

Analysis notebook

A notebook with some exercises

Joseph Mehdiyev

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Chapter 1

Real Numbers

1.1 Algebraic Objects: Fields and Order properties

I already studied the algebraic topics before (Linear Algebra notes). So I will skip this section.

1.2 Absolute value, epsilon-neighborhood

Absolute value is a function $f : \mathbb{R} \rightarrow \mathbb{R}_0$ such that,

$$\begin{aligned} f(x) &= x & \text{if } x \geq 0 \\ f(x) &= -x & \text{if } x < 0 \end{aligned}$$

Absolute value describes **Distance** between two values. It is important to think this function as distance more than some function that “makes negative values positive”

Proposition 1.2.1. $\forall x, y \in \mathbb{R}$,

1. $|x| \geq 0$
2. $|-x| = |x|$
3. $|xy| = |x||y|$
4. $|x|^2 = x^2$
5. $|x| \leq y \iff -y \leq x \leq y$
6. $-|x| \leq x \leq |x|$

Proof. Proofs are rather simple, so I will not bother writing here. □

Theorem 1.2.1 (Triangle Inequality). $\forall x, y \in \mathbb{R}$,

$$|x + y| \leq |x| + |y|$$

Proof. From the proposition we have,

$$\begin{aligned} -|x| &\leq x \leq |x| \\ -|y| &\leq y \leq |y| \end{aligned}$$

Adding these equations we get

$$-|x| - |y| \leq x + y \leq |x| + |y| \Rightarrow |x + y| \leq |x| + |y|$$

□

Corollary 1.2.1. $\forall x, y \in \mathbb{R}$,

1. $||x| - |y|| \leq |x - y|$
2. $|x - y| \leq |x| + |y|$
3. $|\sum_{i=1}^n a_i| \leq \sum_{i=1}^n |a_i|$

Proof. These Corollaries are direct consequence of triangle inequality, with third inequality using the proof with induction. I will not provide proofs since they are kind of boring and time consuming. \square

Definition 1.2.1 (epsilon neighborhood). . The ϵ – neighborhood of a is defined as a set

$$V_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\}$$

Which is equivalent to open interval

$$(a - \epsilon, a + \epsilon)$$

Analysis heavily uses epsilon definitions and epsilon neighborhood for rigorous proofs. Therefore this definition is an useful tool.

1.3 Axiom of Completeness, Infimum and Supremum

Definition 1.3.1. A set $A \subseteq \mathbb{R}$ is **bounded above** if $\exists b \in \mathbb{R}$ s.t $a \leq b \forall a \in A$. The number b is the **upper bound of A**. We denote set of upper bounds of A as A^u . Similarly, we define lower bounds and the set as A^ℓ .

Definition 1.3.2 (supremum). A upper bound a of a set S is called **supremum** if,

$$a = \min A^u$$

Mathematically we show the notation as $a = \sup S$.

In Similar fashion, we define $b = \inf S$ for lower bounds.

Axiom of Completeness (AoC). Every non-empty subsets of \mathbb{R} that is bounded above have supremum. The Axiom also deduces the existence of infimum in a similar fashion.

Lemma 1.3.1 (Epsilon Definition of supremum). $s \in \mathbb{R}$ is a supremum of a set $A \subseteq \mathbb{R}$ iff

$$\forall \epsilon > 0 \exists a \in A | s - \epsilon < a$$

Proof sketch. The both ways of the lemma can be proven by definition of the supremum. \square

We use similar lemma for infimum.

Proposition 1.3.1 (Maximum and Supremum). If maximum of $A \neq \{\emptyset\} \subseteq \mathbb{R}$ exists, then

$$\max A = \sup A$$

Proof. Denote $s = \sup A$ and $m = \max A$. By definition, $s = \min A^u$ and $m = A^u \cap A$. The result is an immediate consequence of the definitions of maximum and supremum.

m is a proper supremum, since $\forall x \in A$ we have $x \leq m$, and since also $m \in A$, $t = \sup A < m$ is impossible. \square

Similarly, we have $\min A = \inf A$.

Proposition 1.3.2 (Uniqueness of Supremum). Supremum and Infimum are unique.

