Problem Set C

Numerical Solutions

In this problem set you will use **ode45** and **plot** to calculate and plot numerical solutions to ordinary differential equations. The use of these commands is explained in Chapters 3, 7, and 8. The solution to Problem 3 appears in the *Sample Solutions*.

1. We are interested in describing the solution $y = \phi(t)$ to the initial value problem

$$\frac{dy}{dt} = 2\frac{t}{y} + e^{y}, \qquad y(0) = 1. \tag{C.1}$$

Observe that $dy/dt \geq e^{y} \geq e$ in the region $t \geq 0$, $y \geq 1$, and therefore $\phi(t)$ increases to ∞ as t increases. But how fast does it increase?

(a) First of all, $\phi(t)$ must increase at least as fast as the solution $y = \phi_0(t)$ to the initial value problem

$$\frac{dy}{dt} = e^{y}, \qquad y(0) = 1.$$

Solve this problem symbolically and conclude that $\phi(t) \to \infty$ as $t \to t^*$ for some $t^* \leq 1/e < 1/2$.

(b) Next, since $2t/y \leq 1$ for $t \leq 1/2$ and $y \geq 1$, it follows that $\phi(t)$ increases at most as fast as the solution $y = \phi_1(t)$ to the initial value problem

$$\frac{dy}{dt} = 1 + e^{y}, \qquad y(0) = 1.$$

Solve this problem symbolically and conclude that $t^* \geq \ln(e+1) - 1$.

(c) Now compute a numerical solution of (C.1) and find an approximate value of t^*.

(d) Finally, plot the numerical solution on the same graph with $\phi_0(t)$ and $\phi_1(t)$ and compare the solutions.

2. Consider the initial value problem

$$y' = y^2 - e^{-t^2}, \qquad y(0) = 2. \tag{C.2}$$

Since **dsolve** is unable to solve this equation, we need to use qualitative and numerical methods. Observe that $0 < e^{-t^2} \le 1$, so that

$$y^2 \ge y^2 - e^{-t^2} \ge y^2 - 1.$$

Therefore the solution $\phi(t)$ to (C.2) must satisfy $\phi_0(t) \ge \phi(t) \ge \phi_1(t)$, where ϕ_0 and ϕ_1 are the solutions to the respective initial value problems

$$\begin{cases} y' = y^2 & y(0) = 2, \\ y' = y^2 - 1 & y(0) = 2. \end{cases}$$

(a) Solve for ϕ_0 and ϕ_1 explicitly (using **dsolve**) and conclude that $\phi(t) \to \infty$ as $t \to t^*$, for some $t^* \in [0.5, 0.5 \ln 3]$.

(b) Compute a numerical solution of (C.2), find an approximate value of t^*, and plot $\phi(t)$ for $0 \le t < t^*$.

3. We shall study solutions $y = \phi_b(t)$ to the initial value problem

$$y' = (y - t)(1 - y^3), \qquad y(0) = b$$

for nonnegative values of t.

(a) Plot numerical solutions $\phi_b(t)$ for several values of b. Include values of b that are less than or equal to 0, between 0 and 1, equal to 1, and greater than 1.

(b) Now, based on these plots, describe the behavior of the solution curves $\phi_b(t)$ for positive t, when $b \le 0, 0 < b < 1, b = 1$, and $b > 1$. Identify limiting behavior and indicate where the solutions are increasing or decreasing.

(c) Next, combine your plots with a plot of the line $y = t$. The graph should suggest that the solution curves for $b > 1$ are asymptotic to this line. Explain from the differential equation why that is plausible. (*Hint*: Use the differential equation to consider the sign of y' on and close to the line $y = t$.)

(d) Finally, superimpose a plot of the direction field of the differential equation to confirm your analysis.

4. We shall study solutions $y = \phi_b(t)$ to the initial value problem

$$y' = (y - \sqrt{t})(1 - y^2), \qquad y(0) = b$$

for nonnegative values of t.

