Analysis notebook

A notebook with some exercises

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Preface

About the Book

This book is more like a notebook for my personal and educational purposes. I prefer taking my notes in latex, and organize all of them in a booklike structure like this one you are reading right now. The book is open source and in public domain.

Book's source

You may find the tex source files in my github account.

References

Majority of times, two books were used to study analysis:

1. Understanding Analysis - Stephen Abbot

2. Introduction to Real Analysis - Robert G. Bartle

Furthermore, other sources such as mathexchange, wikipedia, and university lecture notes are used generally. If a specific source is used, it is usually listed in the end of the chapter.

How to use the Book

I use this for fast fact checking purposes (that is what notes are for, right?).

Contents

Chapter 1

Real Numbers

1.1 Algebraic Objects: Fields and Order properties

I already studied the algebraic topics before (Linear Algebra notes). So I will skip this section.

1.2 Absolute value, epsilon-neighborhood

Absolute value is a function $f : \mathbb{R} \to \mathbb{R}_0$ such that,

$$f(x) = x \qquad \text{if } x \geq 0$$
$$f(x) = -x \qquad \text{if } x < 0$$

Absolute value describes **Distance** between two values. It is important to think this function as distance more than some function that "makes negative values positive"

Proposition 1.2.1. $\forall x, y \in \mathbb{R}$,

1. $|x| \geq 0$
2. $|-x| = |x|$
3. $|xy| = |x||y|$
4. $|x|^2 = x^2$
5. $|x| \leq y \iff -y \leq x \leq y$
6. $-|x| \leq x \leq |x|$

Proof. Proofs are rather simple, so I will not bother writing here. □

Theorem 1.2.1 (Triangle Inequality). $\forall x, y \in \mathbb{R}$,

$$|x + y| \leq |x| + |y|$$

Proof. From the proposition we have,

$$-|x| \leq x \leq |x|$$
$$-|y| \leq y \leq |y|$$

Adding these equations we get

$$-|x| - |y| \leq x + y \leq |x| + |y| \Rightarrow |x + y| \leq |x| + |y|$$

□

Corollary 1.2.1. $\forall x, y \in \mathbb{R}$,

1. $||x| - |y|| \leq |x - y|$
2. $|x - y| \leq |x| + |y|$
3. $|\sum_{i=1}^{n} a_i| \leq \sum_{i=1}^{n} |a_i|$

Proof. These Corollaries are direct consequence of triangle inequality, with third inequality using the proof with induction. I will not provide proofs since they are kind of boring and time comsuming. □

Definition 1.2.1 (epsilon neighborhood). . The $\epsilon - neighborhood$ of a is defined as a set

$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\}$$

Which is equivalent to open interval

$$(a - \epsilon, a + \epsilon)$$

Analysis heavily uses epsilon definitions and epsilon neighborhood for rigirous proofs. Therefore this definition is an useful tool.

1.3 Axiom of Completeness, Infimum and Supremum

Definition 1.3.1. A set $A \subseteq R$ is **bounded above** if $\exists b \in R$ s.t $a \leq b$ $\forall a \in A$. The number b is the **upper bound of A**. We denote set of upper bounds of A as A^u.
Similarly, we define lower bounds and the set as A^ℓ.

Definition 1.3.2 (supremum). A upper bound a of a set S is called **supremum** if,

$$a = \min A^u$$

Mathematically we show the notation as $a = \sup S$.
In Similar fashion, we define $b = \inf S$ for lower bounds.

Axiom of Completeness (AoC). Every non-empty subsets of $\mathbb{R}$ that is bounded above have supremum. The Axiom also deduces the existence of infimum in a similar fashion.

Lemma 1.3.1 (Epsilon Definition of supremum). $s \in \mathbb{R}$ is a supremum of a set $A \subseteq \mathbb{R}$ iff

$$\forall\ \epsilon > 0 \exists a \in A | s - \epsilon < a$$

Proof sketch. The both ways of the lemma can be proven by definition of the supremum. □

We use similar lemma for infimum.

