Analysis notebook

A notebook with some exercises

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Preface

About the Book

This book is more like a notebook for my personal and educational purposes. I prefer taking my notes in latex, and organize all of them in a booklike structure like this one you are reading right now. The book is open source and in public domain.

Book's source

You may find the tex source files in my github account.

References

Majority of times, two books were used to study analysis:

1. Understanding Analysis - Stephen Abbot

2. Introduction to Real Analysis - Robert G. Bartle

Furthermore, other sources such as mathexchange, wikipedia, and university lecture notes are used generally. If a specific source is used, it is usually listed in the end of the chapter.

How to use the Book

I use this for fast fact checking purposes (that is what notes are for, right?).

Contents

Chapter 1

Real Numbers

1.1 Absolute value, epsilon-neighborhood

Absolute value is a function $f : \mathbb{R} \to \mathbb{R}_0$ such that,

$$f(x) = x \qquad \text{if } x \geq 0$$
$$f(x) = -x \qquad \text{if } x < 0$$

Absolute value describes **Distance** between two values. It is important to think this function as distance more than some function that "makes negative values positive"

Proposition 1.1.1: Properties of absolute value

$(\forall x \in \mathbb{R}), (\forall y \in \mathbb{R})$,

1. $|x| \geq 0$
2. $|-x| = |x|$
3. $|xy| = |x||y|$
4. $|x|^2 = x^2$
5. $|x| \leq y \iff -y \leq x \leq y$
6. $-|x| \leq x \leq |x|$

Theorem 1.1.1: Triangle Inequality

$(\forall x \in \mathbb{R})$,$(\forall y \in \mathbb{R})$,

$$|x + y| \leq |x| + |y|$$

Proof. From the proposition we have,

$$-|x| \leq x \leq |x|$$
$$-|y| \leq y \leq |y|$$

Adding these equations we get

$$-|x| - |y| \leq x + y \leq |x| + |y| \Rightarrow |x + y| \leq |x| + |y|$$

□

Corollary 1.1.1. $(\forall x \in \mathbb{R})$,$(\forall y \in \mathbb{R})$,

1. $||x| - |y|| \leq |x - y|$
2. $|x - y| \leq |x| + |y|$

3. $\left|\sum_{i=1}^{n} a_i\right| \leq \sum_{i=1}^{n} |a_i|$

Proof. These Corollaries are direct consequence of triangle inequality, with third inequality using the proof with induction. I will not provide proofs since they are kind of boring and time comsuming. $\square$

Definition 1.1.1: Epsilon Neighborhood

The $\epsilon - neighborhood$ of a is defined as a set

$$V_\epsilon(a) = \{x \in \mathbb{R} \mid |x - a| < \epsilon\}$$

Which is equivalent to open interval

$$(a - \epsilon, a + \epsilon)$$

Analysis heavily uses epsilon definitions and epsilon neighborhood for rigirous proofs. Therefore this definition is an useful tool.

1.2 Axiom of Completeness, Infimum and Supremum

Definition 1.2.1: Upper Bounds

A set $A \subseteq R$ is **bounded above** if $(\exists b \in R)$ s.t $a \leq b$ $(\forall a \in A)$. The number b is the **upper bound of A**. We denote set of upper bounds of A as A^u.
Similarly, we define lower bounds and the set as A^ℓ.

Definition 1.2.2: Supremum

upper bound s of a set S is called supremum if,

$$s = \min A^u$$

Mathematically we show the notation as $s = \sup S$.
In Similar fashion, we define $\inf S$ for lower bounds.

Axiom of Completeness (AoC). Every non-empty subsets of $\mathbb{R}$ that is bounded above have supremum. The Axiom also deduces the existence of infimum in a similar fashion.

Definition 1.2.3: Epsilon definition of supremum

$s \in \mathbb{R}$ is a supremum of a set $A \subseteq \mathbb{R}$ iff

$$(\forall \epsilon > 0)(\exists a \in A) \mid s - \epsilon < a$$

Proof sketch. The both ways of the lemma can be proven by definition of the supremum. $\square$

We use similar lemma for infimum.

Proposition 1.2.1: Maximum and Supremum

If maximum of $A \neq \{\emptyset\} \subseteq \mathbb{R}$ exists, then

$$\max A = \sup A$$

Proof. Denote $s = \sup A$ and $m = \max A$. By definition, $s = \min A^u$ and $m = A^u \cap A$. The result is an immediate consequence of the definitions of maximum and supremum.

m is a proper supremum, since $\forall x \in A$ we have $x \leq m$, and since also $m \in A$, $t = \sup A < m$ is impossible. □

Similarly, we have $\min A = \inf A$.

Proposition 1.2.2: Uniqueness of Supremum

Supremum and Infimum are unique.

Proof. For the sake of the contradiction, assume there exists two supremum s_1, s_2. Then by definition of supremum, we have

$$s_1 \geq s_2 \;\wedge\; s_2 \geq s_1 \Rightarrow s_1 = s_2$$

Infimum follows the similar proof. □

Proposition 1.2.3: Existence of Infimum

oC implies the existence of infimum for $A \subseteq \mathbb{R}$ such that $A^\ell \neq \emptyset$,

$$\inf A = -\sup(-A)$$

Proof. Since $A^l \neq \emptyset$, it follows that

$$(\exists x \in A^\ell) \mid x \leq a$$

Then,

$$-x \geq -a \Rightarrow -x \in (-A)^u \neq \emptyset$$

By AoC, $\sup(-A)$ exists. Rest is trivial. □

Proposition 1.2.4: Operations on Supremum

he supremum holds these properties,

$$\sup(A+B) = \sup(A)+\sup(B) \tag{1.1}$$
$$\sup(A\cdot B) = \sup(A)\cdot\sup(B) \tag{1.2}$$
$$\text{if } c \geq 0, \quad \sup(cA) = c\sup(A) \tag{1.3}$$
$$\text{if } c \leq 0, \quad \sup(cA) = c\inf(A) \tag{1.4}$$

