Analysis notebook

A notebook with some exercises

Joseph Mehdiyev

If you want to talk whatever reason you wish, please contact via my e-mail: yusifmehdiyev55@gmail.com

Preface

About the Book

Book's source

How to use the Book

and other data science purposes, I used python library **pytorch** and related libraries. I use Neovim/Vim for Python.

Contents

Chapter 1

Real Numbers

1.1 Axiom of Completeness, Infimum and Supremum

Definition 1.1.1. A set $A \subseteq R$ is **bounded above** if $\exists b \in R$ s.t $a \leq b \; \forall a \in A$. The number b is the **upper bound of A**. We denote set of upper bounds of A as A^u.
Similarly, we define lower bounds and the set as A^ℓ.

Definition 1.1.2 (supremum). A upper bound a of a set S is called **supremum** if,

$$a = \min A^u$$

Mathematically we show the notation as $a = \sup S$.
In Similar fashion, we define $b = \inf S$ for lower bounds.

Axiom of Completeness (AoC). Every non-empty subsets of $\mathbb{R}$ that is bounded above have supremum. The Axiom also deduces the existence of infimum in a similar fashion.
Questions: Can infinity be supremum? Does this axiom imply existence of infimum.

Lemma 1.1.1 (Epsilon Definition of supremum). $s \in \mathbb{R}$ is a supremum of a set $A \subseteq \mathbb{R}$ iff

$$\forall \; \epsilon > 0 \; \exists \; a \in A \mid s - \epsilon < a$$

Proof sketch. The both ways of the lemma can be proven by definition of the supremum. That is, lemma 1.1.1 and definition 1.1.2 are equivalent. □

We use similar lemma for infimum.

Proposition 1.1.1 (Maximum and Supremum). If maximum of $A \neq \{\emptyset\} \subseteq \mathbb{R}$ exists, then

$$\max A = \sup A$$

Proof. Denote $s = \sup A$ and $m = \max A$. By definition, $s = \min A^u$ and $m = A^u \cap A$.
The result is an immediate consequence of the definitions of maximum and supremum.

m is a proper supremum, since $\forall x \in A$ we have $x \leq m$, and since also $m \in A$, $t = \sup A < m$ is impossible. □

Similarly, we have $\min A = \inf A$.

Proposition 1.1.2 (Uniqueness of Supremum). Supremum and Infimum are unique.

Proof. For the sake of the contradiction, assume there exists two supremum s_1, s_2. Then by definition of supremum, we have

$$s_1 \geq s_2 \ \wedge \ s_2 \geq s_1 \Rightarrow s_1 = s_2$$

Infimum follows the similar proof. □

Proposition 1.1.3 (**Existence of Infimum**)**.** AoC implies the existence of infimum for $A \subseteq \mathbb{R}$ such that $A^\ell \neq \emptyset$,

$$\inf A = -\sup(-A)$$

Proof. Since $A^l \neq \emptyset$, it follows that

$$\exists x \in A^\ell \mid x \leq a$$

Then,

$$-x \geq -a \Rightarrow -x \in (-A)^u \neq \emptyset$$

By AoC, $\sup(-A)$ exists. Rest is trivial.

□

Proposition 1.1.4 (**Operations on Supremum**)**.** The supremum holds these properties,

$$\sup(A+B) = \sup(A) + \sup(B) \tag{1.1}$$
$$\sup(A \cdot B) = \sup(A) \cdot \sup(B) \tag{1.2}$$
$$\text{if } c \geq 0, \quad \sup(cA) = c\sup(A) \tag{1.3}$$
$$\text{if } c \leq 0, \quad \sup(cA) = c\inf(A) \tag{1.4}$$

Proof. These properties directly follow from the epsilon definition of the supremum. That is, $\forall \epsilon_a, \epsilon_b, \exists a, b \in A, B$ such that,

$$\sup(A) - a < \epsilon_a \ \wedge \sup(B) - b < \epsilon_b$$

adding these equations to each other, we have

$$\sup(A) + \sup(B) - (a+b) < \epsilon_a + \epsilon_b \tag{1.5}$$

Note that $(a+b) \in A+B$, and let $\epsilon_a + \epsilon_b = \epsilon_{a+b}$. Also we know that,

$$\forall \epsilon_c \exists c \in A+B \mid \sup(A+B) - c < \epsilon_c \tag{1.6}$$

but 1.5 and 1.6 both are valid, hence the conclusion.
We can similarly prove other propositions, even for inf. □

References

1. https://math.colorado.edu/~nita/12_Axiom_of_Completeness.pdf