Name : NGUEPONWOUO Joseph-Marie

NUMERICAL METHODS FOR PDE IN FINANCE : AMERICAN OPTIONS

In [2]:

```
import warnings
warnings.filterwarnings('ignore')
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd
from scipy.stats import norm
import time
import numpy.linalg as lng
from scipy.sparse import csr_matrix as sparse
from scipy.sparse.linalg import spsolve
```

$$min(\partial_t v + \mathbb{A}v, v-\varphi) = 0$$
$$v(t, S_{min}) = v_l(t)$$
$$v(t, S_{max} = v_r(t) = 0$$
$$v(0, s) = \varphi(s)$$
$$\mathbb{A}v = -\frac{\sigma^2}{2}s^2\partial_{s,s}v - rs\partial_s v + rv$$

1. Explicit Euler Scheme

$min(\frac{U_j^{n+1} - U_j^n}{\Delta t} + \frac{\sigma^2}{2}s_j^2\frac{-U_{j-1}^n + 2U_j^n - U_{j+1}^n}{h^2} - rs_j\frac{U_{j+1}^n - U_{j-1}^n}{2h} + rU_j^n, U_j^{n+1} - \varphi(s_j)) = 0$
$n = 0, \ldots N-1; j = 1, \ldots, I$

$U_0^n = v_l(t_n) = Ke^{-rt_n} - S_{min}; n = 0, \ldots, N$

$U_{I+1}^n = v_r(t_n) = 0; n = 0, \ldots, N$

$U_j^0 = \varphi(s_j) = (K - s_j)_+; j = 1, \ldots, I$

- Vector form

$$U^{n+1} = max(U^n - \Delta t(AU^n + q(t_n)), g)$$
$$g \in \mathbb{R}^I,\ g_j = \varphi(s_j)$$

Code corresponding to the Euler Forward scheme

In [9]:

```
# All the methods have been added in the class below
class AmericanOptionEuler:
    def __init__(self, Smin = 50, Smax = 250, K = 100, I = 20, T = 1, N =
```

```python
        self.r = r
        self.sigma = sigma
        self.K = K
        self.T = T
        self.Smin = Smin
        self.Smax = Smax
        self.I = I
        self.N = N
        self.scheme = scheme
        self.dt = T / N
        self.h = (Smax - Smin) / (I + 1)
        self.s = Smin + self.h * np.arange(1, I + 1)
        self.g = self.phi(self.s)
        self.U = self.g
        self.alpha = sigma**2 * self.s**2 / (2 * self.h**2)
        self.beta = r * self.s / (2 * self.h)


    def phi(self,s):
      return np.maximum(self.K - s, 0).reshape(self.I, 1)

    def phi_2(self, s):
      t = [0] * len(s)
      for i in range (len(s)):
        if self.K/2 <= s[i] <= self.K:
          t[i] = self.K
        else:
          t[i] = 0
      return np.array(t).reshape(self.I, 1)

    def v_left(self, t):
      return self.K - self.Smin

    def v_right(self, t):
      return 0

    def construct_A(self):
        A = np.zeros((self.I, self.I))
        for j in range(0, self.I):
          A[j, j] = 2 * self.alpha[j] + self.r
          if j-1 >= 0:
            A[j, j-1] = - self.alpha[j] + self.beta[j]
          if j+1 < self.I:
            A[j, j+1] = - self.alpha[j] - self.beta[j]
        return A

    def q(self, t):
        y = np.zeros((self.I, 1))
        y[0] = (-self.alpha[0] + self.beta[0]) * self.v_left(t)
        y[-1] = (-self.alpha[-1] - self.beta[-1]) * self.v_right(t)
        return y

    def explicit_euler(self):
        A = self.construct_A()
        for n in range(self.N):
            t = n * self.dt
            self.U = np.maximum(self.U - self.dt * (A @ self.U + self.q(t
        return self.U

    def explicit_euler_one_iter(self):
```