(a) Plot numerical solutions $\phi_b(t)$ for several values of b. Include values of b that are less than -1, equal to -1, between -1 and 1, equal to 1, and greater than 1.

(b) Now, based on these plots, describe the behavior of the solution curves $\phi_b(t)$ for positive t, when $b < -1$, $b = -1$, $-1 < b < 1$, $b = 1$, and $b > 1$. Identify limiting behavior and indicate where the solutions are increasing or decreasing.

(c) By combining your plots with a plot of the parabola $y = \sqrt{t}$, show that the solution curves for $b > 1$ are asymptotic to this parabola. Explain from the differential equation why that is plausible.

(d) Finally, superimpose a plot of the direction field of the differential equation to confirm your analysis.

5. In this problem, we analyze the Gompertz-threshold model from Problem 17 of Problem Set B. That is, consider the differential equation

$$y' = y(1 - \ln y)(y - 3).$$

Using various nonnegative values of $y(0)$, find and plot several numerical solutions on the interval $0 \le t \le 6$. By examining the differential equation and analyzing your plots, identify all equilibrium solutions and discuss their stability.

6. Consider the initial value problem

$$\frac{dy}{dt} = \frac{t - e^{-t}}{y + e^y}, \qquad y(1.5) = 0.5.$$

(a) Use **ode45** to find approximate values of the solution at $t = 0, 1, 1.8$, and 2.1. Then plot the solution.

(b) In this part you should use the results from parts (c) and (d) of Problem 5 in Problem Set B (which appears in the *Sample Solutions*). Compare the values of the actual solution and the numerical solution at the four specified points. Plot the actual solution and the numerical solution on the same graph.

(c) Now plot the numerical solution on several large intervals (*e.g.*, $1.5 \le t \le 10$ or $1.5 \le t \le 100$). Make a guess about the nature of the solution as $t \to \infty$. Try to justify your guess on the basis of the differential equation.

7. Consider the initial value problem

$$e^y + (te^y - \sin y)\frac{dy}{dt} = 0, \qquad y(2) = 1.5.$$

(a) Use **ode45** to find approximate values of the solution at $x = 1, 1.5$, and 3. Then plot the solution on the interval $0.5 \le t \le 4$.

(b) In this part you should use the results from parts (c) and (d) of Problem 6 in Problem Set B. Compare the values of the actual solution and the numerical solution at the three specified points. Plot the actual solution and the numerical solution on the same graph.

(c) Now plot the numerical solution on several large intervals (*e.g.*, $2 \le t \le 10$, or $2 \le t \le 100$, or even $2 \le t \le 1000$). Make a guess about the nature of the solution as $t \to \infty$. Try to justify your guess on the basis of the differential equation. Similarly, plot the numerical solution on intervals $\epsilon \le t \le 2$ for different choices of ϵ as a small positive number. What do you think is happening to the solution as $t \to 0+$?

8. The differential equation

$$y' = e^{-t^2}$$

cannot be solved in terms of elementary functions, but it can be solved with **dsolve**.

(a) Use **dsolve** to solve this equation.

(b) The answer is in terms of **erf**, a special function (*cf.* Chapter 5), known as the *error function*. Use the MATLAB differentiation operator **diff** to see that

$$\frac{d}{dx}(\mathrm{erf}(t)) = \frac{2}{\sqrt{\pi}} e^{-t^2}.$$

In fact,

$$\mathrm{erf}(t) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2}\, dt.$$

Use **help erf** to confirm this formula.

(c) Although one does not have an elementary formula for this function, the numerical capabilities of MATLAB mean that we "know" this function as well as we "know" elementary functions like $\tan t$. To illustrate this, evaluate $\mathrm{erf}(t)$ at $t = 0, 1$, and 10.5, and plot $\mathrm{erf}(t)$ on $-10 \le t \le 10$.