Proof. For the sake of the contradiction, assume there exists two supremum s_1, s_2 . Then by definition of supremum, we have

$$s_1 \geq s_2 \wedge s_2 \geq s_1 \Rightarrow s_1 = s_2$$

Infimum follows the similar proof. \square

Proposition 1.3.3 (Existence of Infimum). AoC implies the existence of infimum for $A \subseteq \mathbb{R}$ such that $A^\ell \neq \emptyset$,

$$\inf A = -\sup(-A)$$

Proof. Since $A^\ell \neq \emptyset$, it follows that

$$\exists x \in A^\ell \mid x \leq a$$

Then,

$$-x \geq -a \Rightarrow -x \in (-A)^u \neq \emptyset$$

By AoC, $\sup(-A)$ exists. Rest is trivial. \square

Proposition 1.3.4 (Operations on Supremum). The supremum holds these properties,

$$\sup(A + B) = \sup(A) + \sup(B) \quad (1.1)$$

$$\sup(A \cdot B) = \sup(A) \cdot \sup(B) \quad (1.2)$$

$$\text{if } c \geq 0, \quad \sup(cA) = c \sup(A) \quad (1.3)$$

$$\text{if } c \leq 0, \quad \sup(cA) = c \inf(A) \quad (1.4)$$

Proof. These properties directly follow from the epsilon definition of the supremum. That is, $\forall \epsilon_a, \epsilon_b, \exists a, b \in A, B$ such that,

$$\sup(A) - a < \epsilon_a \wedge \sup(B) - b < \epsilon_b$$

adding these equations to each other, we have

$$\sup(A) + \sup(B) - (a + b) < \epsilon_a + \epsilon_b \quad (1.5)$$

Note that $(a + b) \in A + B$, and let $\epsilon_a + \epsilon_b = \epsilon_{a+b}$. Also we know that,

$$\forall \epsilon_c \exists c \in A + B \mid \sup(A + B) - c < \epsilon_c \quad (1.6)$$

but 1.5 and 1.6 both are valid, hence the conclusion.

We can similarly prove other propositions, even for inf. \square

1.4 Applications of Completeness, Archimedean Property (A.P)

Theorem 1.4.1 (Archimedean Property, A.P). $\forall x \in \mathbb{R} \exists n_x \in \mathbb{N} \mid x \leq n_x$.

Proof. For the sake of contradiction, assume otherwise. Then $n \leq x \forall n \in \mathbb{N}$, by AoC \mathbb{N} has

supremum, s . Since $s - 1 < s$, $s - 1$ is not an upper bound, therefore $\exists m \in \mathbb{N}$ such that $s - 1 < m \Rightarrow s < m + 1$. but $m + 1 \in \mathbb{N}$. Therefore s cannot be a supremum. \square

Theorem 1.4.2 (Density of Rationals in \mathbb{R}). $\forall a, b \in \mathbb{R}, \exists r \in \mathbb{Q}$ such that

$$a < r < b$$

Proof. Since r must be rational, we want to find $m, n \in \mathbb{Z}$ such that $\frac{m}{n} = r$. From Archimedean property,

$$\exists n \in \mathbb{N} : n(y - x) \geq 1$$

Again from Archimedean property,

$$\forall t \in \mathbb{R}, \exists m \in \mathbb{Z} : m - 1 \leq t \leq m$$

In other words, for any real numbers, there are two consecutive integers that lie in the each boundary of the real numbers.

Let $t = nx$. Combining the inequalities, we get

$$nx \leq m \leq 1 + nx \leq ny \Rightarrow x \leq \frac{m}{n} \leq y$$

\square

Theorem 1.4.3 (Density of Irrationals in \mathbb{R}). $\forall x, y \in \mathbb{R}$ such that $x < y$, $\exists z \in \mathbb{I}$ such that

$$x < z < y$$

Proof. It is direct consequence of density of Rationals. We apply density theorem on $\frac{x}{\sqrt{2}}$ and $\frac{y}{\sqrt{2}}$, which we will get $z = r\sqrt{2}, r \in \mathbb{Q}$, hence we are done. \square