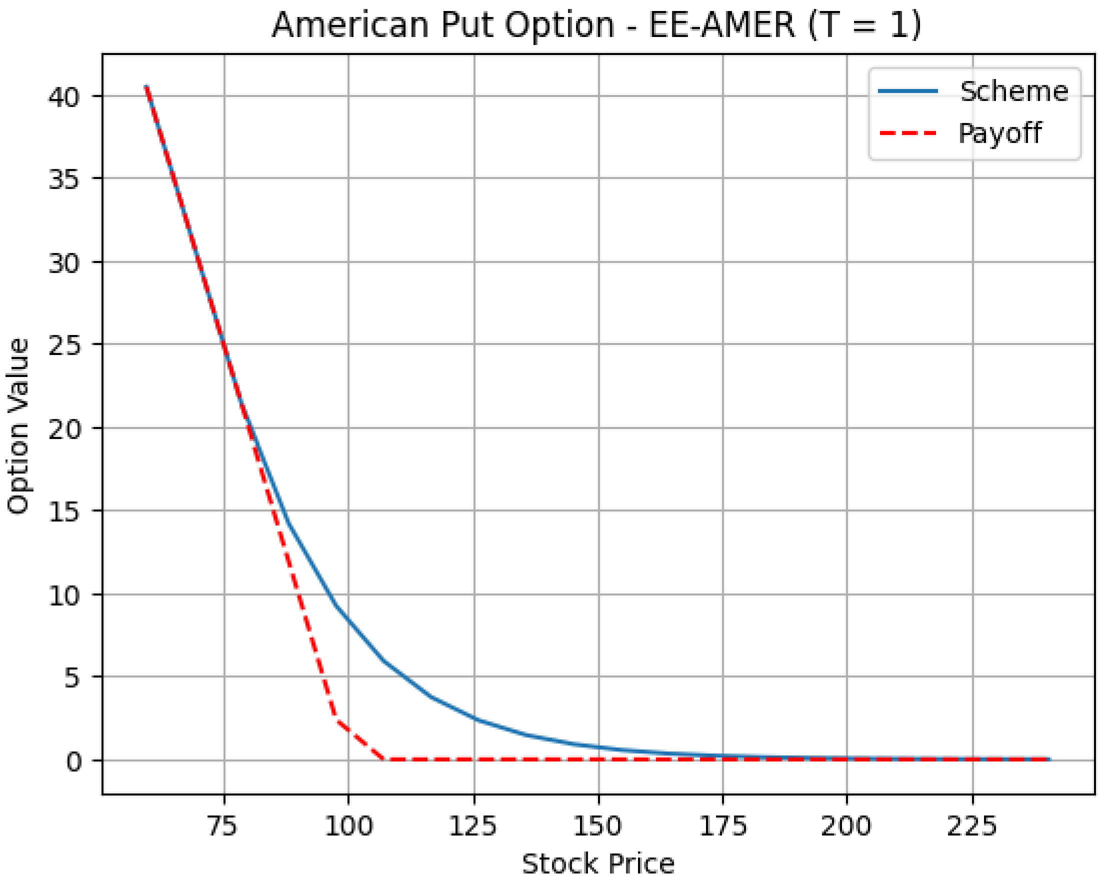
```python
        A = self.construct_A()
        for n in range(1):
            t = n * self.dt
            self.U = np.maximum(self.U - self.dt * (A @ self.U + self.q(t
        return self.U

    def implicit_euler_split(self):
        A = self.construct_A()
        Id = np.identity(self.I)
        for n in range(self.N):
            t = (n+1) * self.dt
            self.U = np.maximum(lng.solve(Id + self.dt * A, self.U - self
        return self.U

    def implicit_euler_psor(self):
        A = self.construct_A()
        Id = np.identity(self.I)
        B = Id + self.dt * A
        for n in range(self.N):
            t = (n+1) * self.dt
            b = self.U - self.dt * self.q(t)
            self.U = PSOR(B, b, self.g, self.U, eta = 1e-1, kmax = 100)
        return self.U

    def implicit_euler_newton(self):
      A = self.construct_A()
      Id = np.identity(self.I)
      B = Id + self.dt * A
      for n in range(self.N):
        t = (n + 1) * self.dt
        b = self.U - self.dt * self.q(t)
        self.U = newton(F_example, F_prime_example, self.U, eta = 1e-6, k
      return self.U

    def implicit_euler_brennan(self):
      A = self.construct_A()
      Id = np.identity(self.I)
      B = Id + self.dt * A
      L, U = ul_decomposition(B)
      #print("L:", L)
      #print("U:", U)
      for n in range(self.N):
        t = (n + 1) * self.dt
        b = self.U - self.dt * self.q(t)
        c = solve_upwind(U, b)
        self.U = descente_p(L, c, self.g)
      return self.U

    def higher_order_bdf_scheme(self):
      A = self.construct_A()
      Id = np.identity(self.I)
      B = (3/2)* Id + self.dt * A
      U_first = self.g
      U_second = self.explicit_euler_one_iter()
      for n in range(self.N):
        t = (n + 1) * self.dt
        b = (4 * U_second - U_first) / 2 - self.dt * self.q(t)
        self.U = np.maximum(lng.solve(B, b), self.g)
        U_first, U_second = U_second, self.U
      return self.U
```

```python
    def implicit_crank_nicolson(self):
        A = self.construct_A()
        Id = np.identity(self.I)
        B = Id + (self.dt / 2) * A
        C = Id - (self.dt / 2) * A
        for n in range(self.N):
            t = (n + 1) * self.dt
            self.U = np.maximum(lng.solve(B, C @ self.U - (self.dt / 2) *
        return self.U

    def interpolate_value(self, s_bar):
        for i in range(len(self.s) - 1):
            if self.s[i] <= s_bar <= self.s[i + 1]:
                s_i, s_ip1 = self.s[i], self.s[i + 1]
                if self.scheme == 'EE-AMER':
                    temp = self.explicit_euler()
                elif self.scheme == 'IE-AMER-SPLIT':
                    temp = self.implicit_euler_split()
                elif self.scheme == 'IE-AMER-CN':
                    temp = self.implicit_crank_nicolson()
                elif self.scheme == 'EI-AMER-PSOR':
                    temp = self.implicit_euler_psor()
                elif self.scheme == 'EI-AMER-NEWTON':
                    temp = self.implicit_euler_newton()
                elif self.scheme == 'EI-AMER-UL':
                    temp = self.implicit_euler_brennan()