(d) Compute $\lim_{t\to\infty} \mathrm{erf}(t)$ and $\int_{-\infty}^{\infty} e^{-t^2}\, dt$.

(e) Next, solve the initial value problem

$$y' = 1 - 2ty, \qquad y(0) = 0$$

using **dsolve**. You should have gotten a formula involving **erf** with a complex argument. Solve this initial value problem by hand-calculation. Your answer will involve an integral that cannot be evaluated by elementary techniques. Since solutions of initial value problems are unique, these two formulas (the one obtained by MATLAB and the one obtained by hand-calculation) must be the same. Explain how they are related. Hint: Use **int** to evaluate $\int_0^t e^{s^2}\, ds$.

9. Consider the initial value problem

$$ty' + (\sin t)y = 0, \qquad y(0) = 1.$$

(a) Use **dsolve** to solve the initial value problem.

(b) Note the occurrence of the built-in function **sinint**, called the *Sine Integral* function and usually written $\mathrm{Si}(t)$. Check that this function is an antiderivative of $\sin t/t$ by differentiating it.

(c) Evaluate $\lim_{t\to\infty} \mathrm{Si}(t)$. Plot $\mathrm{Si}(t)$ and discuss the features of the graph.

(d) Do the same with the solution to the initial value problem.

(e) Now solve the initial value problem using **ode45**, and plot the computed solution. Compare your plot to the one obtained in part (d). You will find that MATLAB cannot evaluate $\sin t/t$ at $t = 0$, even though the singularity is removable. One way to get around this is to give the initial condition $y = 1$ at a value of t extremely close to, but not equal to, zero. Since we are only finding an approximate solution with **ode45** anyhow, there shouldn't be much harm done as long as the amount by which we move the initial condition is small compared with the error we expect from the numerical procedure.

(f) Discuss the stability of the differential equation. Illustrate your conclusions by graphing solutions with different initial values on the interval $[-10, 10]$.

Note: In this problem some of your plots may take a long time to generate.

10. Solve the following initial value problems numerically, then plot the solutions. Based on your plots, predict what happens to each solution as t increases. In particular, if there is a limiting value for y, either finite or infinite, find it. If it is unclear from the plot you've made, try replotting on a larger interval. Another possibility is that the solution blows up in finite time. If so, estimate the time when the solution blows up. Try to use the qualitative methods of Chapter 6 to confirm your answers.

(a) $y' = e^{-3t} + \dfrac{1}{1+y^2}, \quad y(0) = -1.$

(b) $y' = e^{-2t} + y^2, \quad y(0) = 1.$

(c) $y' = \cos t - y^3, \quad y(0) = 0.$

(d) $y' = (\sin t)y - y^2, \quad y(0) = 2.$

11. Solve the following initial value problems numerically, then plot the solutions. Based on your plots, predict what happens to each solution as t increases. See Problem 10 for additional instructions.

(a) $y' = 3t - 5\sqrt{y}, \quad y(0) = 4.$

(b) $y' = (t^3 - y^3)\cos y, \quad y(0) = -1.$

(c) $y' = \dfrac{y^2 + 2ty}{t^2 + 3}, \quad y(1) = 2.$

(d) $y' = -2t + e^{-ty}, \quad y(0) = 1.$

12. Construct a Simulink model to represent the initial value problem

$$y' = \sin y + C \sin t, \qquad y(0) = 0.$$

(*Hint*: For the forcing term $C \sin t$, use the Sine Wave block. When you left-click on this block to bring up the **Block Parameters** window, you can enter the amplitude C in the appropriate box.) Run your model for $0 \leq t \leq 35$ for various values of C (starting with $C = 1$ and $C = -1$) and see what you observe. You can run the model with many values of C simultaneously by entering a vector value for the Sine Wave amplitude. Set the amplitude to the vector **`-4:4`** and observe the results of the simulation in a Scope window. How can you interpret the results in view of an analysis of the equilibrium solutions of the autonomous equation $y' = \sin y$?