Proposition 1.3.1 (Maximum and Supremum). If maximum of $A \neq \{\emptyset\} \subseteq \mathbb{R}$ exists, then

$$\max A = \sup A$$

Proof. Denote $s = \sup A$ and $m = \max A$. By definition, $s = \min A^u$ and $m = A^u \cap A$.
The result is an immediate consequence of the definitions of maximum and supremum.

m is a proper supremum, since $\forall x \in A$ we have $x \leq m$, and since also $m \in A$, $t = \sup A < m$ is impossible. □

Similarly, we have $\min A = \inf A$.

Proposition 1.3.2 (Uniqueness of Supremum). Supremum and Infimum are unique.

Proof. For the sake of the contradiction, assume there exists two supremum s_1, s_2. Then by definition of supremum, we have

$$s_1 \geq s_2 \;\wedge\; s_2 \geq s_1 \Rightarrow s_1 = s_2$$

Infimum follows the similar proof. □

Proposition 1.3.3 (Existence of Infimum). AoC implies the existence of infimum for $A \subseteq \mathbb{R}$ such that $A^\ell \neq \emptyset$,

$$\inf A = -\sup(-A)$$

Proof. Since $A^l \neq \emptyset$, it follows that

$$\exists x \in A^\ell \mid x \leq a$$

Then,

$$-x \geq -a \Rightarrow -x \in (-A)^u \neq \emptyset$$

By AoC, $\sup(-A)$ exists. Rest is trivial.

□

Proposition 1.3.4 (Operations on Supremum). The supremum holds these properties,

$$\sup(A+B) = \sup(A) + \sup(B) \tag{1.1}$$
$$\sup(A \cdot B) = \sup(A) \cdot \sup(B) \tag{1.2}$$
$$\text{if } c \geq 0, \quad \sup(cA) = c\sup(A) \tag{1.3}$$
$$\text{if } c \leq 0, \quad \sup(cA) = c\inf(A) \tag{1.4}$$

Proof. These properties directly follow from the epsilon definition of the supremum. That is, $\forall \epsilon_a, \epsilon_b, \exists a, b \in A, B$ such that,

$$\sup(A) - a < \epsilon_a \;\wedge \sup(B) - b < \epsilon_b$$

adding these equations to each other, we have

$$\sup(A) + \sup(B) - (a+b) < \epsilon_a + \epsilon_b \tag{1.5}$$

Note that $(a+b) \in A+B$, and let $\epsilon_a + \epsilon_b = \epsilon_{a+b}$. Also we know that,

$$\forall \epsilon_c \exists c \in A+B \mid \sup(A+B) - c < \epsilon_c \tag{1.6}$$

but 1.5 and 1.6 both are valid, hence the conclusion.
We can similarly prove other propositions, even for inf. □

1.4 Applications of Completeness, Archimedean Property (A.P)

Theorem 1.4.1 (Archimedean Property, A.P). $\forall x \in \mathbb{R}\ \exists n_x \in \mathbb{N} \mid x \leq n_x$.

Proof. For the sake of contradiction, assume otherwise. Then $n \leq x\ \forall n \in \mathbb{N}$, by AoC $\mathbb{N}$ has supremum, s. Since $s-1<s$, $s-1$ is not a upper bound, therefore $\exists m \in \mathbb{N}$ such that $s-1<m \Rightarrow s<m+1$. but $m+1 \in \mathbb{N}$. Therefore s cannot be a supremum. □

Theorem 1.4.2 (Density of Rationals in $\mathbb{R}$). $\forall a, b \in \mathbb{R}, \exists r \in \mathbb{Q}$ such that

$$a<r<b$$

Proof. Since r must be rational, we want to find $m, n \in \mathbb{Z}$ such that $\frac{m}{n}=r$.
From Archimedean property,

$$\exists n \in \mathbb{N}: n(y-x) \geq 1$$

Again from Archimedean property,

$$\forall t \in \mathbb{R}, \exists m \in \mathbb{Z}: m-1 \leq t \leq m$$

In other words, for any real numbers, there are two consecutive integers that lies in the each boundary of the real numbers.
Let $t=nx$. Combining the inequalities, we get

