Project 2 – Phase Plane Analysis

I. Lotka-Volterra System

The Lotka-Volterra (LV) System is given by the system

$$\frac{dx}{dt} = f_1(x,y) = x(\alpha - \beta y)$$
$$\frac{dy}{dt} = f_2(x,y) = -y(\varepsilon - \delta x)$$

where the parameters $\alpha, \beta, \varepsilon, \delta$ are all assumed to be positive.

a) **Critical Points.** The critical points are found by solving the equations

$$f_1(x,y) = 0, f_2(x,y) = 0.$$

This system is easy to solve, and there are only two possible critical points – given by:

$$(x,y) = (0,0)$$
$$(x,y) = (\frac{\varepsilon}{\delta}, \frac{\alpha}{\delta})$$

b) **Classification.** The Jacobian of the system is given by

$$J = \begin{bmatrix} \frac{\partial f_1(x,y)}{\partial x} & \frac{\partial f_1(x,y)}{\partial y} \\ \frac{\partial f_2(x,y)}{\partial x} & \frac{\partial f_2(x,y)}{\partial y} \end{bmatrix} = \begin{bmatrix} \alpha - \beta y & -\beta x \\ \delta y & -\varepsilon + \delta x \end{bmatrix}$$

Evaluating Jacobian at the first critical point, (0,0), we get

$$J(0,0) = \begin{bmatrix} \alpha & 0 \\ 0 & -\varepsilon \end{bmatrix}$$

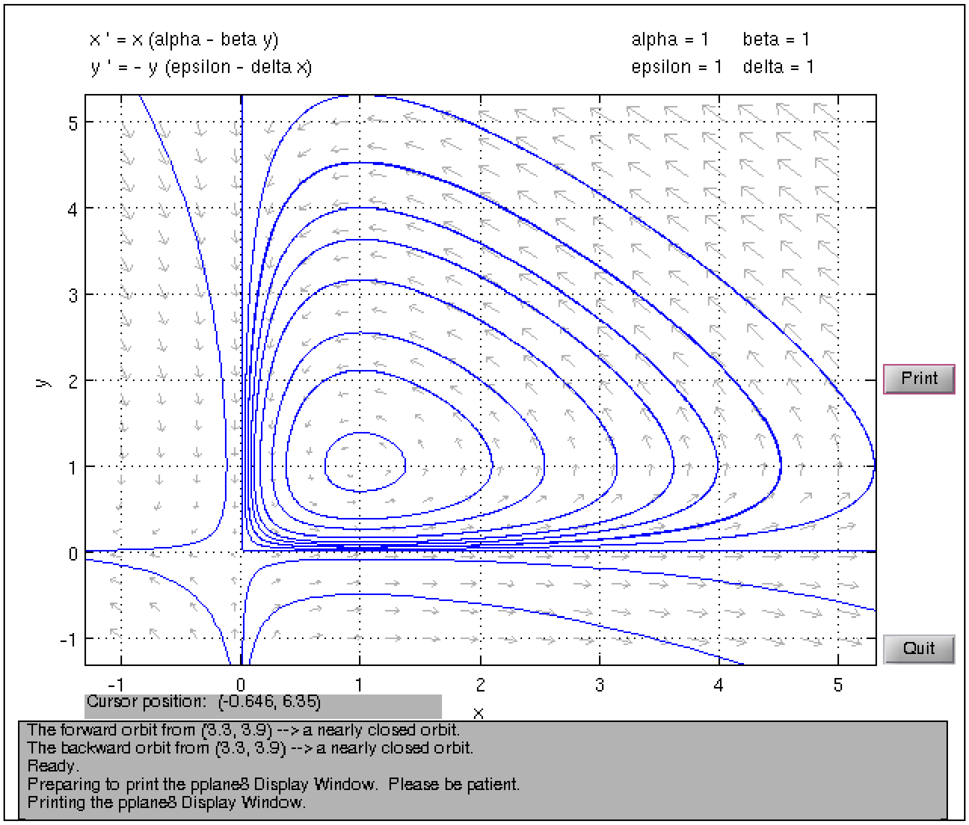
which has eigenvalues $\lambda = \alpha, -\varepsilon$ which are real and of opposite sign. The critical point is therefore a saddle point, and unstable.