                elif self.scheme == 'HO-AMER-BDF':
                    temp = self.higher_order_bdf_scheme()
                else:
                    raise ValueError("Invalid scheme. Choose 'EE', 'IE',
                U_i, U_ip1 = temp[i], temp[i + 1]
                interpolated_value = ((s_ip1 - s_bar) / self.h) * U_i + (
                return interpolated_value[-1]
        raise ValueError("s_bar is out of bounds.")

    def compute_error_table(self, Sval):
        alphas = []
        errex = []
        h_vals = []
        Uvals = []
        alphas = [0.0000, 0.00]
        tab = []
        for I_val in [20, 40, 80, 160, 320]:
            #N_val = 2 * (I_val ** 2) // 10
            N_val = I_val
            dt_val = self.T / N_val
            h_val = (self.Smax - self.Smin) / (I_val + 1)
            s_val = self.Smin + h_val * np.arange(1, I_val + 1)

            # Update parameters
            self.dt = dt_val
            self.h = h_val
            self.s = s_val
            self.I = I_val
            self.N = N_val
            self.alpha = self.sigma**2 * self.s**2 / (2 * self.h**2)
            self.beta = self.r * self.s / (2 * self.h)
            self.g = self.phi(self.s)
```

```
            self.U = self.g
            exact = np.maximum(self.K - Sval, 0)


            start_time = time.time()
            U_ = self.interpolate_value(Sval)
            end_time = time.time()
            h_vals.append(h_val)
            Uvals.append(U_)
            errex.append(abs(U_ - exact))
            tab.append([I_val, N_val, U_, abs(U_ - exact), end_time - sta

        tab[0][3] = 0
        errors = [abs(Uvals[j+1] - Uvals[j]) for j in range(len(Uvals)-1)
        errors.insert(0, 0)
        orders = []
        for k in range(2, len(Uvals)):
            alpha_k = (np.log(abs(Uvals[k-2]- Uvals[k-1]) / abs(Uvals[k]
            alphas.append(alpha_k)
            beta_k = alpha_k / 2
            orders.append([k, alpha_k, beta_k])
        tab = np.insert(tab, 3, errors, axis=1)
        tab = np.insert(tab, 4, alphas, axis=1)
        tab = np.array(tab)
        df = pd.DataFrame(tab, columns=['I', 'N', 'U(s)', 'error', 'alpha
        return df


    def run(self):
        if self.scheme == 'EE-AMER':
            self.explicit_euler()
        elif self.scheme == 'IE-AMER-SPLIT':
            self.implicit_euler_split()
        elif self.scheme == 'IE-AMER-CN':
            self.implicit_crank_nicolson()
        elif self.scheme == 'EI-AMER-PSOR':
            self.implicit_euler_psor()
        elif self.scheme == 'EI-AMER-NEWTON':
            self.implicit_euler_newton()
        elif self.scheme == 'EI-AMER-UL':
            self.implicit_euler_brennan()
        elif self.scheme == 'HO-AMER-BDF':
            self.higher_order_bdf_scheme()
        else:
            raise ValueError("Invalid scheme. Choose 'EE', 'IE', or 'CN'.

        plt.plot(self.s, self.U, label = 'Scheme')
        plt.plot(self.s, [max(self.K - p, 0) for p in self.s], 'r--', lab
        plt.xlabel('Stock Price')
        plt.ylabel('Option Value')
        plt.title(f'American Put Option - {self.scheme} (T = {self.T})')
        plt.legend()
        plt.grid()
        plt.show()
```