13. This problem illustrates one of the possible pitfalls of blindly applying numerical methods without paying attention to the theoretical aspects of the differential equation itself. Consider the equation

$$ty' + 3y - 9t^2 = 0.$$

(a) Use the MATLAB program in Chapter 7 to compute the Euler Method approximation to the solution with initial condition $y(-0.5) = 3.15$, using step size $h = 0.2$ and $n = 10$ steps. The program will generate a list of ordered pairs (x_i, y_i). Use **`plot`** to graph the piecewise linear function connecting the points (x_i, y_i).

(b) Now modify the program to implement the Improved Euler Method. Can you make sense of your answers?

(c) Next, use **`ode45`** to find an approximate solution on the interval $(-0.5, 0.5)$, and plot it with **`plot`**. Print out the values of the solution at the points **`-0.06:0.02:0.06`**. What is the interval on which the approximate solution is defined?

(d) Solve the equation explicitly and graph the solutions for the initial conditions $y(0) = 0$, $y(-0.5) = 3.15$, $y(0.5) = 3.15$, $y(-0.5) = -3.45$, and $y(0.5) = -3.45$. Now explain your results in (a)–(c). Could we have known, without solving the equation, whether to expect meaningful results in parts (a) and (b)? Why? Can you explain how **`ode45`** avoids making the same mistake?

14. Consider the initial value problem

$$\frac{dy}{dt} = e^{-t} - 3y, \qquad y(-1) = 0.$$

(a) Use the MATLAB program `myeuler.m` from Chapter 7 to compute the Euler Method approximation to $y(t)$ with step size $h = 0.5$ and $n = 4$ steps. The program will generate a list of ordered pairs (t_i, y_i). Use **`plot`** to graph the piecewise linear function connecting the points (t_i, y_i). Repeat with $h = 0.2$ and $n = 10$.

(b) Now modify the program to implement the Improved Euler Method, and repeat part (a) using the modified program. Find the exact solution of the initial value problem and plot it, the two Euler approximations, and the two Improved Euler approximations on the same graph. Label the five curves.

(c) Now use the Euler Method program with $h = 0.5$ to approximate the solution on the interval $[-1, 9]$. Plot both the approximate and exact solutions on this interval. How close is the approximation to the exact solution as t increases? In light of the discussion of stability in Chapters 5 and 7, explain your results in parts (a)–(c).

15. Consider the initial value problem

$$\frac{dy}{dt} = 2y + \cos t, \qquad y(0) = -2/5.$$

(a) Use the MATLAB program in Chapter 7 to compute the Euler Method approximation to $y(t)$ with step size $h = 0.5$ and $n = 12$ steps. The program will generate a list of ordered pairs (t_i, y_i). You do not need to print out these numbers, but graph the piecewise linear function connecting the points (t_i, y_i). What appears to be happening to y as t increases?

(b) Repeat part (a) with $h = 0.2$ and $n = 30$, and then with $h = 0.1$ and $n = 60$, each time plotting the results on the same set of axes as before. How are the approximate solutions changing as the step size decreases? Can you make a reliable prediction about the long-term behavior of the solution?

(c) Use **`ode45`** to find an approximate solution and again plot it on the interval $[0, 6]$ on the same set of axes as before. Now what does it look like y is doing as t increases? Next, plot the solution from **`ode45`** on a larger interval (going to $t = 15$) on a new set of axes. Again, what is happening to y as t increases?

(d) Solve the initial value problem exactly and compare the exact solution to the approximations found above. In light of the discussion of stability in Chapters 5 and 7, explain your results in parts (a)–(c).

16. Consider the initial value problem

$$\frac{dy}{dx} = 2y - 2 + 3e^{-t}, \qquad y(0) = 0.$$

(a) Use the MATLAB program in Chapter 7 to compute the Euler Method approximation to $y(t)$ with step size $h = 0.2$ and $n = 10$ steps. The program will generate a list of ordered pairs (t_i, y_i). Use **`plot`** to graph the piecewise linear function connecting the points (t_i, y_i). What appears to be happening to y as t increases?