Evaluating the Jacobian at the second critical point, $(x,y) = (\frac{\varepsilon}{\delta}, \frac{\alpha}{\delta})$, we get

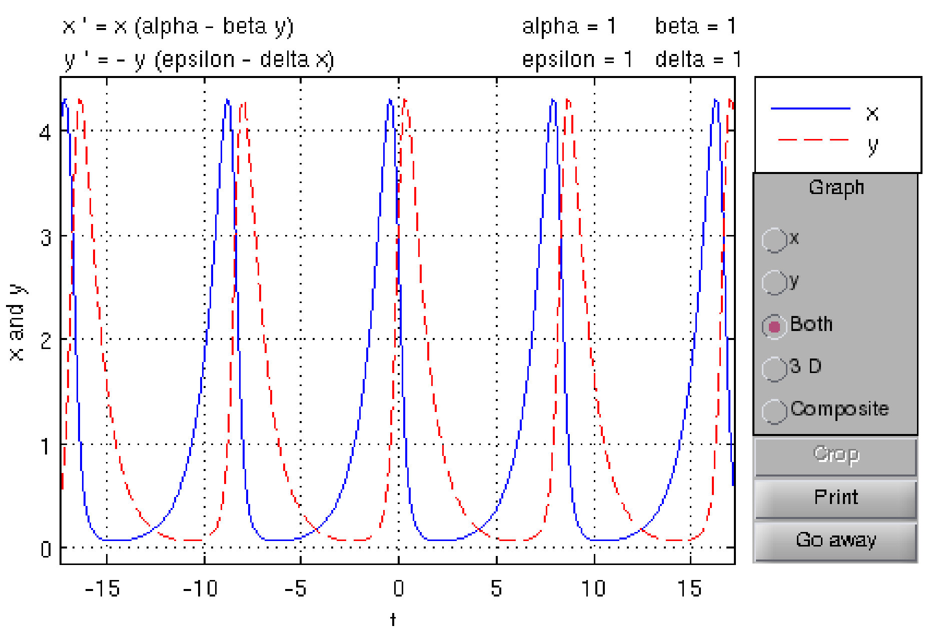
$$J(\frac{\varepsilon}{\delta}, \frac{\alpha}{\delta}) = \begin{bmatrix} 0 & -\frac{\varepsilon\beta}{\delta} \\ \frac{\alpha\delta}{\beta} & 0 \end{bmatrix}$$