Proof. These properties directly follow from the epsilon definition of the supremum. That is, $\forall\epsilon_a, \epsilon_b, \exists a, b \in A, B$ such that,

$$\sup(A) - a < \epsilon_a \ \wedge \sup(B) - b < \epsilon_b$$

adding these equations to each other, we have

$$\sup(A) + \sup(B) - (a+b) < \epsilon_a + \epsilon_b \tag{1.5}$$

Note that $(a+b) \in A+B$, and let $\epsilon_a + \epsilon_b = \epsilon_{a+b}$. Also we know that,

$$(\forall\epsilon_c)(\exists c \in A+B) \mid \sup(A+B) - c < \epsilon_c \tag{1.6}$$

but 1.5 and 1.6 both are valid, hence the conclusion.
We can similarly prove other propositions, even for inf. □

1.3 Applications of Completeness, Archimedean Property (A.P)

Theorem 1.3.1: Archimedean Property, A.P

$$(\forall x \in \mathbb{R})\ (\exists n_x \in \mathbb{N}) \mid x \leq n_x.$$

Proof. For the sake of contradiction, assume otherwise. Then $n \leq x \ \forall n \in \mathbb{N}$, by AoC $\mathbb{N}$ has supremum, s. Since $s-1 < s$, $s-1$ is not a upper bound, therefore $\exists m \in \mathbb{N}$ such that $s-1 < m \Rightarrow s < m+1$. but $m+1 \in \mathbb{N}$. Therefore s cannot be a supremum. □

Theorem 1.3.2: Density of Rationals in R

$(\forall a, b \in \mathbb{R}), (\exists r \in \mathbb{Q})$ such that

$$a < r < b$$

Proof. Since r must be rational, we want to find $m, n \in \mathbb{Z}$ such that $\frac{m}{n} = r$.
From Archimedean property,

$$(\exists n \in \mathbb{N}) : n(y-x) \geq 1$$

Again from Archimedean property,

$$(\forall t \in \mathbb{R}), (\exists m \in \mathbb{Z}) : m-1 \leq t \leq m$$

In other words, for any real numbers, there are two consecutive integers that lies in the each boundary of the real numbers.
Let $t = nx$. Combining the inequalities, we get

$$nx \leq m \leq 1 + nx \leq ny \Rightarrow x \leq \frac{m}{n} \leq y$$

□

Theorem 1.3.3: Density of Irratioanls in R

$(\forall x, y \in \mathbb{R})$ such that $x < y$, $(\exists z \in \mathbb{I})$ such that

$$x < z < y$$

Proof. It is direct consequence of density of Rationals. We apply density theorem on $\frac{x}{\sqrt{2}}$ and $\frac{y}{\sqrt{2}}$, which we will get $z = r\sqrt{2}, r \in \mathbb{Q}$, hence we are done. □

1.4 Intervals

Theorem 1.4.1: Closed and Open Intervals

If $a, b \in \mathbb{R}$ and $a < b$, then **open interval** is defined by,

$$(a, b) = \{x \in \mathbb{R} | a < x < b\}$$

Similarly, we define **closed interval** as,

$$[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}$$

Definition 1.4.1: Nested Intervals

$I_n, n \in \mathbb{N}$ is nested if

$$I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n \subseteq \ldots$$

Theorem 1.4.2: Nested Interval Property

For nested intervals $\{I_n\} = [a_n, b_n], n \in \mathbb{N}$,

$$\bigcap_{i=1}^{\infty} I_n \neq \emptyset$$

Proof. Since intervals are nested intervals, $b_1 \geq a_n$ $(\forall n \in \mathbb{N})$. Hence by AoC supremum α of $\{a_n\}$ exists.
We know that $a_n \leq \alpha$. But since b_n is also a upper bound bigger than α, we have $a_n \leq \alpha \leq b_n$, which means $\alpha \in \bigcap_{i=1}^{\infty} I_n$ □

Remark: Intervals must be closed. Consider $A_n = (0, \frac{1}{n})$. Any element of intersection must be bigger than 0, while smaller than $\frac{1}{n}$. By Archimedian property of real numbers, this is a contradiction, hence $\bigcap_{n=1}^{\infty} A_n = \emptyset$

1.5 Cardinality

Definition 1.5.1: Cardinality

The sets A, B have the same **cardinality** if there exists a bijective function such that $f : A \to B$. We donate cardinal equality with $A \sim B$.
Cardinality mathematically describes the size of the set.
The $\sim$ operation is an equivalence relation.

Definition 1.5.2: Countable Sets

The set A is said to be **countable** if $A \sim \mathbb{N}$. Otherwise the set is called **an uncountable set**.

Theorem 1.5.1: Countability of $\mathbb{Q}$

The set $\mathbb{Q}$ is countable, that is, $\mathbb{Q} \sim \mathbb{N}$.

Proof. There is a proof with visual construction, which maps the rational numbers to natural numbers. □

Theorem 1.5.2: Uncountability of $\mathbb{R}$

The set $\mathbb{R}$ is uncountable.

Proof. Assume otherwise. Then subset $[0, 1] \subseteq \mathbb{R}$ must be also countable □

Definition 1.5.3: Power Set

The powerset $\mathcal{P}(A)$, is the set of all subsets of A.

Theorem 1.5.3

Every infinite subset of a countable set is a countable set.

Theorem 1.5.4

Let $\{A_n\}, n = 1, 2, 3, \ldots$ be sequence of countable sets. Then,

$$S = \bigcup_{n=1}^{\infty} A_n$$

is also countable.