(b) Repeat part (a) with $h = 0.1$ and $n = 20$, and then with $h = 0.05$ and $n = 40$. How are the approximate solutions changing as the step size decreases? Can you make a reliable prediction about the long-term behavior of the solution?

(c) Use **ode45** to find an approximate solution and plot it on the interval $[0, 2]$. Now what does it look like y is doing as t increases? Next, plot the solution from **ode45** on a larger interval (going at least to $t = 10$). Again, what is happening to y as t increases?

(d) Solve the initial value problem exactly and compare the exact solution to the approximations found above. In light of the discussion of stability in Chapters 5 and 7, explain your results in parts (a)–(c).

17. The function **erf**, discussed in Chapter 5 and in Problem 8 in this set, is the solution to the initial value problem

$$\frac{dy}{dt} = \frac{2}{\sqrt{\pi}} e^{-t^2}, \qquad y(0) = 0,$$

so if we solve this initial value problem numerically we get approximate values for the built-in function **erf**. Use **ode45**, employing the accuracy options discussed in Chapter 7, to calculate values for $\mathrm{erf}(0.1), \mathrm{erf}(0.2), \ldots, \mathrm{erf}(1)$ having at least 10 correct digits. Present your results in a table. In a second column print the values of $\mathrm{erf}(x)$ for $x = 0.1, 0.2, \ldots, 1$, obtained by using the built-in function **erf**. Compare the two columns of values.

18. Consider the Gompertz-threshold model,

$$y' = y(1 - \log y)(y - 3).$$

From the direction field and the qualitative approach for this equation, one learns that the solution with initial condition, $y(0) = 5$, approaches 3 as t approaches ∞. Use the event detection feature of **ode45** to find the time t at which $y = 3.1$.

19. Build a Simulink model for the initial value problem

$$y' = y^3 + t, \quad y(0) = 0,$$

which cannot be solved by **dsolve**. Display the graph of $y(t)$ in a Scope block.

(a) When you run the model with the default parameters (t going from 0 to 10), what error message do you get? What do you think is responsible for this, mathematically?

(b) The solution to this IVP only exists for $t < t^*$, for some positive number t^*. What does the model tell you about the (approximate) value of this number?

(c) How does the graph of $y(t)$ for $0 \le t < t^*$ (as displayed in the Scope window) compare with what you expected?

Chapter 10

Solving and Analyzing Second Order Linear Equations

Newton's second law of dynamics—force is equal to mass times acceleration—tells physicists that, in order to understand how the world works, they must pay attention to forces. Since acceleration is a second derivative, the law also tells us that second order differential equations are likely to appear when we apply mathematics to study the real world.

We note that in Chapters 5, 6 and 7 we discussed three different approaches to solving first order differential equations: searching for exact formula solutions; using geometric methods to study qualitative properties of solutions—typically when we cannot find a formula solution; and invoking numerical methods to produce approximate solutions—again when no solution formula is attainable.

In this chapter we shall bring each of these methods to bear on second order equations. The most basic second order differential equations are linear equations with *constant coefficients*:

$$ay'' + by' + cy = g(t).$$

These equations model a wide variety of physical situations, including oscillations of springs, simple electric circuits, and the vibrations of tuning forks to produce sound and of electrons to produce light. In other situations, such as the motion of a pendulum, we may be able to approximate the resulting differential equation reasonably well by a linear differential equation with constant coefficients. Fortunately, we know (and MATLAB can apply) several techniques for finding explicit solution formulas to linear differential equations with constant coefficients.

Unfortunately, we often cannot find solution formulas for more general second order equations, even for linear equations with *variable coefficients*:

$$y'' + p(t)y' + q(t)y = g(t).$$

Such equations have important applications to physics. For example, Airy's equation,

$$y'' - ty = 0, \tag{10.1}$$