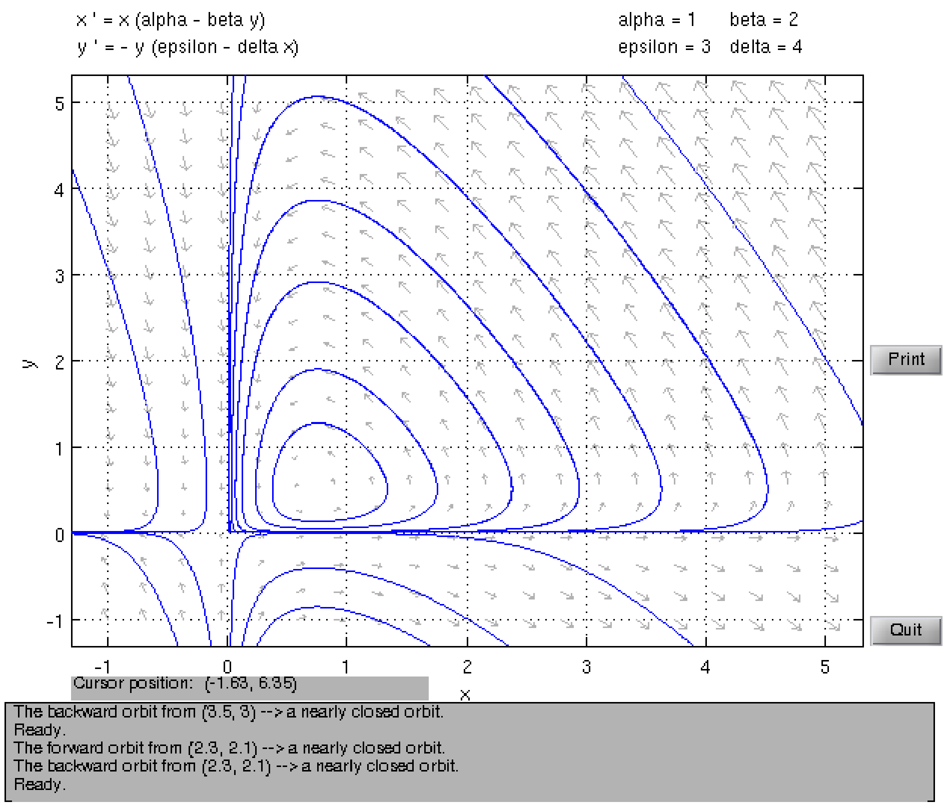
which has eigenvalues $\lambda = \pm i\sqrt{\varepsilon\alpha}$, which are pure imaginary. The critical point is therefore a periodic orbit, and stable.

Examples of the two dimensional phase plane portrait are shown below, for various values of the parameters:

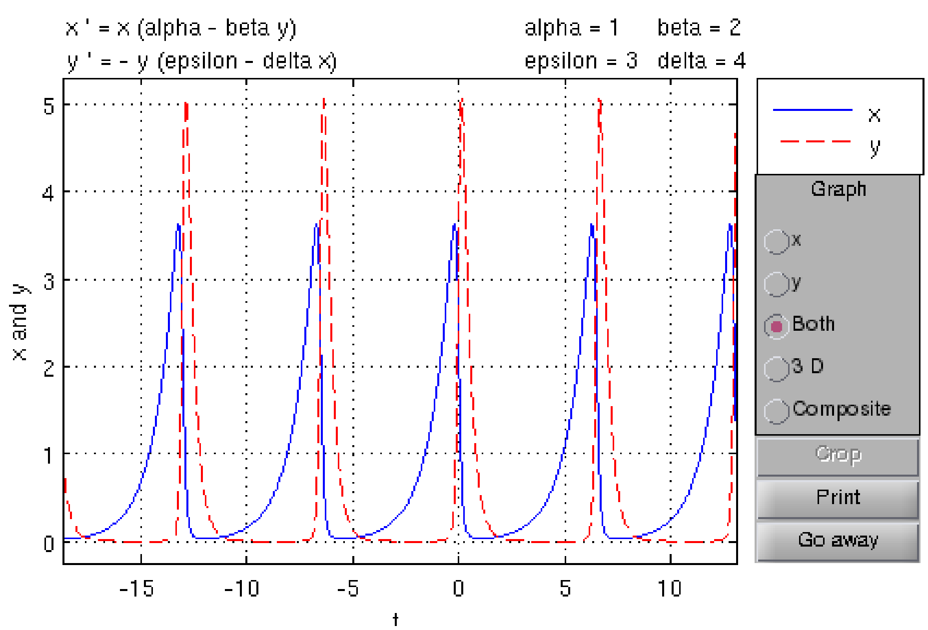


A representative component plot is shown below. Note the characteristic oscillations which lag one another.





A representative component plot is shown below.



You can see clearly that the origin is a saddle point, and that the remaining critical point generates periodic orbits in the first quadrant.

c) **Extinction.** As long as the initial values are both positive, there will always be a periodic solution. The values may eventually be less than 1 (which may indicate extinction). Since the population may be in units of thousands (or even millions), dropping below 1 may not indicate anything unusual. The only other possibility is for x=0, in which case y(t) goes exponentially to zero (extinction). In other words, if there is no prey, the predators will become extinct.