Proof. Diagonalization method (graphical) □

1.6 Exercises

1.1 Show that

$$\sup\{x \in \mathbb{R} \mid x^2 < 2\} = \sqrt{2}$$

Proof. Let the set be A, the set is bounded, since if $(x \in A)\ x > 2 \Rightarrow x^4 > 4$, contradiction. Hence 2 is an upper bound. Let $\alpha = \sup A$. If $\alpha < 2$,

$$(\exists n \in \mathbb{N}) \mid \alpha^2 < (\alpha + \frac{1}{n})^2 < 2$$

This is true because,

$$(\alpha + \frac{1}{n})^2 = \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} < \alpha^2 + \frac{2\alpha + 1}{n}$$

and by Archimedean Property,

$$(\exists n \in \mathbb{N}) \mid \frac{\alpha^2 - 2}{2\alpha + 1} > \frac{1}{n}$$

Similarly, if $\alpha > 2$,

$$(\exists n \in \mathbb{N}) \mid 2 < (\alpha - \frac{1}{n})^2 < \alpha^2$$

Simplifying, we get

$$(\alpha - \frac{1}{n})^2 = \alpha^2 + \frac{1}{n^2} - \frac{2\alpha}{n} < \alpha^2 - \frac{2\alpha}{n}$$

and by Archimedean Property,

$$(\exists n \in \mathbb{N}) \mid \frac{\alpha^2 - 2}{2\alpha} > \frac{1}{n}$$

Therefore $\alpha^2 = 2$. □

1.2 Let $A \subset \mathbb{R}$ be a nonempty set Define $-A = \{x \mid -x \in A\}$. Show that

$$\sup(-A) = -\inf A.$$

Proof. Let $\alpha = \sup(-A)$. Suppose $-A$ is bounded above. By definition of supremum, $(\forall \epsilon > 0)(\exists x \in -A)$,

$$\alpha - \epsilon < x \Rightarrow -\alpha + \epsilon > -x$$

However, $-x \in A$ and the last inequality is the definition of the infimum, hence $-\alpha = \inf A$. If $-A$ is not bounded above, then A is not bounded below, hence $\sup -A = -\inf A$. □

1.3 Let $A, B \subset \mathbb{R}$ be nonepty. Define

$$A + B = \{z = x + y \mid x \in A \wedge y \in B\}$$

Show that

$$\sup(A + B) = \sup A + \sup B$$

Proof. If A or B is unbounded, then $A + B$ is unbounded. Assume Both of them are bounded. Let $\alpha = \sup A$, $\beta = \sup B$. Then by definition of supremum, $(\forall \epsilon > 0)(\exists a \in A)(\exists b \in B)$

$$\alpha - \frac{\epsilon}{2} < a$$

$$\beta - \frac{\epsilon}{2} < b$$

Adding these inequalities, we get

$$(\alpha + \beta) - \epsilon < (a + b) \in A + B$$

Which means $(\alpha + \beta) = \sup(A + B)$. □

1.4 Let $A, b \subset \mathbb{R}$ be nonempty. Define

$$A \cdot B = \{z = x \cdot y \mid x \in A \wedge y \in B\}$$

Show that

$$\sup(A \cdot B) = \sup A \cdot \sup B$$

Proof. Let α, β be supremum of A, B respectively. If A or B is unbounded above, then $A \cdot B$ is unbounded above.Otherwise, by definition of supremum, $(\forall epsilon > 0)(\exists a \in A)(\exists b \in B)$

$$\alpha - \epsilon < a$$

$$\beta - \epsilon < b$$

Multiplying these inequalities, we get,

$$\alpha \cdot \beta - \epsilon(\beta + \alpha - \epsilon) < a \cdot b$$

Since $\epsilon(\beta + \alpha - \epsilon)$ is arbitrary, we see that $\alpha \cdot \beta = \sup(A \cdot B)$. □

1.5 Let A and B be nonempty subsets of real numbers. Show that

$$\sup(A \cup B) = \max\{\sup A, \sup B\}$$

Proof. Let α, β be supremum of A, B respectively. If A or B is unbounded above, then the union is also unbounded above, equality is trivial. Otherwise, without loss of generality, assume $\alpha \leq \beta$. Then, $(\forall x \in A \cup B), x \leq \beta$. Since $x \in B$, we can find $b \in B$ satisfying $(\forall \epsilon > 0)$,

$$\beta - \epsilon > b$$

Hence $\beta = \sup A \cup B$.

□

1.7 References

1. https://math.colorado.edu/~nita/12_Axiom_of_Completeness.pdf

Chapter 2

Sequences and Series

2.1 Sequences and limits

Definition 2.1.1 (Sequences). A sequence is a function with its domain as $\mathbb{N}$.

Definition 2.1.2 (Converge). . A sequence (x_n) is said to conerge to $x \in \mathbb{R}$, or x is said to be limit of (x_n) if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \ : \ |x_n - x| < \epsilon, \ \forall n \geq N$$

If limit exists, sequence is **convergent**, otherwise it is **divergent**.

Definition 2.1.3 (Epsilon Neighborhood definition of convergence). . Below definition with neighborhood is equivalent to the definition above

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \ : \ x_n \in V_\epsilon(x), \ \forall n > N$$

Theorem 2.1.1 (Uniqueness of Limits). The limit of a sequence is **unique**.

Proof. For the sake of the contradiction, let $x = x^{'} = \lim_{n \to \infty}(x_n)$. with the definiton of the limit, $\forall \epsilon > 0, \exists n \in \mathbb{N}$ such that for all $n \geq N, N^{'}$,

$$|x - x_n| < \epsilon/2 \ \forall n \geq N$$

$$|x^{'} - x_n| < \epsilon/2 \ \forall n \geq N^{'}$$

However, by the triangle inequality, we have

$$|x - x^{'}| \leq |x - x_n| + |x^{'} - x_n| < \epsilon/2 + \epsilon/2 = \epsilon, \ \forall n \geq K = \max(N, N^{''})$$

Since this is $\forall \epsilon > 0$, we conclude that $x = x^{''}$. □

2.2 Limit Theorems

Definition 2.2.1. A sequence (x_n) is **bounded** if there exists $U > 0$ such that

$$|x_n| \leq U \ \forall n \in \mathbb{N}$$

A sequence is bounded **iff** the set $\{x_n \ : \ n \in \mathbb{N}\}$ is bounded.