1.5 Intervals

Theorem 1.5.1 (Closed and Open Intervals). If $a, b \in \mathbb{R}$ and $a < b$, then **open interval** is defined by,

$$(a, b) = \{x \in \mathbb{R} | a < x < b\}$$

Similarly, we define **closed interval** as,

$$[a, b] = \{x \in \mathbb{R} | a \leq x \leq b\}$$

Theorem 1.5.2 (Nested Intervals). The sequence of intervals $I_n, n \in \mathbb{N}$ is nested if

$$I_1 \subseteq I_2 \subseteq \dots \subseteq I_n \subseteq \dots$$

Theorem 1.5.3 (Nested Interval Property). For nested intervals $\{I_n\} = [a_n, b_n], n \in \mathbb{N}$, the below is true

$$\bigcap_{i=1}^{\infty} I_n \neq \emptyset$$

Proof. Since intervals are nested intervals, $b_1 \geq a_n \forall n \in \mathbb{N}$. Hence by AoC supremum s of $\{a_n\}$ exists.

We know that $a_n \leq s$. But since b_n is also an upper bound bigger than s , we have $a_n \leq s \leq b_n$, which means $s \in \bigcap_{i=1}^{\infty} I_n$ \square

Remark: Intervals must be closed. Consider $A_n = (0, \frac{1}{n})$. Any element of intersection must be bigger than 0, while smaller than $\frac{1}{n}$. By Archimedean property of real numbers, this

is a contradiction, hence $\bigcap_{n=1}^{\infty} A_n = \emptyset$

1.6 Cardinality

Definition 1.6.1 (Cardinality). The sets A, B have the same **cardinality** if there exists a bijective function such that $f : A \rightarrow B$. We denote cardinal equality with $A \sim B$. Cardinality mathematically describes the size of the set.

The \sim operation is an equivalence relation.

Definition 1.6.2 (Countable Sets). The set A is said to be **countable** if $A \sim \mathbb{N}$. Otherwise the set is called **uncountable sets**.

Theorem 1.6.1 (Countability of \mathbb{Q}). The set \mathbb{Q} is countable, that is, $\mathbb{Q} \sim \mathbb{N}$.

Proof. There is a proof with visual construction, which maps the rational numbers to natural numbers. \square

Theorem 1.6.2 (Uncountability of \mathbb{R}). The set \mathbb{R} is uncountable.

Proof. Assume otherwise. Then subset $[0, 1] \subseteq \mathbb{R}$ must be also countable \square

Definition 1.6.3 (Power set). The powerset $\mathcal{P}(A)$, is the set of all subsets of A .

Theorem 1.6.3. Every infinite subset of a countable set is a countable set.

Theorem 1.6.4. Let $\{A_n\}, n = 1, 2, 3, \dots$ be sequence of countable sets. Then,

$$S = \bigcup_{n=1}^{\infty} A_n$$

Proof. Diagonalization method (graphical) \square

1.7 Exercises

- * Show that for $A = \{1 - \frac{1}{n} : n \in \mathbb{N}\}$, $\sup A = 1$.

Proof. A is bounded above since clearly $\forall a \in A, a < 1$. Then by AoC, supremum exists. Let $u = \sup A$. We will show that $u = 1$.

Clearly, 1 is an upper bound, since $1 > 1 - \frac{1}{n}$ is trivial.

if $u < 1$, we will show that there exists some $a \in A$ such that $u < a$.

$$\forall \epsilon > 0, \exists a \in A \mid 1 - \epsilon < a = 1 - \frac{1}{n} \Rightarrow \epsilon > \frac{1}{n}$$

But, by Archimedean, $\exists n_0 \in \mathbb{N}$ contradicting,

$$u - \epsilon < 1 - \frac{1}{n} \in A$$

Therefore $u = 1$. \square

2. If $S = \{1/n - 1/m : n, m \in \mathbb{N}\}$, find $\inf S$ and $\sup S$.

Proof. Clearly, S is bounded above and below, therefore supremum and infimum exists by AoC. We will show that $\sup S = 1$, and we can find $\inf S = -\sup(-S) = -1$. Clearly 1 is an upper bound. By definition of supremum

$$\exists \epsilon > 0, \forall s \in S \mid 1 - \epsilon < s = 1/n - 1/m \Rightarrow 1 - \epsilon < 1 - \frac{1}{m}$$

Which is equivalent to showing $\exists m \in \mathbb{N} \mid \epsilon > \frac{1}{m}$, which is evident from Archimedean. \square

3. * Let S be a set of nonnegative real numbers that is bounded above and let $T = \{x^2 : x \in S\}$. Prove that if $u = \sup S$, then $u^2 = \sup T$.