II. Lorenz Dynamical System.

The Lorenz Dynamical System is given by the following system of three autonomous ordinary differential equations

$$\frac{dx}{dt} = f_1(x, y, z) = \sigma(y - x)$$
$$\frac{dy}{dt} = f_2(x, y, z) = x(\rho - z) - y$$
$$\frac{dz}{dt} = f_3(x, y, z) = xy - \beta z$$

a) **Critical points.** Aside from the obvious critical point (x,y,z)=(0,0,0), there is one other critical point at

$$(x, y, z) = (\pm\sqrt{\beta(\rho - 1)}, \pm\sqrt{\beta(\rho - 1)}, \rho - 1).$$

The Jacobian of this system is given by

$$J = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -\sqrt{\beta(\rho-1)} \\ \sqrt{\beta(\rho-1)} & \sqrt{\beta(\rho-1)} & -\beta \end{bmatrix}$$

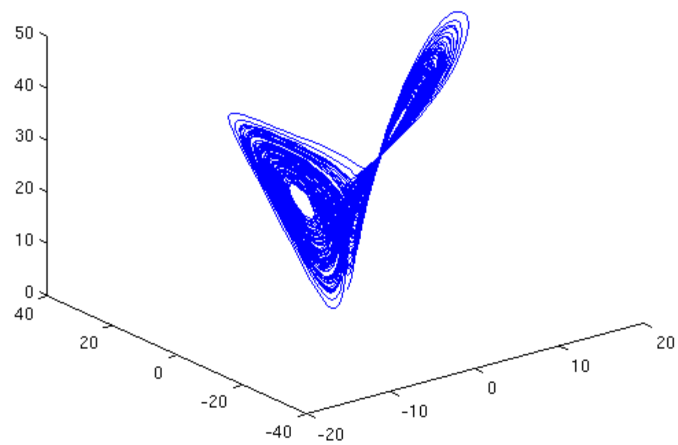
The determinant of this matrix is $\det(J) = 2\beta\sigma(1-\rho)$. This is negative if $\beta > 0, \sigma > 0, \rho > 1$. Since the determinant is the product of the three eigenvalues, this can occur n the following cases

a) 3 real eigenvalues: 1 negative, 2 positive, which implies a saddle point, or
b) 3 real negative eigenvalues , which implies a stable node; or
c) 1 real (negative) and 2 complex eigenvalues, which implies a spiral into a periodic orbit.

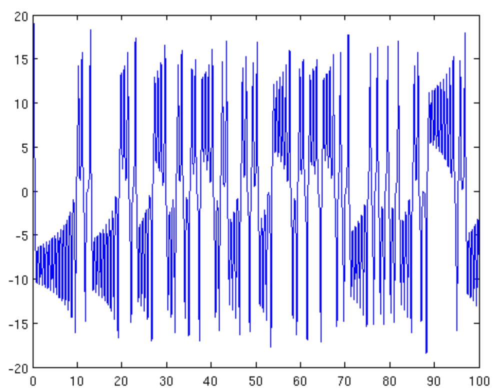
More complex behavior can occur as it flips back and forth between surfaces (classic butterfly), which is indicative of "chaos."

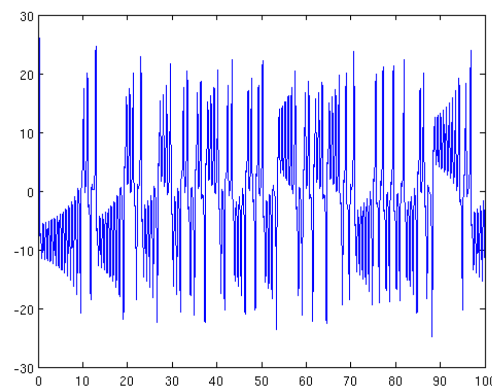
b) **Parameter Case Studies**: Below are shown some different kinds of behaviors for the Lorenz system:

