Analysis notebook

A notebook with some exercises

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Preface

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Chapter 1

Real Numbers

1.1 Algebraic Objects: Fields and Order properties

I already studied the algebraic topics before (Linear Algebra notes). So I will skip this section.

1.2 Absolute value, epsilon-neighborhood

Absolute value is a function $f : \mathbb{R} \to \mathbb{R}_0$ such that,

$$\begin{aligned} f(x) &= x \qquad \text{if } x \geq 0 \\ f(x) &= -x \qquad \text{if } x < 0 \end{aligned}$$

Absolute value describes **Distance** between two values. It is important to think this function as distance more than some function that “makes negative values positive”

Proposition 1.2.1. $\forall x, y \in \mathbb{R}$,

1. $|x| \geq 0$
2. $|-x| = |x|$
3. $|xy| = |x||y|$
4. $|x|^2 = x^2$
5. $|x| \leq y \iff -y \leq x \leq y$
6. $-|x| \leq x \leq |x|$

Proof. Proofs are rather simple, so I will not bother writing here. □

Theorem 1.2.1 (Triangle Inequality). $\forall x, y \in \mathbb{R}$,

$$|x + y| \leq |x| + |y|$$

Proof. From the proposition we have,

$$\begin{aligned} -|x| \leq x \leq |x| \\ -|y| \leq y \leq |y| \end{aligned}$$

Adding these equations we get

$$-|x| - |y| \leq x + y \leq |x| + |y| \Rightarrow |x + y| \leq |x| + |y|$$

□

Corollary 1.2.1. $\forall x, y \in \mathbb{R}$,

1. $||x| - |y|| \leq |x - y|$
2. $|x - y| \leq |x| + |y|$
3. $|\sum_{i=1}^{n} a_i| \leq \sum_{i=1}^{n} |a_i|$

Proof. These Corollaries are direct consequence of triangle inequality, with third inequality using the proof with induction. I will not provide proofs since they are kind of boring and time comsuming. □

Definition 1.2.1 (epsilon neighborhood). . The $\epsilon - \textit{neighborhood}$ of a is defined as a set

$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\}$$

Which is equivalent to open interval

$$(a - \epsilon, a + \epsilon)$$

Analysis heavily uses epsilon definitions and epsilon neighborhood for rigirous proofs. Therefore this definition is an useful tool.

1.3 Axiom of Completeness, Infimum and Supremum

Definition 1.3.1. A set $A \subseteq R$ is **bounded above** if $\exists b \in R$ s.t $a \leq b \ \forall a \in A$. The number b is the **upper bound of A**. We denote set of upper bounds of A as A^u.
Similarly, we define lower bounds and the set as A^ℓ.

Definition 1.3.2 (supremum). A upper bound a of a set S is called **supremum** if,

$$a = \min A^u$$

Mathematically we show the notation as $a = \sup S$.
In Similar fashion, we define $b = \inf S$ for lower bounds.

Axiom of Completeness (AoC). Every non-empty subsets of $\mathbb{R}$ that is bounded above have supremum. The Axiom also deduces the existence of infimum in a similar fashion.

Lemma 1.3.1 (Epsilon Definition of supremum). $s \in \mathbb{R}$ is a supremum of a set $A \subseteq \mathbb{R}$ iff

$$\forall \ \epsilon > 0 \exists a \in A | s - \epsilon < a$$

Proof sketch. The both ways of the lemma can be proven by definition of the supremum. □

We use similar lemma for infimum.

Proposition 1.3.1 (Maximum and Supremum). If maximum of $A \neq \{\emptyset\} \subseteq \mathbb{R}$ exists, then

$$\max A = \sup A$$

Proof. Denote $s = \sup A$ and $m = \max A$. By definition, $s = \min A^u$ and $m = A^u \cap A$.
The result is an immediate consequence of the definitions of maximum and supremum.

m is a proper supremum, since $\forall x \in A$ we have $x \leq m$, and since also $m \in A$, $t = \sup A < m$ is impossible. $\square$

Similarly, we have $\min A = \inf A$.