Proof. Since S is bounded above, T is also bounded above. By AoC, supremum of T exists. Let $t = \sup T$. Clearly, u^2 is upper bound of T , that is,

$$s \in S \mid s^2 \leq u^2 \Rightarrow y = s^2 \in T \mid y \leq u^2$$

Now, we will show that u^2 the least upper bound, that is,

$$\forall \epsilon > 0 \exists s \in S \mid u^2 - s^2 < \epsilon \implies (u - s)(u + s) < \epsilon$$

Since $u = \sup S$, we have

$$u - s < \epsilon_0 \quad \epsilon_0 > 0$$

Moreover, $u + s \leq 2u$. Combining these inequalities, we have

$$(u - s)(u + s) < 2u\epsilon_0$$

Then we just choose some $\epsilon > 2u\epsilon_0$. \square

Second proof.

$$a = \sup A \Rightarrow a^2 = \sup A \cdot \sup A = \sup A^2 = \sup T$$

\square

4. Given any $x \in \mathbb{R}$, show that there exists a unique $n \in \mathbb{Z}$ such that $x \leq n < n + 1$.

Proof. By definition of floor function, we have

$$\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$$

Clearly, $n - \lfloor x \rfloor$ satisfies our property. Assume two $m, n \in \mathbb{Z}$ exists. WLOG $n > m$. Then,

$$m < n \Rightarrow m + 1 \leq n \implies m + 1 \leq n \leq x < m + 1 < n + 1$$

Clearly, $m + 1 < m + 1$ is a contradiction. \square

5. * Show that there exists $y \in \mathbb{R}$ such that $y^2 = 3$.

Proof. Let $S = \{s \in \mathbb{R} : 0 \leq s, s^2 < 3\}$. Clearly, S is bounded, by AoC, $\sup S = u$ exists. We will show that $u^2 = 3$.

Clearly $u^2 = 3$ is an upper bound.

If $u^2 < 3$, we will show that $\exists n \in \mathbb{N} : u + \frac{1}{n} \in S$

$$\left(u + \frac{1}{n}\right)^2 < 3 \Rightarrow u^2 + \frac{1}{n^2} + \frac{2u}{n} \leq u^2 + \frac{1}{n}(2u + \frac{1}{n}) \implies \frac{1}{n} < \frac{3 - u^2}{2u + 1}$$

By Archimedean, such n exists satisfying our last inequality, hence contradiction. \square

6. Let $I_n = [0, 1/n]$ for $n \in \mathbb{N}$. Prove that $\cap_{n=1}^{\infty} I_n = \{0\}$.

Proof. For all $n \in \mathbb{N}$, clearly $0 \in I_n$. For any $x > 0$, by Archimedean there exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < x$, hence conclusion. \square

1.8 Notes and Mistakes on Exercises

1. Avoid “intuitive” proofs, prove every part of the proof rigorously. For example, the last exercise section, question 1, I also should prove $1 > 1 - \frac{1}{n}$ regardless of trivality.
2. The “steps” in the proofs usually should be **reversed**. In a scratch paper, for example, find and construct an epsilon/ natural number(?) and write it formally in the proof.
3. Using floor function is wrong in the last exercise. A.P should be used.

1.9 References

1. https://math.colorado.edu/~nita/12_Axiom_of_Completeness.pdf

Chapter 2

Sequences and Series

2.1 Sequences and limits

Definition 2.1.1 (Sequences). A sequence is a function with its domain as \mathbb{N} .

Definition 2.1.2 (Converge). . A sequence (x_n) is said to converge to $x \in \mathbb{R}$, or x is said to be limit of (x_n) if

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : |x_n - x| < \epsilon, \forall n \geq N$$

If limit exists, sequence is **convergent**, otherwise it is **divergent**.

Definition 2.1.3 (Epsilon Neighborhood definition of convergence). . Below definition with neighborhood is equivalent to the definition above

$$\forall \epsilon > 0, \exists N \in \mathbb{N} : x_n \in V_\epsilon(x), \forall n > N$$

Theorem 2.1.1 (Uniqueness of Limits). The limit of a sequence is **unique**.