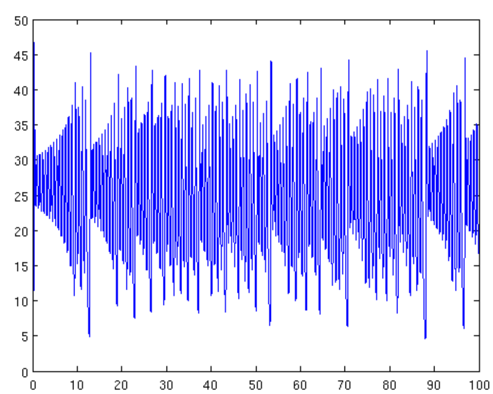
With the default parameters: $\sigma = 10, \rho = 28, \beta = 8/3$, we get the classic "butterfly"



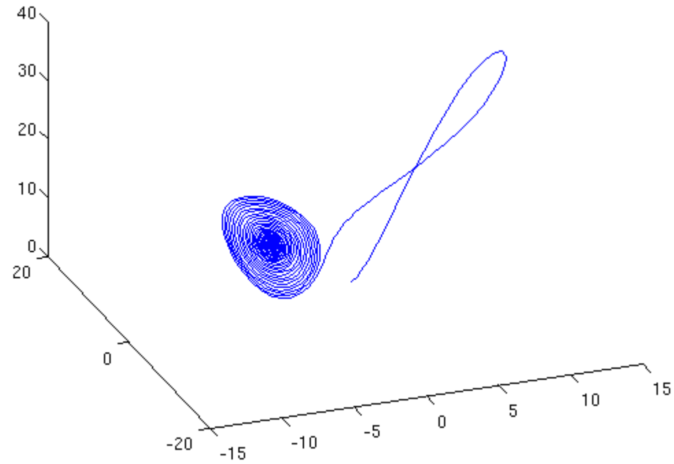
with the component plots x(t) vs t, y(t) vs t, and z(t) vs t

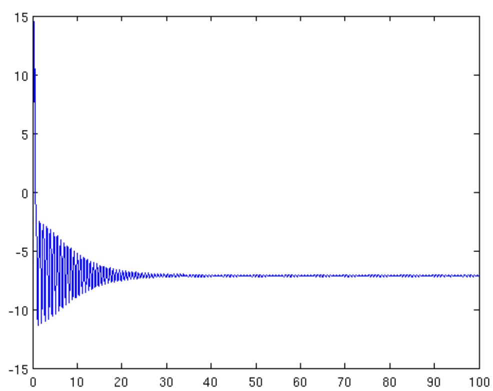


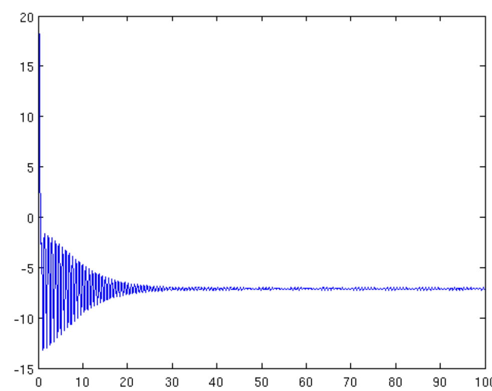


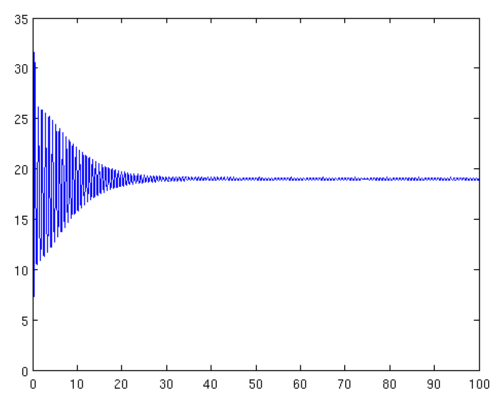


With the default parameters: $\sigma = 10, \rho = 20, \beta = 8/3$, we get a tight spiral into a critical point $(\sqrt{19 \cdot 8/3}, \sqrt{19 \cdot 8/3}, 19)$