Theorem 2.2.1. A convergent sequence is bounded.

Proof. If a sequence converges, then all but finite number of terms of the sequence belongs to $V_\epsilon(x)$. Since $V_{\epsilon/2}(x)$ is bounded, the sequence itself is bounded. □

Theorem 2.2.2 (Algebra of limits). let $X = (x_n), Y = (y_n)$ converge to x, y respectively. Then sequences $X + Y, X - Y, X \cdot Y, cX$ converge to $x + y, x - y, xy, cx$ respectively.
If $y \neq 0$, X/Y converges to x/y.

Proof. We will show that $X+Y$ property only, others are similar. By definition of convergence, $\forall \epsilon > 0, \exists N, N^{'} \in \mathbb{N}$ such that

$$|x - x_n| < \epsilon/2, \forall n \geq N$$

$$|y - y_n| < \epsilon/2, \forall n \geq N^{'}$$

□

However, notice that $\forall n \geq \max N, N^{'}$

$$|(x + y) - (x_n + y_n)| = |(x - x_n) + (y - y_n)| \leq |x - x_n| + |y - y_n| < \epsilon/2 + \epsilon/2 = \epsilon$$

Which proves our theorem

Theorem 2.2.3. If (x_n) is convergent sequence and $x_n \geq 0$ for all $n \in \mathbb{N}$, then $x = \lim(x_n) \geq 0$.

Theorem 2.2.4. if $(x_n), (y_n)$ are convergent sequences and $x_n \leq y_n$ for all $n \in \mathbb{N}$, then $x \leq y$.

Theorem 2.2.5. If (x_n) is a convergent sequence and $a \leq x_n \leq b$ for all $n \in \mathbb{N}$, then $a \leq x \leq b$.

Theorem 2.2.6 (Squeeze theorem). Let $(x_n), (y_n), (z_n)$ be sequences such that

$$x_n \leq y_n \leq z_n$$

And $x = z$. Then (y_n) converges and

$$x = y = z$$

All above theorems are proven similarly, the idea is the same.

2.3 Monotone Sequences

Definition 2.3.1. (x_n) is **monotone** if it is either increasing or decreasing.

Theorem 2.3.1 (Monotone Convergence Theorem). A monotone sequence is convergent iff it is bunded. Furthermore, if x_n is a bounded increasing sequence, then

$$\lim(x_n) = \sup\{x_n : n \in \mathbb{N}\}$$

Similarly, if y_n is a bounded decreasing sequence, then

$$\lim(y_n) = \inf\{y_n : n \in \mathbb{N}\}$$

2.4 Subsequences

Definition 2.4.1 (Subsequences). Let $\{n_k\}$ be strict monotone increasing sequence of real numbers, then the sequence $X' = (x_{n_k})$ is called **subsequence**

Theorem 2.4.1. If a sequence (x_n) converge to x, then the subsequence (x_{n_k}) also converge to x.

Proof. By definition, $\forall \epsilon > 0,\ \exists N(\epsilon) \in \mathbb{N}$ such that $\forall n \geq N(\epsilon)$,

$$|x_n - x| < \epsilon$$

Because $n_k \geq k$ (induction), then we can find such $k \geq N(\epsilon)$, then $n_k \geq N(\epsilon)$, which means

$$|x_{n_K} - x| < \epsilon$$

□

Theorem 2.4.2 (Monotone subsequence theorem). If (x_n) is a sequence, then there exists a monotone subsequence.

Theorem 2.4.3 (The Bolzano-Weierstrass Theorem). A bounded sequence has a convergent subsequence.

Proof. It is direct consequence of monotone subsequence theorem. Since we can find a monotone subsequence, and is bounded, we can conclude it is convergent. □

2.5 The Cauchy Criterion

Definition 2.5.1 (Cauchy Sequence). A sequence (x_n) is said to be a **Cauchy sequence** if $\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}$,

$$|x_n - x_m| < \epsilon \quad \forall m > n > N$$

Theorem 2.5.1. A sequence is convergent if and only if it is a cauchy sequence

Proof. □

2.6 Exercises

1. Show that sequence of (2^n) does not converge.

   Proof. It suffices to prove that (2^n) is unbounded. Assume otherwise that there exists $M \in \mathbb{R}$ such that $2^n \leq M$ for all $n \in \mathbb{N}$. Then,

   $$n \leq \log_2(M) = c$$

   However by the unboundness of $\mathbb{N}$, we can find n_0 such that $n_0 > c$ for any $c \in \mathbb{R}$, contradicting our claim. □

2. * Show that $z_n = (a^n + b^n)^{1/n}$ where $0 < a < b$ converge to b.

   Proof. Since $a > 0$, we have
   $$(a^n + b^n)^{1/n} > (b^n)^{1/n} = b$$
   Since $a < b$, we have
   $$(a^n + b^n)^{1/n} < (2b^n)^{1/n} = 2^{1/n}b$$
   Then,
   $$b \leq z_n \leq 2^{1/n}b$$
   Using the squeeze theorem and the fact that $2^{1/n}$ converges to 1, we can see that $\lim z_n = b$. ☐

3. * Let $x_1 = 8$ and let $x_{n+1} = \frac{1}{2}x_n + 2$ for $n \in \mathbb{N}$. Show that x_n converges, and find the limit.

   Proof. We will show that (x_n) is monotone and bounded.

   1) $x_n \geq 4$ for all $n \in \mathbb{N}$.