Proposition 1.3.2 (Uniqueness of Supremum). Supremum and Infimum are unique.

Proof. For the sake of the contradiction, assume there exists two supremum s_1, s_2. Then by definition of supremum, we have

$$s_1 \geq s_2 \ \wedge \ s_2 \geq s_1 \Rightarrow s_1 = s_2$$

Infimum follows the similar proof. $\square$

Proposition 1.3.3 (Existence of Infimum). AoC implies the existence of infimum for $A \subseteq \mathbb{R}$ such that $A^\ell \neq \emptyset$,

$$\inf A = -\sup(-A)$$

Proof. Since $A^l \neq \emptyset$, it follows that

$$\exists x \in A^\ell \mid x \leq a$$

Then,

$$-x \geq -a \Rightarrow -x \in (-A)^u \neq \emptyset$$

By AoC, $\sup(-A)$ exists. Rest is trivial. $\square$

Proposition 1.3.4 (Operations on Supremum). The supremum holds these properties,

$$\sup(A+B) = \sup(A) + \sup(B) \tag{1.1}$$
$$\sup(A \cdot B) = \sup(A) \cdot \sup(B) \tag{1.2}$$
$$\text{if } c \geq 0, \quad \sup(cA) = c\sup(A) \tag{1.3}$$
$$\text{if } c \leq 0, \quad \sup(cA) = c\inf(A) \tag{1.4}$$

Proof. These properties directly follow from the epsilon definition of the supremum. That is, $\forall \epsilon_a, \epsilon_b, \exists a, b \in A, B$ such that,

$$\sup(A) - a < \epsilon_a \ \wedge \sup(B) - b < \epsilon_b$$

adding these equations to each other, we have

$$\sup(A) + \sup(B) - (a+b) < \epsilon_a + \epsilon_b \tag{1.5}$$

Note that $(a+b) \in A+B$, and let $\epsilon_a + \epsilon_b = \epsilon_{a+b}$. Also we know that,

$$\forall \epsilon_c \exists c \in A+B \mid \sup(A+B) - c < \epsilon_c \tag{1.6}$$

but 1.5 and 1.6 both are valid, hence the conclusion.
We can similarly prove other propositions, even for inf. $\square$

1.4 Applications of Completeness, Archimedean Property (A.P)

Theorem 1.4.1 (Archimedean Property, A.P). $\forall x \in \mathbb{R}\ \exists n_x \in \mathbb{N} \mid x \leq n_x$.

Proof. For the sake of contradiction, assume otherwise. Then $n \leq x\ \forall n \in \mathbb{N}$, by AoC $\mathbb{N}$ has supremum, s. Since $s-1<s$, $s-1$ is not a upper bound, therefore $\exists m \in \mathbb{N}$ such that $s-1<m \Rightarrow s<m+1$. but $m+1 \in \mathbb{N}$. Therefore s cannot be a supremum. □

Theorem 1.4.2 (Density of Rationals in $\mathbb{R}$). $\forall a, b \in \mathbb{R}, \exists r \in \mathbb{Q}$ such that

$$a<r<b$$

Proof. Since r must be rational, we want to find $m, n \in \mathbb{Z}$ such that $\frac{m}{n}=r$.
From Archimedean property,

$$\exists n \in \mathbb{N}: n(y-x) \geq 1$$

Again from Archimedean property,

$$\forall t \in \mathbb{R}, \exists m \in \mathbb{Z}: m-1 \leq t \leq m$$

In other words, for any real numbers, there are two consecutive integers that lies in the each boundary of the real numbers.
Let $t=nx$. Combining the inequalities, we get

$$nx \leq m \leq 1+nx \leq ny \Rightarrow x \leq \frac{m}{n} \leq y$$

□

Theorem 1.4.3 (Density of Irrationals in $\mathbb{R}$). $\forall x, y \in \mathbb{R}$ such that $x<y$, $\exists z \in \mathbb{I}$ such that

$$x<z<y$$

Proof. It is direct consequence of density of Rationals. We apply density theorem on $\frac{x}{\sqrt{2}}$ and $\frac{y}{\sqrt{2}}$, which we will get $z=r\sqrt{2}, r \in \mathbb{Q}$, hence we are done. □