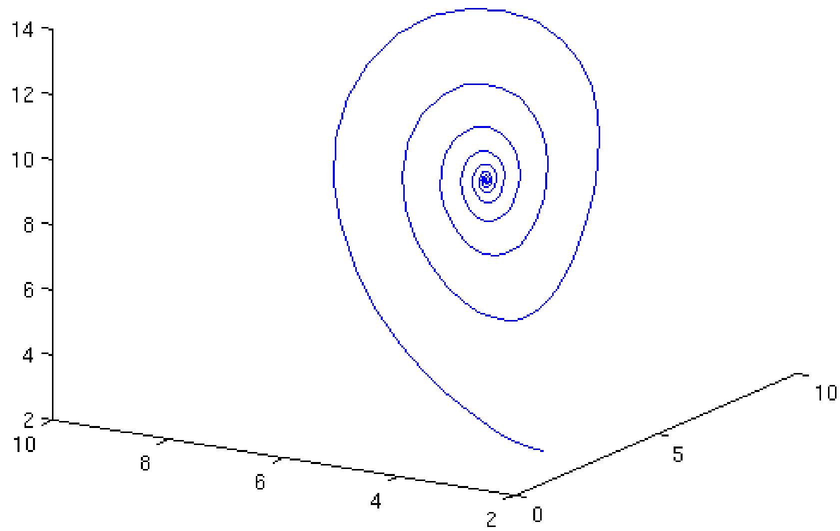


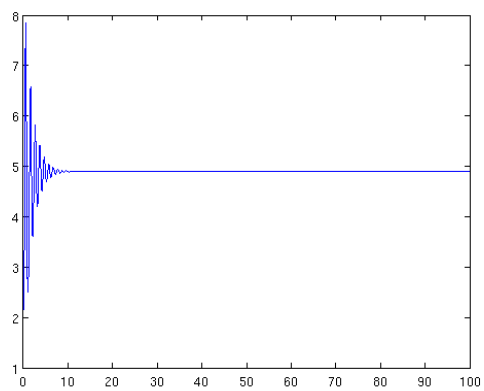


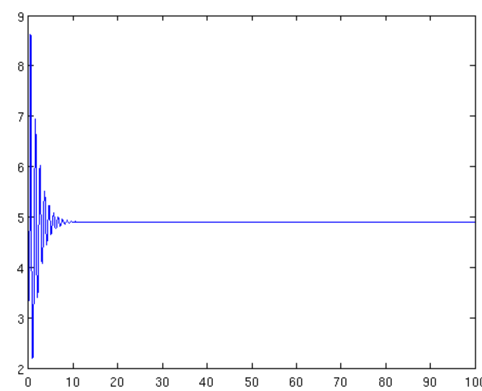


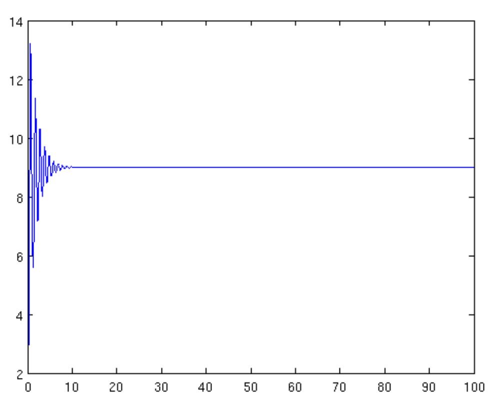


With the default parameters: $\sigma = 10, \rho = 10, \beta = 8/3$, we get a slow spiral into a critical point $(\sqrt{9 \cdot 8/3}, \sqrt{9 \cdot 8/3}, 9)$