$$nx \leq m \leq 1+nx \leq ny \Rightarrow x \leq \frac{m}{n} \leq y$$

□

Theorem 1.4.3 (Density of Irrationals in $\mathbb{R}$). $\forall x, y \in \mathbb{R}$ such that $x<y$, $\exists z \in \mathbb{I}$ such that

$$x<z<y$$

Proof. It is direct consequence of density of Rationals. We apply density theorem on $\frac{x}{\sqrt{2}}$ and $\frac{y}{\sqrt{2}}$, which we will get $z=r\sqrt{2}, r \in \mathbb{Q}$, hence we are done. □

1.5 Intervals

Theorem 1.5.1 (Closed and Open Intervals). If $a, b \in \mathbb{R}$ and $a<b$, then **open interval** is defined by,

$$(a, b)=\{x \in \mathbb{R} | a<x<b\}$$

Similarly, we define **closed interval** as,

$$[a, b]=\{x \in \mathbb{R} | a \leq x \leq b\}$$

Theorem 1.5.2 (Nested Intervals). The sequence of intervals $I_n, n \in \mathbb{N}$ is nested if

$$I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n \subseteq \ldots$$

Theorem 1.5.3 (Nested Interval Property). For nested intervals $\{I_n\}=[a_n, b_n], n \in \mathbb{N}$, the below is true

$$\bigcap_{i=1}^{\infty} I_n \neq \emptyset$$

Proof. Since intervals are nested intervals, $b_1 \geq a_n \forall n \in \mathbb{N}$. Hence by AoC supremum s of $\{a_n\}$ exists.
We know that $a_n \leq s$. But since b_n is also a upper bound bigger than s, we have $a_n \leq s \leq b_n$, which means $s \in \bigcap_{i=1}^{\infty} I_n$ □

Remark: Intervals must be closed. Consider $A_n=(0, \frac{1}{n})$. Any element of intersection must be bigger than 0, while smaller than $\frac{1}{n}$. By Archimedian property of real numbers, this is a contradiction, hence $\bigcap_{n=1}^{\infty} A_n = \emptyset$

1.6 Cardinality

Definition 1.6.1 (Cardinality). The sets A, B have the same **cardinality** if there exists a bijective function such that $f : A \to B$. We donate cardinal equality with $A \sim B$.
Cardinality mathematically describes the size of the set.
The $\sim$ operation is an equivalence relation.

Definition 1.6.2 (Countable Sets). The set A is said to be **countable** if $A \sim \mathbb{N}$. Otherwise the set is called **uncountable sets**.

Theorem 1.6.1 (Countability of $\mathbb{Q}$.). The set $\mathbb{Q}$ is countable, that is, $\mathbb{Q} \sim \mathbb{N}$.

Proof. There is a proof with visual construction, which maps the rational numbers to natural numbers. □

Theorem 1.6.2 (Uncountability of $\mathbb{R}$). The set $\mathbb{R}$ is uncountable.

Proof. Assume otherwise. Then subset $[0, 1] \subseteq \mathbb{R}$ must be also countable □

Definition 1.6.3 (Power set). The powerset $\mathcal{P}(A)$, is the set of all subsets of A.

Theorem 1.6.3. Every infinite subset of a countable set is a countable set.

Theorem 1.6.4. Let $\{A_n\}, n = 1, 2, 3, \ldots$ be sequence of countable sets. Then,

$$S = \bigcup_{n=1}^{\infty} A_n$$

Proof. Diagonalization method (graphical) □

1.7 Exercises

1. * Show that for $A = \{1 - \frac{1}{n} \ : \ n \in \mathbb{N}\}, \sup A = 1$.