   By induction, for $n = 1, 2$ we have $8 > 4$ and $6 > 4$. Now assume it is true for $n = k$. Then,
   $$x_{k+1} = \frac{1}{2}x_k + 2 > 4$$
   2) $x_{n+1} < x_n$ for all $n \in \mathbb{N}$. By induction, for $n = 1, 2$ we have $6 < 8$. Now assume it is true for $n = k$. Then,
   $$x_{k+1} = \frac{1}{2}x_k + 2 < \frac{1}{2}x_{k-1} + 2 = x_k$$
   Then sequence is monotone and bounded, therefore it is convergent to the $\inf\{x_n : n \in \mathbb{N}\} = 4$, which we already know how to prove. ☐

4. Prove that $e_n = \left(1 + \frac{1}{n}\right)^{1/n}$ is convergent.

   Proof. Direct consequence of monotone convergence theorem. ☐

5. * Prove that $\lim(c^{1/n}) = 1$ for $0 < c < 1$.

   Proof. The sequence $(c^{1/n})$ is monotone:
   $$c^{1/n} < c^{1/(n+1)} \Leftrightarrow \frac{1}{n}\ln c < \frac{1}{n+1}\ln c \Rightarrow \frac{1}{n} > \frac{1}{n+1} \forall n \in \mathbb{N}$$
   Which is true, since $n + 1 > n \Rightarrow \frac{1}{n+1} < \frac{1}{n}$ for all natural numbers.

   The sequence is bounded:
   $$c^{1/n} < 1 \Rightarrow c < 1$$
   Which is true since $0 < c < 1$. Then, by monotone convergence theorem, our sequence converges. Let limit be L. But, the subsequence $x_{2n} = c^{1/2n} = \sqrt{c^{1/n}}$ also converges to the same limit, which means
   $$L = \sqrt{L} \Rightarrow L \in \{0, 1\}$$
   $L = 0$ is impossible, since $a^x = 0$ iff $a = 0$, but $0 < c$. Then, $L = 1$. ☐

6. * Let (f_n) be the Fibonacci sequence, and let $x_n := f_{n+1}/f_n$. Given that $\lim(x_n) = L$ exists, find L.

   Proof.
   $$x_n = f_{n+1}/f_n = (f_n + f_{n-1})/f_n = 1 + f_{n-1}/f_n \Rightarrow L = 1 + 1/L$$
   Solving the quadratic equation, we have $L = \frac{1}{2}(1 + \sqrt{5})$ ☐

7. * Let (x_n) be a bounded sequence and for each $n \in \mathbb{N}$, let $s_n := \sup\{x_k : k \geq n\}$ and $S := \inf\{s_n\}$. Show that there exists a subsequence of (x_n) that converges to S.

8. * Show that the sequence $\left(\dfrac{n+1}{n}\right)$ is a Cauchy Sequence.

Proof. Choose $M > 2/\epsilon$, then $\forall \epsilon > 0, m > n \geq M$, $\dfrac{1}{m} < \dfrac{1}{n} \leq \dfrac{1}{M} < \epsilon/2$, and,

$$\left|1 + \frac{1}{n} - 1 - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} < \epsilon/2 + \epsilon/2 = \epsilon$$

Which shows that our sequence is a cauchy sequence. □

9. * Show that if (x_n) and (y_n) are cauchy sequences, then $(x_n + y_n)$ is also a cauchy sequence.

Proof. By definition of cauchy sequence, $\forall \epsilon > 0$,

$$\exists N_1 \in \mathbb{N} \ : \ |x_m - x_n| < \epsilon/2 \quad \forall m > n \geq N_1$$

$$\exists N_2 \in \mathbb{N} \ : \ |y_m - y_n| < \epsilon/2 \quad \forall m > n \geq N_2$$

Choose $N = \max(N_1, N_2)$. Then for all $m > n \geq N, \epsilon > 0$,

$$|(x_n + y_m) - (x_m + y_m)| \leq |(x_n - x_m) + (y_n + y_m)| < |x_n - x_m| + |y_n - y_m| < \epsilon/2 + \epsilon/2 = \epsilon$$

□

Chapter 3

Basic Topology in R

3.1 Open and Closed sets

Definition 3.1.1 (Open Sets). A set is open if for $\forall a \in A$ $\exists V_\epsilon(a) \subseteq A$.

Theorem 3.1.1. The union of open sets is open.

Proof. Let $\{O_i : i \in I\}$ be collection of open sets and let $O = \bigcup_{i \in I} O_i$.

$\forall a \in O_i$, we can choose such $\epsilon > 0$ such that $V_\epsilon(a) \subseteq O_i$. But $O_i \subseteq O$, which implies $V_\epsilon(a) \subseteq O$. Since a is arbitrary, we are done. □

Theorem 3.1.2. The intersection of a finite collection of open sets is open.

Proof. Let $\{O_i\}$ be finite collection of open sets, and let $a \in \bigcap_{i \in I} O_i$. Since these sets are open, we can find $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ such that $V_{\epsilon_i}(a) \subseteq O_i$ $\forall a \in O_i$

Then, chose $\epsilon = \min\{e_i\}$, then

$$V_\epsilon(a) \subseteq V_{\epsilon_i}(a) \ \forall i \in I$$

Then it follows that the intersection will contain $V_\epsilon(a)$, hence we are done. □

Definition 3.1.2 (Limit points). A point x is a limit point of the set A if $\forall \epsilon > 0$,

$$V_\epsilon(x) \cap A$$

. Contains some point other than x.

Limit points are also called cluster points, accumulation points and so on.

Theorem 3.1.3. A point x is a limit point of the set A iff $x = \lim a_n$ for some sequence $a_n \neq x$ contained in A.

Proof. ($\Rightarrow$). x is a limint point iff

$$(V_\epsilon(x) \cap A) \backslash \{x\} \neq \{0\}$$

Then $\forall n \in \mathbb{N}$, choose $\epsilon = 1/n$, then $\exists a_n \in A$ such that

$$a_n \in (V_{1/n}(x) \cap A) \backslash \{x\}$$

Which implies $\lim a_n = x$.