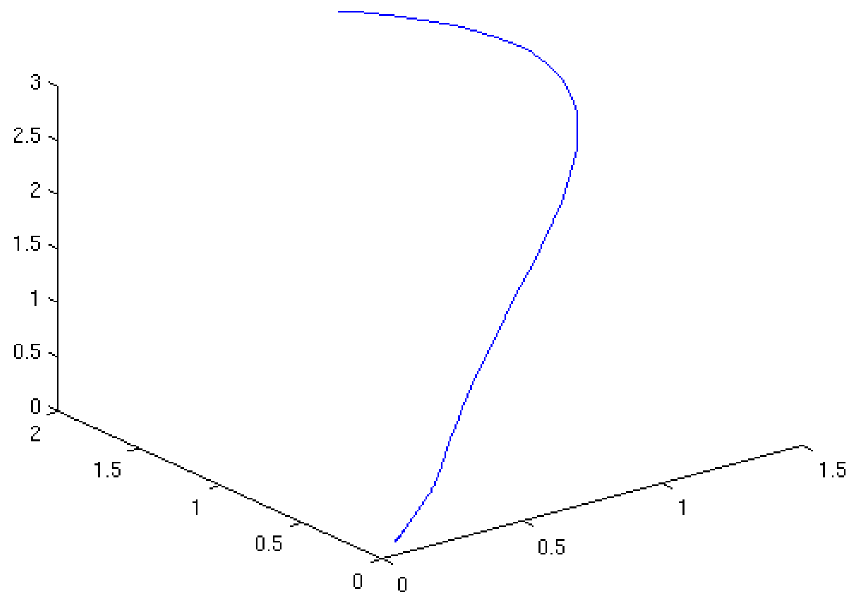


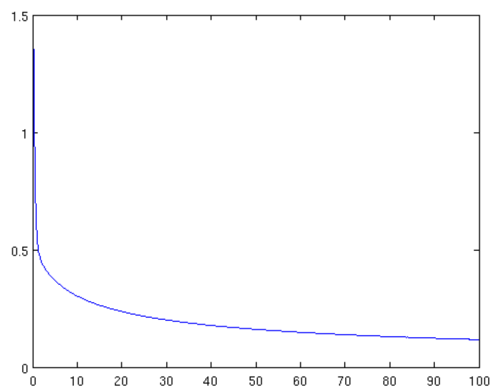


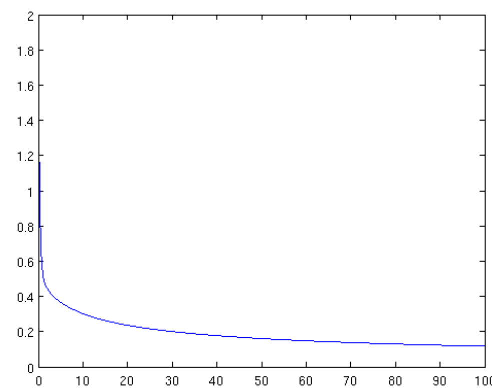


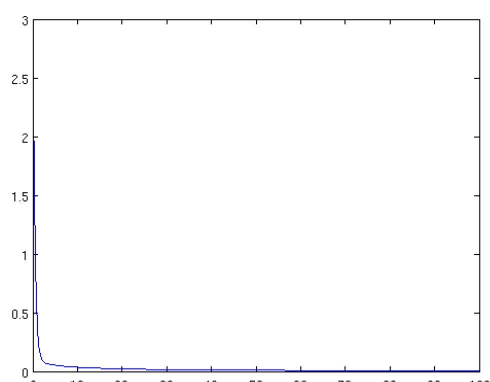


With the default parameters: $\sigma = 10, \rho = 1, \beta = 8/3$, we get a slow spiral into a critical point $(0,0,0)$.