   Proof. A is bounded above since clearly $\forall a \in A, a < 1$. Then by AoC, supremum exists. Let $u = \sup A$. We will show that $u = 1$.
   Clearly, 1 is a upper bound, since $1 > 1 - \frac{1}{n}$ is trivial.
   if $u < 1$, we will show that there exists some $a \in A$ such that $u < a$.

   $$\forall \epsilon > 0, \ \exists a \in A \mid 1 - \epsilon < a = 1 - \frac{1}{n} \Rightarrow \epsilon > \frac{1}{n}$$

   But, by Archimedean, $\exists n_0 \in \mathbb{N}$ contradicting,

   $$u - \epsilon < 1 - \frac{1}{n} \in A$$

   Therefore $u = 1$. □

2. If $S = \{1/n - 1/m : n, m \in \mathbb{N}\}$, find $\inf S$ and $\sup S$.

   Proof. Clearly, S is bounded above and below, therefore supremum and infimum exists by AoC. We will show that $\sup S = 1$, and we can find $\inf S = -\sup(-S) = -1$. Clearly 1 is an upper bound. By definition of supremum

$$\exists \epsilon > 0,\ \forall s \in S \mid 1 - \epsilon < s = 1/n - 1/m \Rightarrow 1 - \epsilon < 1 - \frac{1}{m}$$

   Which is equivalent to showing $\exists m \in \mathbb{N} \mid \epsilon > \frac{1}{m}$, which is evident from Archimedean. □

3. * Let S be a set of nonnegative real numbers that is bounded above and let $T = \{x^2 : x \in S\}$. Prove that if $u = \sup S$, then $u^2 = \sup T$.

   Proof. Since S is bounded above, T is also bounded above. By AoC, supremum of T exists. Let $t = \sup T$. Clearly, u^2 is upper bound of T, that is,

$$s \in S \mid s^2 \leq u^2 \Rightarrow y = s^2 \in T \mid y \leq u^2$$

   Now, we will show that u^2 the least upper bound, that is,

$$\forall \epsilon > 0\ \exists s \in S \mid u^2 - s^2 < \epsilon \Longrightarrow (u - s)(u + s) < \epsilon$$

   Since $u = \sup S$, we have

$$u - s < \epsilon_0\ \epsilon_0 > 0$$

   Moreover, $u + s \leq 2u$. Combining these inequalities, we have

$$(u - s)(u + s) < 2u\epsilon_0$$

   Then we just choose some $\epsilon > 2u\epsilon_0$. □

   Second proof.

$$a = \sup A \Rightarrow a^2 = \sup A \cdot \sup A = \sup A^2 = \sup T$$

   □

4. Given any $x \in \mathbb{R}$, show that there exists a unique $n \in \mathbb{Z}$ such that $x \leq x < n + 1$.

   Proof. By definition of floor function, we have

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$$

   Clearly, $n - \lfloor x \rfloor$ satisfies our property. Assume two $m, n \in \mathbb{Z}$ exists. WLOG $n > m$. Then,

$$m < n \Rightarrow m + 1 \leq n \Longrightarrow m + 1 \leq n \leq x < m + 1 < n + 1$$

   Clearly, $m + 1 < m + 1$ is a contradiction. □

5. * Show that there exists $y \in \mathbb{R}$ such that $y^2 = 3$.

   Proof. Let $S = \{s \in \mathbb{R} : 0 \leq s, s^2 < 3\}$. Clearly, S is bounded, by AoC, $\sup S = u$ exists. We will show that $u^2 = 3$.

   Clearly $u^2 = 3$ is an upper bound.

   If $u^2 < 3$, we will show that $\exists n \in \mathbb{N} : u + \frac{1}{n} \in S$

$$\left(u + \frac{1}{n}\right)^2 < 3 \Rightarrow u^2 + \frac{1}{n^2} + \frac{2u}{n} \leq u^2 + \frac{1}{n}(2u + \frac{1}{n}) \Longrightarrow \frac{1}{n} < \frac{3 - u^2}{2u + 1}$$

   By Archimedean, such n exists satisfying our last inequality, hence contradiction.