($\Leftarrow$). Assume a sequence (a_n) exists in A, then $\exists N(\epsilon)$ such that $\forall n \geq N(\epsilon)$,

$$a_n \in V_\epsilon(x)$$

Then, neighborhood of x contains a element distinct from a_n,and belongs to A. □

Definition 3.1.3 (Isolated Points). A point is called isolated point of A if it is not a limit point of A.

Definition 3.1.4 (Closed sets). A set is closed if it contains its limit points.

Theorem 3.1.4. A set $A \subseteq \mathbb{R}$ is closed iff every Cauchy sequence contained in A has a limit that is also in A.

Proof. Since every cauchy sequence is convergent sequence, this theorem is equivalent to definition of closed sets. □

3.2 Closure

Definition 3.2.1 (Closure). Given a set $A \subseteq \mathbb{R}$, let L be the set of all limit points of A. Then closure of A is defined as $\overline{A} = A \cup L$.

Theorem 3.2.1. 1 For any $A \subseteq RR$, $\overline{A}$ is the smallest closed set containing A.

3.3 Completements

Definition 3.3.1 (Complement of a set). The complement of a set $A \subseteq \mathbb{R}$ is defined as follows,

$$A^c = \{x \in \mathbb{R} : x \notin A\}$$

Theorem 3.3.1. A set $A \subseteq \mathbb{R}$ is open iff A^c is closed. Consequently, B is closed iff B^c is open.

Proof. Exercise. □

Theorem 3.3.2. Intersection of infinitely many closed sets and union of finitely many closed sets is closed

Proof. Using the De Morgan's laws to the similar theorem of open sets, it is a direct consequence. □

3.4 Compact sets

Definition 3.4.1 (Compact sets). A set $A \subseteq \mathbb{R}$ is compact if every sequence in A has a subsequence that also converges in A.

Theorem 3.4.1. A set $A \subseteq \mathbb{R}$ is Compact iff it is bounded and closed.

Proof. Assume A is unbounded. Then, we can choose a sequence that is also unbounded. Then their subsequence is also unbounded. But sequences have to be bounded to converge, hence contradiction

To prove A is closed, we use the fact that subsequence converge to the same value as its sequence. But the definition requires to the value to be in A, hence A is closed. $\square$

3.5 Perfect Sets

Definition 3.5.1 (Perfect sets). A set is called perfect if it is closed and contains no isolated points.

Theorem 3.5.1. A nonempty perfect set is uncountable.

3.6 Connected sets

Definition 3.6.1. Two non-empty sets $A, B \subseteq \mathbb{R}$ are separated if $\overline{A} \cap B$ and $A \cap \overline{B}$ are both empty. A set $C = A \cup B$ is then called connected.

3.7 Exercises

1. Let A be a non-empty and bounded above, so that $s = \sup A$ exists. Show that $s \in \overline{A}$.

   Proof. By definition of supremum, $\exists a \in A$ such that

   $$a \in V_\epsilon(s)$$

   Which means, neighborhood of s contains atleast one element, hence s is a limit point. By definition of closure, $a \in \overline{A}$. $\square$

2. Given that $A \subseteq RR$, let L be set of all limit points of A. Show that L is closed.

   Proof. Let x_0 be a limit point of L. Then by definition, $\forall \epsilon > 0, \exists x \in L$ such that

   $$|x_0 - x| < \epsilon/2$$

   However, x is also a limit point of A, then $\exists x' \in A$ such that

   $$|x - x'| < \epsilon/2$$

   But notice that

   $$|x_0 - x'| \leq |x_0 - x| + |x - x'| < \epsilon/2 + \epsilon/2 = \epsilon$$

   Which means that x_0 is limit point of A, then $x_0 \in L$ for all limit points, hence we are finished. $\square$

3. Show that if $A \subseteq$ is compact and non-empty, then $\sup A$ and $\inf A$ both exists and are elements of A.

   Proof. Since A is bounded, by axiom of completeness, their supremum and infimum exists. Let $s = \sup A$. Then by definition of supremum, $\exists x \in A$ such that

   $$x \in V_\epsilon(s)$$

   Which means s is a limit point. However, since A is also closed, $s \in A$. Similarly infimum can be shown. $\square$

4. Open cover definition is equivalent to closed and bounded definition

Proof. We will show that K is bounded. Choose a set $O_x = V_c(x), \forall x \in K$ for some $c \in \mathbb{R}$. By axiom of completeness,

$$\sup V_c(x), \inf V_c(x)$$

exists. Moreover, by our assumption,

$$\{O_x : x \in K\}$$

is finite. Then,

$$\sup\{O_x : x \in K\} = \max\{\sup V_c(x) : x \in K\}$$
$$\inf\{O_x : x \in K\} = \min\{\inf V_c(x) : x \in K\}$$

Therefore the set is bounded.

Now we will show that K is closed. Let (y_n) be cauchy sequence such that $\forall n \in \mathbb{N}, y_n \in K$. We want to show that $\lim y_n = y \in K$. For the sake of contradiction, assume $y \notin K$. Choose sets $O_x = V_{\frac{|y-x|}{2}}(x), x \in K$. Then

$$\{V_{\frac{|y-x|}{2}}(x) : x \in K\} \text{ is finite}$$

Choose $\epsilon_0 = \min\{\frac{|y-x|}{2}\}$ However, by definition of convergence,

$$|y - y_n| < \epsilon_0 \ \forall n \geq N$$

But it contradicts the fact that

$$|y - x| \leq |y - y_n| + |y_n - x| < \epsilon_0 + \frac{|y-x|}{2} \leq |y - x|$$

Hence contradiction, which means O_x actually does not cover our set.