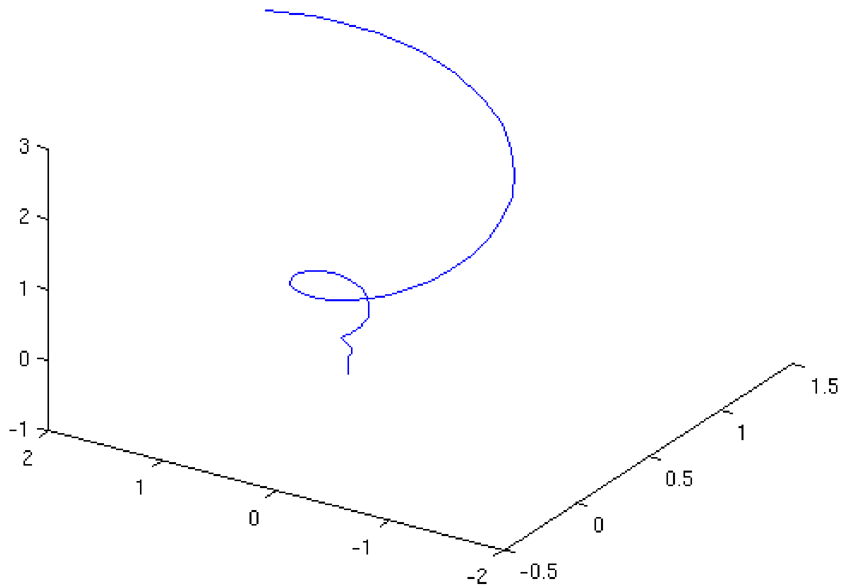


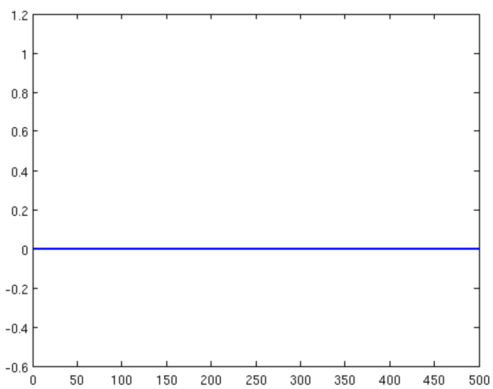


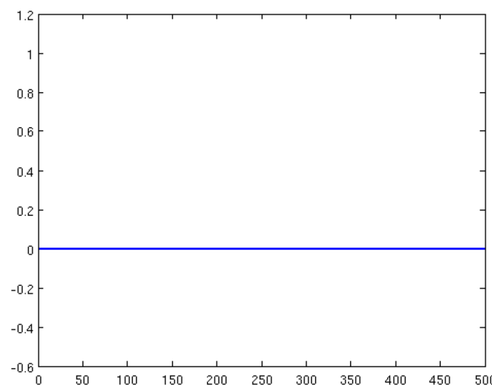


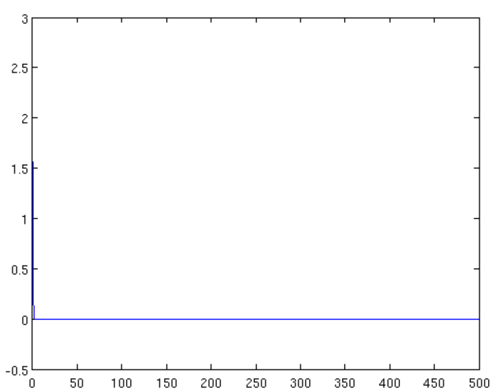


With the default parameters: $\sigma = 10, \rho = 1, \beta = 8/3$, we get a "corkscrew"









c) **Summary and Conclusions**. The behavior of the Lorenz system in the neighborhood of the critical points is not all that unusual. What is more unusual, and not present in the two-d LV system, is the phenomenon of the orbit "re-visiting" surfaces more or less randomly. If two points start close together, they will eventually diverge dramatically – when one or the other "jumps" to the other surface. It is this extreme sensitivity to initial conditions that is an indicator of what is known as "chaos."

One can calculate the dimension of these surfaces (which form the butterfly "wings"), and the amazing conclusion is that they have **fractional** dimension!

This project is a gateway into what is know as the study of "dynamical systems and chaos."