   □

6. Let $I_n = [0, 1/n]$ for $n \in \mathbb{N}$. Prove that $\cap_{n=1}^{\infty} I_n = \{0\}$.

   Proof. For all $n \in \mathbb{N}$, clearly $0 \in I_n$. For any $x > 0$, by Archimedean there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < x$, hence conclusion. □

1.8 Notes and Mistakes on Exercises

1. Avoid "intuitive" proofs, prove every part of the proof rigorously. For example, the last exercise section, question 1, I also should prove $1 > 1 - \frac{1}{n}$ regardless of trivality.

2. The "steps" in the proofs usually should be **reversed**. In a scratch paper, for example, find and construct an epsion/ natural number(?) and write it formally in the proof.

3. Using floor function is wrong in the last exercise. A.P should be used.

1.9 References

1. https://math.colorado.edu/~nita/12_Axiom_of_Completeness.pdf

Chapter 2

Sequences and Series

2.1 Sequences and limits

Definition 2.1.1 (Sequences). A sequence is a function with its domain as $\mathbb{N}$.

Definition 2.1.2 (Converge). . A sequence (x_n) is said to conerge to $x \in \mathbb{R}$, or x is said to be limit of (x_n) if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \ : \ |x_n - x| < \epsilon, \ \forall n \geq N$$

If limit exists, sequence is **convergent**, otherwise it is **divergent**.

Definition 2.1.3 (Epsilon Neighborhood definition of convergence). . Below definition with neighborhood is equivalent to the definition above

$$\forall \epsilon > 0, \exists N \in \mathbb{N} \ : \ x_n \in V_\epsilon(x), \ \forall n > N$$

Theorem 2.1.1 (Uniqueness of Limits). The limit of a sequence is **unique**.

Proof. For the sake of the contradiction, let $x = x^{'} = \lim_{n\to\infty}(x_n)$. with the definiton of the limit, $\forall \epsilon > 0, \exists n \in \mathbb{N}$ such that for all $n \geq N, N^{'}$,

$$|x - x_n| < \epsilon/2 \ \forall n \geq N$$

$$|x^{'} - x_n| < \epsilon/2 \ \forall n \geq N^{'}$$

However, by the triangle inequality, we have

$$|x - x^{'}| \leq |x - x_n| + |x^{'} - x_n| < \epsilon/2 + \epsilon/2 = \epsilon, \ \forall n \geq K = \max(N, N^{''})$$

Since this is $\forall \epsilon > 0$, we conclude that $x = x^{''}$. $\square$

2.2 Limit Theorems

Definition 2.2.1. A sequence (x_n) is **bounded** if there exists $U > 0$ such that

$$|x_n| \leq U \ \forall n \in \mathbb{N}$$

A sequence is bounded **iff** the set $\{x_n \ : \ n \in \mathbb{N}\}$ is bounded.

Theorem 2.2.1. A convergent sequence is bounded.

Proof. If a sequence converges, then all but finite number of terms of the sequence belongs to $V_\epsilon(x)$. Since $V_{\epsilon/2}(x)$ is bounded, the sequence itself is bounded. □

Theorem 2.2.2 (Algebra of limits). let $X = (x_n), Y = (y_n)$ converge to x, y respectively. Then sequences $X + Y, X - Y, X \cdot Y, cX$ converge to $x + y, x - y, xy, cx$ respectively.
If $y \neq 0$, X/Y converges to x/y.

Proof. We will show that $X+Y$ property only, others are similar. By definition of convergence, $\forall \epsilon > 0, \exists N, N^{'} \in \mathbb{N}$ such that

$$|x - x_n| < \epsilon/2, \forall n \geq N$$

$$|y - y_n| < \epsilon/2, \forall n \geq N^{'}$$

□

However, notice that $\forall n \geq \max N, N^{'}$

$$|(x + y) - (x_n + y_n)| = |(x - x_n) + (y - y_n)| \leq |x - x_n| + |y - y_n| < \epsilon/2 + \epsilon/2 = \epsilon$$

Which proves our theorem

Theorem 2.2.3. If (x_n) is convergent sequence and $x_n \geq 0$ for all $n \in \mathbb{N}$, then $x = \lim(x_n) \geq 0$.

Theorem 2.2.4. if $(x_n), (y_n)$ are convergent sequences and $x_n \leq y_n$ for all $n \in \mathbb{N}$, then $x \leq y$.