1.5 Intervals

Theorem 1.5.1 (Closed and Open Intervals). If $a, b \in \mathbb{R}$ and $a<b$, then **open interval** is defined by,

$$(a, b)=\{x \in \mathbb{R} \mid a<x<b\}$$

Similarly, we define **closed interval** as,

$$[a, b]=\{x \in \mathbb{R} \mid a \leq x \leq b\}$$

Theorem 1.5.2 (Nested Intervals). The sequence of intervals $I_n, n \in \mathbb{N}$ is nested if

$$I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n \subseteq \ldots$$

Theorem 1.5.3 (Nested Interval Property). For nested intervals $\{I_n\}=[a_n, b_n], n \in \mathbb{N}$, the below is true

$$\bigcap_{i=1}^{\infty} I_n \neq \emptyset$$

Proof. Since intervals are nested intervals, $b_1 \geq a_n \forall n \in \mathbb{N}$. Hence by AoC supremum s of $\{a_n\}$ exists.
We know that $a_n \leq s$. But since b_n is also a upper bound bigger than s, we have $a_n \leq s \leq b_n$, which means $s \in \bigcap_{i=1}^{\infty} I_n$ □

Remark: Intervals must be closed. Consider $A_n=(0, \frac{1}{n})$. Any element of intersection must be bigger than 0, while smaller than $\frac{1}{n}$. By Archimedian property of real numbers, this is a contradiction, hence $\bigcap_{n=1}^{\infty} A_n = \emptyset$

1.6 Cardinality

Definition 1.6.1 (Cardinality). The sets A, B have the same **cardinality** if there exists a bijective function such that $f : A \to B$. We donate cardinal equality with $A \sim B$.
Cardinality mathematically describes the size of the set.

The $\sim$ operation is an equivalence relation.

Definition 1.6.2 (Countable Sets). The set A is said to be **countable** if $A \sim \mathbb{N}$. Otherwise the set is called **uncountable sets**.

Theorem 1.6.1 (Countability of $\mathbb{Q}$.). The set $\mathbb{Q}$ is countable, that is, $\mathbb{Q} \sim \mathbb{N}$.

Proof. There is a proof with visual construction, which maps the rational numbers to natural numbers. □

Theorem 1.6.2 (Uncountability of $\mathbb{R}$). The set $\mathbb{R}$ is uncountable.

Proof. Assume otherwise. Then subset $[0, 1] \subseteq \mathbb{R}$ must be also countable □

Definition 1.6.3 (Power set). The powerset $\mathcal{P}(A)$, is the set of all subsets of A.

Theorem 1.6.3. Every infinite subset of a countable set is a countable set.

Theorem 1.6.4. Let $\{A_n\}, n = 1, 2, 3, \ldots$ be sequence of countable sets. Then,

$$S = \bigcup_{n=1}^{\infty} A_n$$

Proof. Diagonalization method (graphical) □

1.7 Exercises

1. * Show that for $A = \{1 - \frac{1}{n} \ : \ n \in \mathbb{N}\}, \sup A = 1$.

   Proof. A is bounded above since clearly $\forall a \in A, a < 1$. Then by AoC, supremum exists. Let $u = \sup A$. We will show that $u = 1$.