Proof. For the sake of the contradiction, let $x = x' = \lim_{n \rightarrow \infty} (x_n)$. with the definition of the limit, $\forall \epsilon > 0, \exists n \in \mathbb{N}$ such that for all $n \geq N, N'$,

$$|x - x_n| < \epsilon/2 \quad \forall n \geq N$$

$$|x' - x_n| < \epsilon/2 \quad \forall n \geq N'$$

However, by the triangle inequality, we have

$$|x - x'| \leq |x - x_n| + |x' - x_n| < \epsilon/2 + \epsilon/2 = \epsilon, \quad \forall n \geq K = \max(N, N'')$$

Since this is $\forall \epsilon > 0$, we conclude that $x = x''$. □

2.2 Limit Theorems

Definition 2.2.1. A sequence (x_n) is **bounded** if there exists $U > 0$ such that

$$|x_n| \leq U \quad \forall n \in \mathbb{N}$$

A sequence is bounded **iff** the set $\{x_n : n \in \mathbb{N}\}$ is bounded.

Theorem 2.2.1. A convergent sequence is bounded.

Proof. If a sequence converges, then all but finite number of terms of the sequence belongs to $V_\epsilon(x)$. Since $V_{\epsilon/2}(x)$ is bounded, the sequence itself is bounded. \square

Theorem 2.2.2 (Algebra of limits). let $X = (x_n), Y = (y_n)$ converge to x, y respectively. Then sequences $X + Y, X - Y, X \cdot Y, cX$ converge to $x + y, x - y, xy, cx$ respectively. If $y \neq 0$, X/Y converges to x/y .

Proof. We will show that $X+Y$ property only, others are similar. By definition of convergence, $\forall \epsilon > 0, \exists N, N' \in \mathbb{N}$ such that

$$|x - x_n| < \epsilon/2, \forall n \geq N$$

$$|y - y_n| < \epsilon/2, \forall n \geq N'$$

\square

However, notice that $\forall n \geq \max N, N'$

$$|(x + y) - (x_n + y_n)| = |(x - x_n) + (y - y_n)| \leq |x - x_n| + |y - y_n| < \epsilon/2 + \epsilon/2 = \epsilon$$

Which proves our theorem

Theorem 2.2.3. If (x_n) is convergent sequence and $x_n \geq 0$ for all $n \in \mathbb{N}$, then $x = \lim(x_n) \geq 0$.

Theorem 2.2.4. if $(x_n), (y_n)$ are convergent sequences and $x_n \leq y_n$ for all $n \in \mathbb{N}$, then $x \leq y$.

Theorem 2.2.5. If (x_n) is a convergent sequence and $a \leq x_n \leq b$ for all $n \in \mathbb{N}$, then $a \leq x \leq b$.

Theorem 2.2.6 (Squeeze theorem). Let $(x_n), (y_n), (z_n)$ be sequences such that

$$x_n \leq y_n \leq z_n$$

And $x = z$. Then (y_n) converges and

$$x = y = z$$

All above theorems are proven similarly, the idea is the same.

2.3 Exercises

1. Show that sequence of (2^n) does not converge.

Proof. It suffices to prove that (2^n) is unbounded. Assume otherwise that there exists $M \in \mathbb{R}$ such that $2^n \leq M$ for all $n \in \mathbb{N}$. Then,

$$n \leq \log_2(M) = c$$

However by A.P, we can find n_0 such that $n_0 > c$ for any $c \in \mathbb{R}$, contradicting our claim. \square

2. Show that $z_n = (a^n + b^n)^{1/n}$ where $0 < a < b$ converge to b .

Proof. Since $a > 0$, we have

$$(a^n + b^n)^{1/n} > (b^n)^{1/n} = b$$

Since $a < b$, we have

$$(a^n + b^n)^{1/n} < (2b^n)^{1/n} = 2^{1/n}b$$

Then,

$$b \leq z_n \leq 2^{1/n}b$$

Using the squeeze theorem and the fact that $2^{1/n}$ converges to 1, we can see that $\lim z_n = b$. \square