Theorem 2.2.5. If (x_n) is a convergent sequence and $a \leq x_n \leq b$ for all $n \in \mathbb{N}$, then $a \leq x \leq b$.

Theorem 2.2.6 (Squeeze theorem). Let $(x_n), (y_n), (z_n)$ be sequences such that

$$x_n \leq y_n \leq z_n$$

And $x = z$. Then (y_n) converges and

$$x = y = z$$

All above theorems are proven similarly, the idea is the same.

2.3 Monotone Sequences

Definition 2.3.1. (x_n) is **monotone** if it is either increasing or decreasing.

Theorem 2.3.1 (Monotone Convergence Theorem). A monotone sequence is convergent iff it is bunded. Furthermore, if x_n is a bounded increasing sequence, then

$$\lim(x_n) = \sup\{x_n : n \in \mathbb{N}\}$$

Similarly, if y_n is a bounded decreasing sequence, then

$$\lim(y_n) = \inf\{y_n : n \in \mathbb{N}\}$$

2.4 Subsequences

Definition 2.4.1 (Subsequences). Let $\{n_k\}$ be strict monotone increasing sequence of real numbers, then the sequence $X^{'} = (x_{n_k})$ is called **subsequence**

Theorem 2.4.1. If a sequence (x_n) converge to x, then the subsequence (x_{n_k}) also converge to x.

Proof. By definition, $\forall \epsilon > 0,\ \exists N(\epsilon) \in \mathbb{N}$ such that $\forall n \geq N(\epsilon)$,

$$|x_n - x| < \epsilon$$

Because $n_k \geq k$ (induction), then we can find such $k \geq N(\epsilon)$, then $n_k \geq N(\epsilon)$, which means

$$|x_{n_K} - x| < \epsilon$$

□

Theorem 2.4.2 (Monotone subsequence theorem). If (x_n) is a sequence, then there exists a monotone subsequence.

Theorem 2.4.3 (The Bolzano-Weierstrass Theorem). A bounded sequence has a convergent subsequence.

Proof. It is direct consequence of monotone subsequence theorem. Since we can find a monotone subsequence, and is bounded, we can conclude it is convergent. □

2.5 The Cauchy Criterion

Definition 2.5.1 (Cauchy Sequence). A sequence (x_n) is said to be a **Cauchy sequence** if $\forall \epsilon > 0$, $\exists N(\epsilon) \in \mathbb{N}$,

$$|x_n - x_m| < \epsilon \quad \forall m > n > N$$

Theorem 2.5.1. A sequence is convergent if and only if it is a cauchy sequence

Proof. □

2.6 Exercises

1. Show that sequence of (2^n) does not converge.

   Proof. It suffices to prove that (2^n) is unbounded. Assume otherwise that there exists $M \in \mathbb{R}$ such that $2^n \leq M$ for all $n \in \mathbb{N}$. Then,

   $$n \leq \log_2(M) = c$$

   However by the unboundness of $\mathbb{N}$, we can find n_0 such that $n_0 > c$ for any $c \in \mathbb{R}$, contradicting our claim. □

2. * Show that $z_n = (a^n + b^n)^{1/n}$ where $0 < a < b$ converge to b.

   Proof. Since $a > 0$, we have
$$(a^n + b^n)^{1/n} > (b^n)^{1/n} = b$$
   Since $a < b$, we have
$$(a^n + b^n)^{1/n} < (2b^n)^{1/n} = 2^{1/n}b$$
   Then,
$$b \leq z_n \leq 2^{1/n}b$$
   Using the squeeze theorem and the fact that $2^{1/n}$ converges to 1, we can see that $\lim z_n = b$. □

3. * Let $x_1 = 8$ and let $x_{n+1} = \frac{1}{2}x_n + 2$ for $n \in \mathbb{N}$. Show that x_n converges, and find the limit.