$\square$

Chapter 4

Limit and Continuity

4.1 Limits of functions

Definition 4.1.1 (limit). Let $A \subset \mathbb{R}$ and c be a cluster point of A. Then, for any function $f : A \to \mathbb{R}$, $L \in \mathbb{R}$ is said to be a **limit of f at c.**, if $\forall \epsilon > 0$, $\exists \delta > 0$ such that if $x \in A$,

$$0 < |x - c| < \delta \quad \to \quad |f(x) - L| < \epsilon$$

Theorem 4.1.1 (Uniqueness of limit). Limit of $f : A \to \mathbb{R}$ to c cluster point of A is unique for c.

Proof. Assume otherwise, then two limits L, L' such that $\forall \epsilon >$, $\exists \delta > 0$ that $|x - c| < \delta$ implies

$$|f(x) - L| < \epsilon/2$$

$$|f(x) - L'| < \epsilon/2$$

Then adding them up, we have

$$|L - L'| \leq |f(x) - L| + |f(x) - L'| < \epsilon$$

Which gives contradiction. □

Theorem 4.1.2 (Sequential criterion of functional limits). For a function $f : A \to \mathbb{R}$ and its limit point c and cauchy sequence $(x_n) \subseteq A$ such that $x_n \to c$,

$$\lim_{x \to c} f(x) = L \Leftrightarrow f(x_n) \to L$$

Proof. ($\Rightarrow$) Assume that $\lim_{x \to c} f(x) = L$. Then, $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in A)$

$$|x - c| < \delta \Rightarrow |f(x) - L| < \epsilon$$

$\forall (x_n)$ cauchy sequence, we know that $(\forall \delta > 0)(\exists N \in \mathbb{N})(\forall n \geq N)$

$$|x_n - c| < \delta$$

But this implies that

$$|x_n - c| < \delta \Rightarrow |f(x_n) - L| < \epsilon$$

($\Leftarrow$) For the sake of contradiction, assume $\lim_{x \to c} f(x) \neq L$, then by definition, $(\exists \epsilon > 0)(\forall \delta > 0)(\forall x \in A)$,

$$x \in \mathbb{V}_\delta(c) \Rightarrow f(x) \notin V_\epsilon(L)$$

Let this ϵ be notated as ϵ_0. We construct a sequence (x_n) such that for $\delta_n = \frac{1}{n}$,

$$x_n \in \mathbb{V}_{\delta_n}$$

Then, clearly $(x_n) \to c$ since $(\delta_n) \to 0$. However,

$$x_n \in \mathbb{V}_{\delta_n} \Rightarrow f(x_n) \notin V_\epsilon(L)$$

So, $f(x_n)$ does not converge to L, but we assumed $f(x_n) \to L$, hence contradiction. □

Theorem 4.1.3 (Algebra operations on limit). Let $A \subset \mathbb{R}$, and let $f, g : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A, and let $b \in \mathbb{R}$.
Then, similar to sequences, if $\lim_{x\to c} f = L$ and $\lim_{x\to c} g = M$.

1. $\lim_{x\to c}(f + g) = L + M$
2. $\lim_{x\to c}(f - g) = L - M$
3. $\lim_{x\to c}(fg) = LM$
4. $\lim_{x\to c}(bf) = bL$
5. $\lim_{x\to c}(f/g) = L/M$ if $g(x) \neq 0 \forall x \in A$ and $M \neq 0$.

Proof. All of these statements can be proven by translating them into sequence equivalent expressions. Since we already know the algebra operations on sequence limits, we are done. □

Definition 4.1.2 (Divergence for functional limits). Let $f : A \to \mathbb{R}$ and c be a limit point of f. If there exists (x_n) and (y_n) such that

$$\lim x_n = \lim y_n = c \wedge \lim f(x_n) \neq f(y_n)$$

We say that the functional limit at c does not exist.

4.2 Continous Functions

Definition 4.2.1 (Continous functions). A function $f : A \to \mathbb{R}$ is **continous** at a limit point $c \in A$ if $\forall \epsilon > 0, \exists \delta > 0$ such that

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$$

Other equivalent definitions are,

$$\lim_{x\to c} f(x) = f(c)$$

$$x \in V_\delta(c) \Rightarrow f(x) \in V_\epsilon(f(c))$$

$$(x_n) \to c \Rightarrow f(x_n) \to f(c)$$

Theorem 4.2.1 (Algebra of continous functions)**.** Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ be continous functions at $c \in A$. Then,

1. $kf(x) \forall k \in \mathbb{R}$
2. $f(x) + g(x)$
3. $f(x)g(x)$

4. $f(x)/g(x), g(x) \neq 0$

Are all continous at c.

Proof. Direct consequence of algebra of functiona limits. ☐

Theorem 4.2.2 (Composition of continous Functions). Given functions f, g where $g(f(x))$ is well defined, if f is continous and g is continous at $c, f(c)$ respectively, then $g(f(x))$ is continous at c.

Proof. We use sequences to prove this theorem, let $(x_n) \to c$ be a cauchy sequence, then by continouty definition,

$$f(x_n) \to f(c) \Rightarrow g(f(x_n)) \to g(f(c))$$

since g is continous at $f(c)$. Hence we are done. ☐

Theorem 4.2.3 (Preservation of Compact sets*). Let $f : A \to \mathbb{R}$ be continous function on A. If $K \subset A$ is compact, then $f(K)$ is compact.

Proof. Let arbitrary $(y_n) \subseteq f(K)$. Then, $\exists (x_n) \subseteq K$ such that $f(x_n) = y_n$. Since K is compact,

$$\left(\exists (x_{n_k}) \subseteq (x_n)\right) \; x_{n_k} \to x \in K$$

☐

We also have a subsequence of (y_n) such that $f(x_{n_k}) = y_{n_k}$. However, since f is continous,

$$y_{n_k} := f(x_{n_k}) \to f(x) \in f(K)$$

Which means that (y_n) has a subsequence that converged to a value inside $f(K)$, hence $f(K)$ is compact.