   Clearly, 1 is a upper bound, since $1 > 1 - \frac{1}{n}$ is trivial.

   if $u < 1$, we will show that there exists some $a \in A$ such that $u < a$.

   $$\forall \epsilon > 0, \ \exists a \in A \mid 1 - \epsilon < a = 1 - \frac{1}{n} \Rightarrow \epsilon > \frac{1}{n}$$

   But, by Archimedean, $\exists n_0 \in \mathbb{N}$ contradicting,

   $$u - \epsilon < 1 - \frac{1}{n} \in A$$

   Therefore $u = 1$. □

2. If $S = \{1/n - 1/m : n, m \in \mathbb{N}\}$, find $\inf S$ and $\sup S$.

   Proof. Clearly, S is bounded above and below, therefore supremum and infimum exists by AoC. We will show that $\sup S = 1$, and we can find $\inf S = -\sup(-S) = -1$. Clearly 1 is an upper bound. By definition of supremum

   $$\exists \epsilon > 0,\ \forall s \in S \mid 1 - \epsilon < s = 1/n - 1/m \Rightarrow 1 - \epsilon < 1 - \frac{1}{m}$$

   Which is equivalent to showing $\exists m \in \mathbb{N} \mid \epsilon > \frac{1}{m}$, which is evident from Archimedean. □

3. * Let S be a set of nonnegative real numbers that is bounded above and let $T = \{x^2 : x \in S\}$. Prove that if $u = \sup S$, then $u^2 = \sup T$.

   Proof. Since S is bounded above, T is also bounded above. By AoC, supremum of T exists. Let $t = \sup T$. Clearly, u^2 is upper bound of T, that is,

   $$s \in S \mid s^2 \leq u^2 \Rightarrow y = s^2 \in T \mid y \leq u^2$$

   Now, we will show that u^2 the least upper bound, that is,

   $$\forall \epsilon > 0\ \exists s \in S \mid u^2 - s^2 < \epsilon \Longrightarrow (u - s)(u + s) < \epsilon$$

   Since $u = \sup S$, we have

   $$u - s < \epsilon_0\ \epsilon_0 > 0$$

   Moreover, $u + s \leq 2u$. Combining these inequalities, we have

   $$(u - s)(u + s) < 2u\epsilon_0$$

   Then we just choose some $\epsilon > 2u\epsilon_0$. □

   Second proof.

   $$a = \sup A \Rightarrow a^2 = \sup A \cdot \sup A = \sup A^2 = \sup T$$

   □

4. Given any $x \in \mathbb{R}$, show that there exists a unique $n \in \mathbb{Z}$ such that $x \leq x < n + 1$.

   Proof. By definition of floor function, we have

   $$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$$

   Clearly, $n - \lfloor x \rfloor$ satisfies our property. Assume two $m, n \in \mathbb{Z}$ exists. WLOG $n > m$. Then,

   $$m < n \Rightarrow m + 1 \leq n \Longrightarrow m + 1 \leq n \leq x < m + 1 < n + 1$$

   Clearly, $m + 1 < m + 1$ is a contradiction. □

5. * Show that there exists $y \in \mathbb{R}$ such that $y^2 = 3$.

   Proof. Let $S = \{s \in \mathbb{R} : 0 \leq s, s^2 < 3\}$. Clearly, S is bounded, by AoC, $\sup S = u$ exists. We will show that $u^2 = 3$.

   Clearly $u^2 = 3$ is an upper bound.

   If $u^2 < 3$, we will show that $\exists n \in \mathbb{N} : u + \frac{1}{n} \in S$

   $$\left(u + \frac{1}{n}\right)^2 < 3 \Rightarrow u^2 + \frac{1}{n^2} + \frac{2u}{n} \leq u^2 + \frac{1}{n}(2u + \frac{1}{n}) \Longrightarrow \frac{1}{n} < \frac{3 - u^2}{2u + 1}$$

   By Archimedean, such n exists satisfying our last inequality, hence contradiction.

   □

6. Let $I_n = [0, 1/n]$ for $n \in \mathbb{N}$. Prove that $\cap_{n=1}^{\infty} I_n = \{0\}$.

   Proof. For all $n \in \mathbb{N}$, clearly $0 \in I_n$. For any $x > 0$, by Archimedean there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < x$, hence conclusion. □

1.8 Notes and Mistakes on Exercises

1. Avoid "intuitive" proofs, prove every part of the proof rigorously. For example, the last exercise section, question 1, I also should prove $1 > 1 - \frac{1}{n}$ regardless of trivality.