   Proof. We will show that (x_n) is monotone and bounded.
   1) $x_n \geq 4$ for all $n \in \mathbb{N}$.
   By induction, for $n = 1, 2$ we have $8 > 4$ and $6 > 4$. Now assume it is true for $n = k$. Then,
$$x_{k+1} = \frac{1}{2}x_k + 2 > 4$$
   2) $x_{n+1} < x_n$ for all $n \in \mathbb{N}$. By induction, for $n = 1, 2$ we have $6 < 8$. Now assume it is true for $n = k$. Then,
$$x_{k+1} = \frac{1}{2}x_k + 2 < \frac{1}{2}x_{k-1} + 2 = x_k$$
   Then sequence is monotone and bounded, therefore it is convergent to the $\inf\{x_n : n \in \mathbb{N}\} = 4$, which we already know how to prove. □

4. Prove that $e_n = \left(1 + \frac{1}{n}\right)^{1/n}$ is convergent.

   Proof. Direct consequence of monotone convergence theorem. □

5. * Prove that $\lim(c^{1/n}) = 1$ for $0 < c < 1$.

   Proof. The sequence $(c^{1/n})$ is monotone:
$$c^{1/n} < c^{1/(n+1)} \Leftrightarrow \frac{1}{n}\ln c < \frac{1}{n+1}\ln c \Rightarrow \frac{1}{n} > \frac{1}{n+1} \forall n \in \mathbb{N}$$
   Which is true, since $n + 1 > n \Rightarrow \frac{1}{n+1} < \frac{1}{n}$ for all natural numbers.
   The sequence is bounded:
$$c^{1/n} < 1 \Rightarrow c < 1$$
   Which is true since $0 < c < 1$. Then, by monotone convergence theorem, our sequence converges. Let limit be L. But, the subsequence $x_{2n} = c^{1/2n} = \sqrt{c^{1/n}}$ also converges to the same limit, which means
$$L = \sqrt{L} \Rightarrow L \in \{0, 1\}$$
   $L = 0$ is impossible, since $a^x = 0$ iff $a = 0$, but $0 < c$. Then, $L = 1$. □

6. * Let (f_n) be the Fibonacci sequence, and let $x_n := f_{n+1}/f_n$. Given that $\lim(x_n) = L$ exists, find L.

   Proof.
$$x_n = f_{n+1}/f_n = (f_n + f_{n-1})/f_n = 1 + f_{n-1}/f_n \Rightarrow L = 1 + 1/L$$
   Solving the quadratic equation, we have $L = \frac{1}{2}(1 + \sqrt{5})$ □

7. * Let (x_n) be a bounded sequence and for each $n \in \mathbb{N}$, let $s_n := \sup\{x_k : k \geq n\}$ and $S := \inf\{s_n\}$. Show that there exists a subsequence of (x_n) that converges to S.

8. * Show that the sequence $\left(\frac{n+1}{n}\right)$ is a Cauchy Sequence.

Proof. Choose $M > 2/\epsilon$, then $\forall \epsilon > 0, m > n \geq M$, $\frac{1}{m} < \frac{1}{n} \leq \frac{1}{M} < \epsilon/2$, and,

$$\left|1 + \frac{1}{n} - 1 - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} < \epsilon/2 + \epsilon/2 = \epsilon$$

Which shows that our sequence is a cauchy sequence. □

9. * Show that if (x_n) and (y_n) are cauchy sequences, then $(x_n + y_n)$ is also a cauchy sequence.

Proof. By definition of cauchy sequence, $\forall \epsilon > 0$,

$$\exists N_1 \in \mathbb{N} \ : \ |x_m - x_n| < \epsilon/2 \quad \forall m > n \geq N_1$$

$$\exists N_2 \in \mathbb{N} \ : \ |y_m - y_n| < \epsilon/2 \quad \forall m > n \geq N_2$$

Choose $N = \max(N_1, N_2)$. Then for all $m > n \geq N, \epsilon > 0$,

$$|(x_n + y_m) - (x_m + y_m)| \leq |(x_n - x_m) + (y_n + y_m)| < |x_n - x_m| + |y_n - y_m| < \epsilon/2 + \epsilon/2 = \epsilon$$