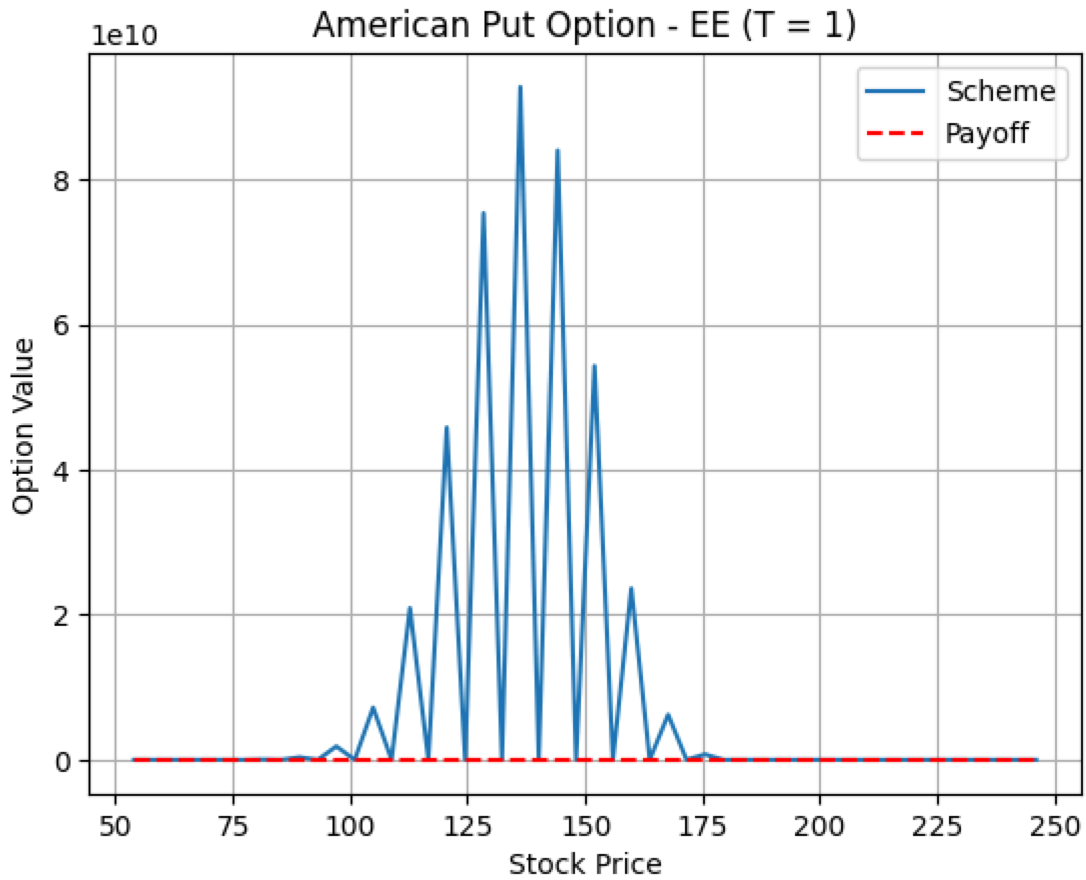
- **Check that the program does give a stable solution for the parameters $I = 20$ and $N = 20$**

```
In [ ]: option = AmericanOptionEuler(r=0.1, sigma=0.3, K=100, T=1, Smin=50, Smax=
        option.run()
```