2. The "steps" in the proofs usually should be **reversed**. In a scratch paper, for example, find and construct an epsion/ natural number(?) and write it formally in the proof.

3. Using floor function is wrong in the last exercise. A.P should be used.

1.9 References

1. https://math.colorado.edu/~nita/12_Axiom_of_Completeness.pdf

Chapter 2

Sequences and Series

2.1 Sequences and limits

Definition 2.1.1 (Sequences). A sequence is a function with its domain as $\mathbb{N}$.

Definition 2.1.2 (Converge). . A sequence (x_n) is said to conerge to $x \in \mathbb{R}$, or x is said to be limit of (x_n) if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \ : \ |x_n - x| < \epsilon, \ \forall n \geq N$$

If limit exists, sequence is **convergent**, otherwise it is **divergent**.

Definition 2.1.3 (Epsilon Neighborhood definition of convergence). . Below definition with neighborhood is equivalent to the definition above

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \ : \ x_n \in V_\epsilon(x), \ \forall n > N$$

Theorem 2.1.1 (Uniqueness of Limits). The limit of a sequence is **unique**.

Proof. For the sake of the contradiction, let $x = x^{'} = \lim_{n \to \infty}(x_n)$. with the definiton of the limit, $\forall \epsilon > 0, \exists n \in \mathbb{N}$ such that for all $n \geq N, N^{'}$,

$$|x - x_n| < \epsilon/2 \ \forall n \geq N$$

$$|x^{'} - x_n| < \epsilon/2 \ \forall n \geq N^{'}$$

However, by the triangle inequality, we have

$$|x - x^{'}| \leq |x - x_n| + |x^{'} - x_n| < \epsilon/2 + \epsilon/2 = \epsilon, \ \forall n \geq K = \max(N, N^{''})$$

Since this is $\forall \epsilon > 0$, we conclude that $x = x^{''}$. □

2.2 Limit Theorems

Definition 2.2.1. A sequence (x_n) is **bounded** if there exists $U > 0$ such that

$$|x_n| \leq U \ \forall n \in \mathbb{N}$$

A sequence is bounded **iff** the set $\{x_n \ : \ n \in \mathbb{N}\}$ is bounded.

Theorem 2.2.1. A convergent sequence is bounded.

Proof. If a sequence converges, then all but finite number of terms of the sequence belongs to $V_\epsilon(x)$. Since $V_{\epsilon/2}(x)$ is bounded, the sequence itself is bounded. □

Theorem 2.2.2 (Algebra of limits). let $X = (x_n), Y = (y_n)$ converge to x, y respectively. Then sequences $X + Y, X - Y, X \cdot Y, cX$ converge to $x + y, x - y, xy, cx$ respectively.
If $y \neq 0$, X/Y converges to x/y.

Proof. We will show that $X+Y$ property only, others are similar. By definition of convergence, $\forall \epsilon > 0, \exists N, N^{'} \in \mathbb{N}$ such that

$$|x - x_n| < \epsilon/2, \forall n \geq N$$

$$|y - y_n| < \epsilon/2, \forall n \geq N^{'}$$

□

However, notice that $\forall n \geq \max N, N^{'}$

$$|(x + y) - (x_n + y_n)| = |(x - x_n) + (y - y_n)| \leq |x - x_n| + |y - y_n| < \epsilon/2 + \epsilon/2 = \epsilon$$

Which proves our theorem

Theorem 2.2.3. If (x_n) is convergent sequence and $x_n \geq 0$ for all $n \in \mathbb{N}$, then $x = \lim(x_n) \geq 0$.

Theorem 2.2.4. if $(x_n), (y_n)$ are convergent sequences and $x_n \leq y_n$ for all $n \in \mathbb{N}$, then $x \leq y$.

Theorem 2.2.5. If (x_n) is a convergent sequence and $a \leq x_n \leq b$ for all $n \in \mathbb{N}$, then $a \leq x \leq b$.