Theorem 4.2.4 (Extreme Value Theorem EVT). If $f : K \to \mathbb{R}$ is continous on a compact set K, then f has minimum and maximum values at K.

Proof. This is direct consequence of preservation of compact sets. We know that $f(K)$ itself is compact, and all compact sets have maximum and minimum. Hence we are done. ☐

Definition 4.2.2. A function $f : A \to \mathbb{R}$ is **uniformly continuous** if $(\forall \epsilon > 0)(\exists \delta > 0)$ such that $(\forall x, y \in A)$,

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

Theorem 4.2.5. A function that is continuous on a compact set K is also uniformly continous at the set K.

Proof. Assume otherwise, then $(\exists \epsilon_0 > 0)$ such that $(\forall (x_n), (y_n) \in K)$,

$$\lim |x_n - y_n| \to 0 \quad \textit{while} \quad |f(x_n) - f(y_n)| \geq \epsilon_0$$

Since K is compact, $(\exists x_{n_k}, y_{n_k})$ such that $x_{n_k} \to x, y_{n_k} \to y$. Then,

$$\lim |x_{n_k} - y_{n_k}| = \lim |x_n - y_n| \to 0$$

Which means x_{n_k}, y_{n_k} converge to a same value $c \in K$. Since f is continous,

$$(f(x_{n_k}) - f(y_{n_k})) \to 0$$

Which contradicts our assumption. □

Theorem 4.2.6 (Connectedness are preserved). Let f be continuous function at A. If $E \subset A$ is connected, then $f(E)$ is also connected.

Definition 4.2.3 (Right hand limit). For any limit point c of a set A and f in domain A, we define right hand limit as

$$(\forall \epsilon > 0)(\exists \delta > 0)(\forall x \in A) 0 < x - c < \delta \Rightarrow |f(x) - L| < \epsilon$$

We usually show this in this notation,

$$\lim_{x \to c^+} f(x) = L$$

Similarly we define for left hand limit

Theorem 4.2.7. For a function f in domain A, and a limit point $c \in A$, limit of f in c exists iff right hand and left hand limits are equivalent.

Theorem 4.2.8 (Intermediate Value Theorem). If $f : [a, b] \to \mathbb{R}$ is continuous and $L \in \mathbb{R}$ lies in $(f(a), f(b))$ interval, then $\exists c\ in(a, b)$ such that $f(c) = L$.

4.3 Exercises

1. Let $f := \mathbb{R} \to \mathbb{R}$ and let $c \in \mathbb{R}$. Show that $\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to 0} f(x + c) = L$.

   Proof. By definition, $\forall \epsilon > 0$, $\exists \delta$ such that

   $$|x - c| < \delta \quad \text{means} \quad |f(x) - L| < \epsilon$$

   Choose $x := x + c$. Then we have,

   $$|x - 0| < \delta \quad \text{means} \quad |f(x + c) - L| < \epsilon$$

   □

2. Let I be an interval in $\mathbb{R}$, let $f : I \to \mathbb{R}$, and let $c \in I$. suppose $\exists K, L$ such that $|f(x) - L| \leq K|x - c|$ for $x \in I$. Show that $\lim_{x \to c} f(x) = L$.

   Proof. $\forall \epsilon > 0$, choose $\delta = \epsilon / K$, then

   $$|x - c| < \epsilon / K \quad \to \quad |f(x) - L| \leq K|x - c| < \epsilon$$

   □

3. Let $I := (0, a)$ where $a > 0$, and let $g(x) = x^2$ for $x \in I$. For any points $x, c \in I$, show that $|g(x) - c^2| \leq 2a|x - c|$. Use this inequality to prove that $\lim_{x \to c} x^2 = c^2$ for any $c \in I$.

   Proof. Since $x \in I$, $0 < x < a$. Similarly, $0 < c < a$. Then,

   $$|g(x) - c^2| = |x^2 - c^2| = |x - c||x + c| \leq 2a|x - c|$$

   From this inequality, we choose $\delta = \epsilon / 2a$. Then $\forall \epsilon > 0$,

   $$|x - c| < \epsilon / 2a \quad \to \quad |x^2 - c^2| \leq 2a|x - c| < \epsilon$$

   □

4. Show that $\lim_{x \to c} x^3 = c^3$ for any $c \in \mathbb{R}$.

Proof. If $x < c$, we can choose $\delta = \epsilon/3c$, then $\forall \epsilon > 0$,

$$|x - c| < \epsilon/3c \quad \rightarrow \quad |x^3 - c^3| = |x - c||x^2 + xc + c^2| \leq |x - c||3c^2| < \epsilon$$

$\square$

Chapter 5

Derivatives

5.1 Differentiability

Definition 5.1.1 (Differentiability). Let $f : A \to \mathbb{R}$. $(\forall c \in A)$, the derivative of f at c is defined as

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Theorem 5.1.1. If f is differentiable at a point $c \in D$,then f is continous on c.

Proof.

$$\lim_{x \to c}(f(x) - f(c)) = \lim_{x \to c} \left(\frac{f(x) - f(c)}{x - c} \right) (x - c) = \frac{\mathrm{d}f}{\mathrm{d}x} \cdot 0 = 0$$

□

Theorem 5.1.2 (Algebra Operations on derivatives). Similarly, derivatives have algebraic operations.

1. $(f + g)'(x) = f(x) + g(x)$
2. $(kf)'(x) = kf'(x)$
3. $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$
4. $(f/g)'(x) = \dfrac{g(x)f'(x) - g'(x)f(x)}{g(x)^2}$

Theorem 5.1.3 (Chain Rule).] Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$. If f is differentiable at A and g is differentiable at $f(x) \in B$, then

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x)$$