- **Check that there is an unstable behavior with other parameters :** $I = 50$ **and** $N = 20$

```
In [ ]: option = AmericanOptionEuler(r=0.1, sigma=0.3, K=100, T=1, Smin=50, Smax=
        option.run()
```



- **Approximation of $\bar{v}$**

In [ ]:
```
option = AmericanOptionEuler(r=0.1, sigma=0.3, K=100, T=1, Smin=50, Smax=
option.compute_error_table(90)
```

Out[ ]:

	I	N	U(s)	error	alpha	errex	tcpu
0	19.0	72.0	12.948855	0.000000	0.000000	0.000000	0.001226
1	39.0	304.0	13.064946	0.116091	0.000000	3.064946	0.006109
2	79.0	1248.0	13.109601	0.044655	1.378354	3.109601	0.013975
3	159.0	5056.0	13.117808	0.008207	2.443830	3.117808	0.128860
4	319.0	20352.0	13.119988	0.002179	1.913023	3.119988	0.996035

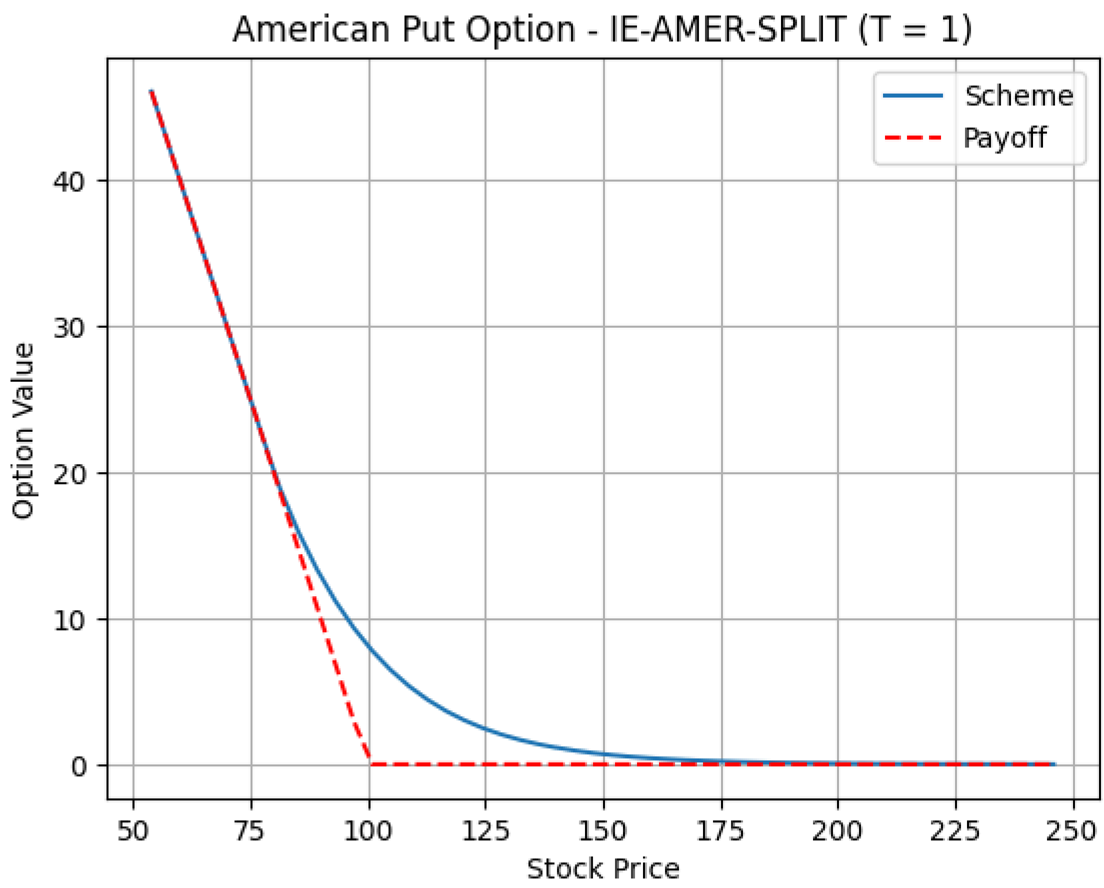
2. A First Implicit Scheme : the splitting scheme