Theorem 2.2.6 (Squeeze theorem). Let $(x_n), (y_n), (z_n)$ be sequences such that

$$x_n \leq y_n \leq z_n$$

And $x = z$. Then (y_n) converges and

$$x = y = z$$

All above theorems are proven similarly, the idea is the same.

2.3 Monotone Sequences

Definition 2.3.1. (x_n) is **monotone** if it is either increasing or decreasing.

Theorem 2.3.1 (Monotone Convergence Theorem). A monotone sequence is convergent iff it is bunded. Furthermore, if x_n is a bounded increasing sequence, then

$$\lim(x_n) = \sup\{x_n : n \in \mathbb{N}\}$$

Similarly, if y_n is a bounded decreasing sequence, then

$$\lim(y_n) = \inf\{y_n : n \in \mathbb{N}\}$$

2.4 Subsequences

Definition 2.4.1 (Subsequences). Let $\{n_k\}$ be strict monotone increasing sequence of real numbers, then the sequence $X^{'} = (x_{n_k})$ is called **subsequence**

Theorem 2.4.1. If a sequence (x_n) converge to x, then the subsequence (x_{n_k}) also converge to x.

Proof. By definition, $\forall \epsilon > 0,\ \exists N(\epsilon) \in \mathbb{N}$ such that $\forall n \geq N(\epsilon)$,

$$|x_n - x| < \epsilon$$

Because $n_k \geq k$ (induction), then we can find such $k \geq N(\epsilon)$, then $n_k \geq N(\epsilon)$, which means

$$|x_{n_K} - x| < \epsilon$$

□

Theorem 2.4.2 (Monotone subsequence theorem). If (x_n) is a sequence, then there exists a monotone subsequence.

Theorem 2.4.3 (The Bolzano-Weierstrass Theorem). A bounded sequence has a convergent subsequence.

Proof. It is direct consequence of monotone subsequence theorem. Since we can find a monotone subsequence, and is bounded, we can conclude it is convergent. □

2.5 The Cauchy Criterion

Definition 2.5.1 (Cauchy Sequence). A sequence (x_n) is said to be a **Cauchy sequence** if $\forall \epsilon > 0$, $\exists N(\epsilon) \in \mathbb{N}$,

$$|x_n - x_m| < \epsilon \quad \forall m > n > N$$

Theorem 2.5.1. A sequence is convergent if and only if it is a cauchy sequence

Proof. □

2.6 Exercises

1. Show that sequence of (2^n) does not converge.

   Proof. It suffices to prove that (2^n) is unbounded. Assume otherwise that there exists $M \in \mathbb{R}$ such that $2^n \leq M$ for all $n \in \mathbb{N}$. Then,

   $$n \leq \log_2(M) = c$$

   However by the unboundness of $\mathbb{N}$, we can find n_0 such that $n_0 > c$ for any $c \in \mathbb{R}$, contradicting our claim. □

2. * Show that $z_n = (a^n + b^n)^{1/n}$ where $0 < a < b$ converge to b.

   Proof. Since $a > 0$, we have
   $$(a^n + b^n)^{1/n} > (b^n)^{1/n} = b$$
   Since $a < b$, we have
   $$(a^n + b^n)^{1/n} < (2b^n)^{1/n} = 2^{1/n}b$$
   Then,
   $$b \le z_n \le 2^{1/n}b$$
   Using the squeeze theorem and the fact that $2^{1/n}$ converges to 1, we can see that $\lim z_n = b$. □

3. * Let $x_1 = 8$ and let $x_{n+1} = \frac{1}{2}x_n + 2$ for $n \in \mathbb{N}$. Show that x_n converges, and find the limit.

   Proof. We will show that (x_n) is monotone and bounded.

   1) $x_n \ge 4$ for all $n \in \mathbb{N}$.