□

Chapter 3

Basic Topology in R

Chapter 4

Limit and Continuity

4.1 Limits of functions

Definition 4.1.1 (limit). Let $A \subset \mathbb{R}$ and c be a cluster point of A. Then, for any function $f : A \to \mathbb{R}$, $L \in \mathbb{R}$ is said to be a **limit of f at c.**, if $\forall \epsilon > 0$, $\exists \delta > 0$ such that if $x \in A$,

$$0 < |x - c| < \delta \quad \to \quad |f(x) - L| < \epsilon$$

Theorem 4.1.1 (Uniqueness of limit). Limit of $f : A \to \mathbb{R}$ to c cluster point of A is unique for c.

Proof. Assume otherwise, then two limits L, L' such that $\forall \epsilon >, \exists \delta > 0$ that $|x - c| < \delta$ implies

$$|f(x) - L| < \epsilon/2$$

$$|f(x) - L'| < \epsilon/2$$

Then adding them up, we have

$$|L - L'| \leq |f(x) - L| + |f(x) - L'| < \epsilon$$

Which gives contradiction. □

Theorem 4.1.2 (Algebra operations on limit). Let $A \subset \mathbb{R}$, and let $f, g : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A, and let $b \in \mathbb{R}$.

Then, similar to sequences, if $\lim_{x \to c} f = L$ and $\lim_{x \to c} g = M$.

1. $\lim_{x \to c}(f + g) = L + M$
2. $\lim_{x \to c}(f - g) = L - M$
3. $\lim_{x \to c}(fg) = LM$
4. $\lim_{x \to c}(bf) = bL$
5. $\lim_{x \to c}(f/g) = L/M$ if $g(x) \neq 0 \forall x \in A$ and $M \neq 0$.

4.2 Exercises

1. Let $f := \mathbb{R} \to \mathbb{R}$ and let $c \in \mathbb{R}$. Show that $\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to 0} f(x + c) = L$.

   Proof. By definition, $\forall \epsilon > 0$, $\exists \delta$ such that

$$|x - c| < \delta \quad \text{means} \quad |f(x) - L| < \epsilon$$

Choose $x \coloneqq x + c$. Then we have,

$$|x - 0| < \delta \quad \text{means} \quad |f(x + c) - L| < \epsilon$$

$\square$

2. Let I be an interval in $\mathbb{R}$, let $f : I \to \mathbb{R}$, and let $c \in I$. suppose $\exists K, L$ such that $|f(x) - L| \leq K|x - c|$ for $x \in I$. Show that $\lim_{x \to c} f(x) = L$.

   Proof. $\forall \epsilon > 0$, choose $\delta = \epsilon/K$, then

   $$|x - c| < \epsilon/K \quad \to \quad |f(x) - L| \leq K|x - c| < \epsilon$$

   $\square$

3. Let $I \coloneqq (0, a)$ where $a > 0$, and let $g(x) = x^2$ for $x \in I$. For any points $x, c \in I$, show that $|g(x) - c^2| \leq 2a|x - c|$. Use this inequality to prove that $\lim_{x \to c} x^2 = c^2$ for any $c \in I$.

   Proof. Since $x \in I$, $0 < x < a$. Similarly, $0 < c < a$. Then,

   $$|g(x) - c^2| = |x^2 - c^2| = |x - c||x + c| \leq 2a|x - c|$$

   From this inequality, we choose $\delta = \epsilon/2a$. Then $\forall \epsilon > 0$,

   $$|x - c| < \epsilon/2a \quad \to \quad |x^2 - c^2| \leq 2a|x - c| < \epsilon$$

   $\square$

4. Show that $\lim_{x \to c} x^3 = c^3$ for any $c \in \mathbb{R}$.

   Proof. If $x < c$, we can choose $\delta = \epsilon/3c$, then $\forall \epsilon > 0$,

   $$|x - c| < \epsilon/3c \quad \to \quad |x^3 - c^3| = |x - c||x^2 + xc + c^2| \leq |x - c||3c^2| < \epsilon$$

   $\square$