1 gallon of water and an undetermined number $S(0)$ of pounds of salt in the solution. Water containing 1 lb/gal

of salt begins flowing into the tank at the rate of 2 gal/min. The well-mixed solution flows out at a rate of 1 gal/min. Derive the differential equation for $S(t)$, the number of pounds of salt in the tank after t minutes, that models this physical situation. (*Note*: At time $t = 0$ there is 1 gallon of solution, but the volume increases with time.) Now draw the direction field of the differential equation on the rectangle $0 \le t \le 10$, $0 \le S \le 10$. From your plot,

(a) find the value A of $S(0)$ below which the amount of salt is a constantly increasing function, but above which the amount of salt will temporarily decrease before increasing;

(b) indicate how the nature of the solution function in case $S(0) = 1$ differs from all other solutions.

Now use **dsolve** to solve the differential equation. Reinforce your conclusions above by

(c) algebraically computing the value of A;

(d) giving the formula for the solution function when $S(0) = 1$;

(e) giving the amount of salt in the tank (in terms of $S(0)$) when it is at the point of overflowing;

(f) computing, for $S(0) > A$, the minimum amount of salt in the tank, and the time it occurs;

(g) explaining what principle guarantees the truth of the following statement: If two solutions S_1, S_2 correspond to initial data $S_1(0)$, $S_2(0)$ with $S_1(0) < S_2(0)$, then for any $t \ge 0$, it must be that $S_1(t) < S_2(t)$.

20. In this problem, we use **dsolve** and **solve** to model some population data. The procedure will be:

(i) Assume a model differential equation involving unknown parameters.

(ii) Use **dsolve** to solve the differential equation in terms of the parameters.

(iii) Use **solve** to find the values of the parameters that fit the given data.

(iv) Make predictions based on the results of the previous steps.

(a) Let's use the model

$$\frac{dp}{dt} = ap + b, \quad p(0) = c,$$

where p represents the population at time t. Check to see that **dsolve** can solve this initial value problem in terms of the unknown constants a, b, c. Then define a function that expresses the solution at time t in terms of a, b, c, and t. Give physical interpretations to the constants a, b and c.

(b) Next, let's try to model the population of Nevada,which was one of the fastest growing states in the U.S. during the second half of the twentieth century. Here is a table of census data:

Year	Population in thousands
1950	160.1
1960	285.3
1970	488.7
1980	800.5
1990	1201.8

We would like to find the values of a, b, and c that fit the data. However, with three unknown constants we will not be able to fit five data points. Use **solve** to find the values of a, b, and c that give the correct population for the years 1960, 1970, and 1980. We will later use the data from 1950 and 1990 to check the accuracy of the model. *Important*: In this part let t represent the time in years since 1950, because MATLAB may get stuck if you ask it to fit the data at such high values of t as 1960–1980.

(c) Now define a function of t that expresses the predicted population in year t using the values of a, b, c found in part (b). Find the population this model gives for 1950 and 1990, and compare with the values in the table above. Use the model to predict the population of Nevada in the year 2000, and to predict when the population will reach 3 million. How would you adjust these predictions based on the 1950 and 1990 data? What adjustment to a and/or b might you make? Finally, graph the population function that the model gives from 1950 to 2050, and describe the predicted future of the population of Nevada, including the limiting population (if any) as $t \to \infty$.

21. Consider $y' = (\alpha - 1)y - y^3$.

(a) Use **solve** to find the roots of $(\alpha - 1)y - y^3$. Explain why $y = 0$ is the only real root when $\alpha \leq 1$, and why there are three distinct real roots when $\alpha > 1$.

(b) For $\alpha = -2, -1, 0$, draw a direction field for the differential equation and deduce that there is only one equilibrium solution. What is it? Is it stable?

(c) Do the same for $\alpha = 1$.

(d) For $\alpha = 1.5$, 2, draw the direction field. Identify all equilibrium solutions, and describe their stability.

(e) Explain the following statement: "As α increases through 1, the stable solution $x = 0$ *bifurcates* into two stable solutions."

22. In Chapter 6, we discussed the equation (6.3) and the fact that its solutions blow up in finite time. In this problem, we explore the solutions in a little more detail.

(a) Use **dsolve** to solve the initial value problem $y' = y^2 + t$ with the initial condition $y(0) = 0$. You may find that MATLAB writes the solution in terms of some mysterious functions that are unfamiliar to you, namely **`AiryAi`** and **`AiryBi`**. You can find out more about these by typing **`mhelp AiryAi`**, at least if you are not using the Student Version of MATLAB. (The command **`mhelp`** retrieves help about commands that are internal to the Maple kernel, used for symbolic computations. Unfortunately, **`mhelp`** is not available in the Student Version.) In fact, **`AiryAi`** and **`AiryBi`** are not defined in MATLAB itself, which makes working with them a bit complicated.

(b) Plot the solution you have obtained over the interval $-1 \leq t \leq 2$. You will find that **`ezplot`** fails in this case, because of the difficulty with **`AiryAi`** and **`AiryBi`** just explained in (a). However, you can plot the solution by defining a vector **`T`** of values of t, letting **`Y = subs(sol, t, T)`**, and using **`plot`**. You may have to experiment with the spacing of the values of t and with the scale on the y-axis to get a good picture. At what value t^* of t does it seem that the solution "blows up"? What happens to your graph past this value of t^*? Do you think it has any validity? Why or why not?

(c) Compare your estimated value of t^* with the upper bound for t^* obtained in Chapter 6 by comparison with the equation $y' = y^2 + 1$ for $t \geq 1$. Note that to make the comparison, you need to use the solution $y = \tan(t + C)$ of $y' = y^2 + 1$ that matches your solution to $y' = y^2 + t$ at $t = 1$, so you will need to estimate the appropriate value of C as well.

(d) Superimpose your graph of the solution to (6.3) on top of the direction field for the equation, and visually verify the tangency of the solution curves to the direction field.