   By induction, for $n = 1, 2$ we have $8 > 4$ and $6 > 4$. Now assume it is true for $n = k$. Then,
   $$x_{k+1} = \frac{1}{2}x_k + 2 > 4$$
   2) $x_{n+1} < x_n$ for all $n \in \mathbb{N}$. By induction, for $n = 1, 2$ we have $6 < 8$. Now assume it is true for $n = k$. Then,
   $$x_{k+1} = \frac{1}{2}x_k + 2 < \frac{1}{2}x_{k-1} + 2 = x_k$$
   Then sequence is monotone and bounded, therefore it is convergent to the $\inf\{x_n : n \in \mathbb{N}\} = 4$, which we already know how to prove. □

4. Prove that $e_n = \left(1 + \frac{1}{n}\right)^{1/n}$ is convergent.

   Proof. Direct consequence of monotone convergence theorem. □

5. * Prove that $\lim(c^{1/n}) = 1$ for $0 < c < 1$.

   Proof. The sequence $(c^{1/n})$ is monotone:
   $$c^{1/n} < c^{1/(n+1)} \Leftrightarrow \frac{1}{n}\ln c < \frac{1}{n+1}\ln c \Rightarrow \frac{1}{n} > \frac{1}{n+1} \forall n \in \mathbb{N}$$
   Which is true, since $n + 1 > n \Rightarrow \frac{1}{n+1} < \frac{1}{n}$ for all natural numbers.

   The sequence is bounded:
   $$c^{1/n} < 1 \Rightarrow c < 1$$
   Which is true since $0 < c < 1$. Then, by monotone convergence theorem, our sequence converges. Let limit be L. But, the subsequence $x_{2n} = c^{1/2n} = \sqrt{c^{1/n}}$ also converges to the same limit, which means
   $$L = \sqrt{L} \Rightarrow L \in \{0, 1\}$$
   $L = 0$ is impossible, since $a^x = 0$ iff $a = 0$, but $0 < c$. Then, $L = 1$. □

6. * Let (f_n) be the Fibonacci sequence, and let $x_n := f_{n+1}/f_n$. Given that $\lim(x_n) = L$ exists, find L.

   Proof.
   $$x_n = f_{n+1}/f_n = (f_n + f_{n-1})/f_n = 1 + f_{n-1}/f_n \Rightarrow L = 1 + 1/L$$
   Solving the quadratic equation, we have $L = \frac{1}{2}(1 + \sqrt{5})$ □

7. * Let (x_n) be a bounded sequence and for each $n \in \mathbb{N}$, let $s_n := \sup\{x_k : k \ge n\}$ and $S := \inf\{s_n\}$. Show that there exists a subsequence of (x_n) that converges to S.

8. * Show that the sequence $\left(\frac{n+1}{n}\right)$ is a Cauchy Sequence.

Proof. Choose $M > 2/\epsilon$, then $\forall \epsilon > 0, m > n \geq M$, $\frac{1}{m} < \frac{1}{n} \leq \frac{1}{M} < \epsilon/2$, and,

$$\left|1 + \frac{1}{n} - 1 - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} < \epsilon/2 + \epsilon/2 = \epsilon$$

Which shows that our sequence is a cauchy sequence. □

9. * Show that if (x_n) and (y_n) are cauchy sequences, then $(x_n + y_n)$ is also a cauchy sequence.

Proof. By definition of cauchy sequence, $\forall \epsilon > 0$,

$$\exists N_1 \in \mathbb{N} \ : \ |x_m - x_n| < \epsilon/2 \quad \forall m > n \geq N_1$$

$$\exists N_2 \in \mathbb{N} \ : \ |y_m - y_n| < \epsilon/2 \quad \forall m > n \geq N_2$$

Choose $N = \max(N_1, N_2)$. Then for all $m > n \geq N, \epsilon > 0$,

$$|(x_n + y_m) - (x_m + y_m)| \leq |(x_n - x_m) + (y_n + y_m)| < |x_n - x_m| + |y_n - y_m| < \epsilon/2 + \epsilon/2 = \epsilon$$

□

Chapter 3

Basic Topology in R

Chapter 4

Limit and Continuity

4.1 Limits of functions

Definition 4.1.1 (limit). Let $A \subset \mathbb{R}$ and c be a cluster point of A. Then, for any function $f : A \to \mathbb{R}$, $L \in \mathbb{R}$ is said to be a **limit of f at c.**, if $\forall \epsilon > 0$, $\exists \delta > 0$ such that if $x \in A$,

$$0 < |x - c| < \delta \quad \to \quad |f(x) - L| < \epsilon$$

Theorem 4.1.1 (Uniqueness of limit). Limit of $f : A \to \mathbb{R}$ to c cluster point of A is unique for c.

Proof. Assume otherwise, then two limits L, L' such that $\forall \epsilon >, \exists \delta > 0$ that $|x - c| < \delta$ implies

$$|f(x) - L| < \epsilon/2$$

$$|f(x) - L'| < \epsilon/2$$

Then adding them up, we have

$$|L - L'| \leq |f(x) - L| + |f(x) - L'| < \epsilon$$

Which gives contradiction. □

Theorem 4.1.2 (Algebra operations on limit). Let $A \subset \mathbb{R}$, and let $f, g : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A, and let $b \in \mathbb{R}$.

Then, similar to sequences, if $\lim_{x \to c} f = L$ and $\lim_{x \to c} g = M$.

1. $\lim_{x \to c}(f + g) = L + M$
2. $\lim_{x \to c}(f - g) = L - M$
3. $\lim_{x \to c}(fg) = LM$
4. $\lim_{x \to c}(bf) = bL$
5. $\lim_{x \to c}(f/g) = L/M$ if $g(x) \neq 0 \forall x \in A$ and $M \neq 0$.

4.2 Exercises

1. Let $f := \mathbb{R} \to \mathbb{R}$ and let $c \in \mathbb{R}$. Show that $\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to 0} f(x + c) = L$.

   Proof. By definition, $\forall \epsilon > 0$, $\exists \delta$ such that

   $$|x - c| < \delta \quad \text{means} \quad |f(x) - L| < \epsilon$$

Choose $x \coloneqq x + c$. Then we have,

$$|x - 0| < \delta \quad \text{means} \quad |f(x + c) - L| < \epsilon$$

□

2. Let I be an interval in $\mathbb{R}$, let $f : I \to \mathbb{R}$, and let $c \in I$. suppose $\exists K, L$ such that $|f(x) - L| \leq K|x - c|$ for $x \in I$. Show that $\lim_{x \to c} f(x) = L$.

   Proof. $\forall \epsilon > 0$, choose $\delta = \epsilon/K$, then

   $$|x - c| < \epsilon/K \quad \to \quad |f(x) - L| \leq K|x - c| < \epsilon$$

   □

3. Let $I \coloneqq (0, a)$ where $a > 0$, and let $g(x) = x^2$ for $x \in I$. For any points $x, c \in I$, show that $|g(x) - c^2| \leq 2a|x - c|$. Use this inequality to prove that $\lim_{x \to c} x^2 = c^2$ for any $c \in I$.

   Proof. Since $x \in I$, $0 < x < a$. Similarly, $0 < c < a$. Then,

   $$|g(x) - c^2| = |x^2 - c^2| = |x - c||x + c| \leq 2a|x - c|$$

   From this inequality, we choose $\delta = \epsilon/2a$. Then $\forall \epsilon > 0$,

   $$|x - c| < \epsilon/2a \quad \to \quad |x^2 - c^2| \leq 2a|x - c| < \epsilon$$

   □

4. Show that $\lim_{x \to c} x^3 = c^3$ for any $c \in \mathbb{R}$.

   Proof. If $x < c$, we can choose $\delta = \epsilon/3c$, then $\forall \epsilon > 0$,

   $$|x - c| < \epsilon/3c \quad \to \quad |x^3 - c^3| = |x - c||x^2 + xc + c^2| \leq |x - c||3c^2| < \epsilon$$

   □