- Compute $\frac{U^{n+1,(1)} - U^n}{\Delta t} + AU^{n+1,(1)} + q(t_{n+1}) = 0$
- $U^{n+1} = max(U^{n+1,(1)}, g)$

So here, the iterative formula is given by:

$$U^{n+1} = max(\ (Id + \Delta t\ A)^{-1}(U^n - \Delta t\ q(t_{n+1})),\ g)$$

In [ ]:
```
option = AmericanOptionEuler(r=0.1, sigma=0.3, K=100, T=1, Smin=50, Smax=
option.run()
```



- **Convergence table**

In [ ]:
```
option.compute_error_table(90)
```

Out[ ]:

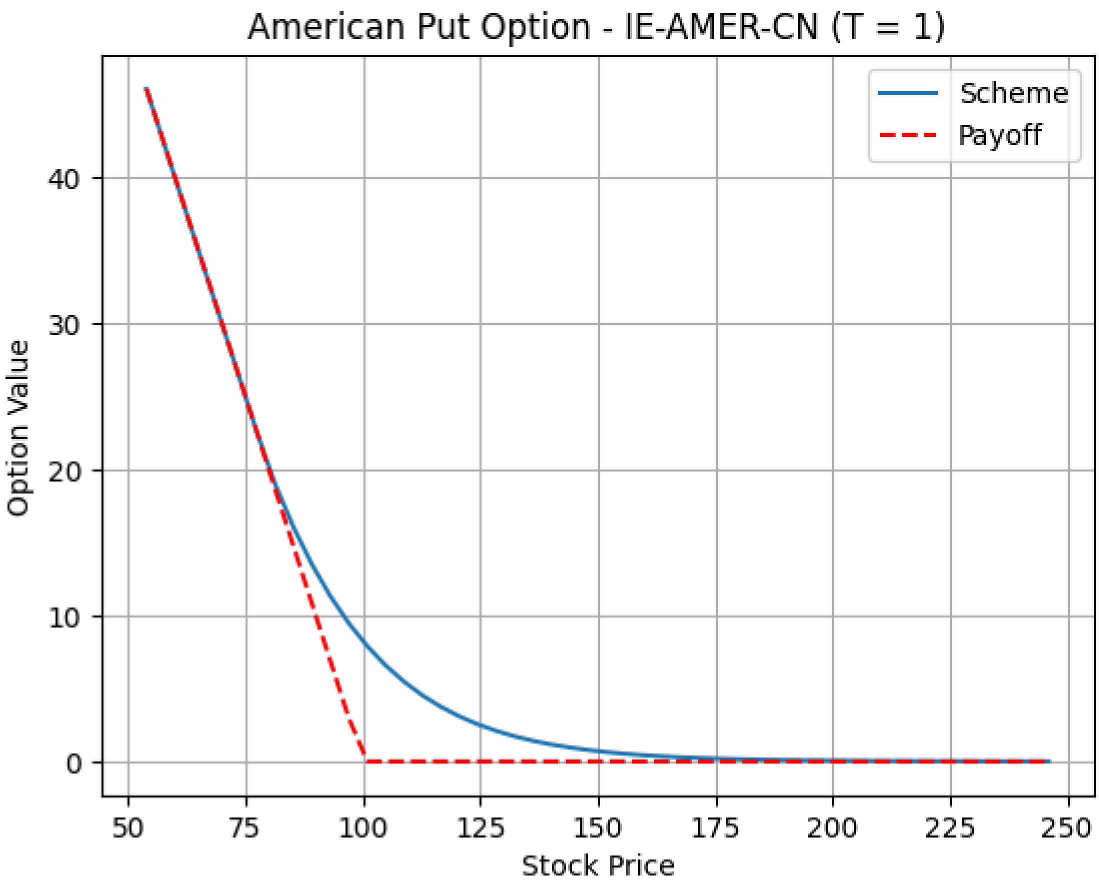
	I	N	U(s)	error	alpha	errex	tcpu
0	19.0	72.0	12.890131	0.000000	0.000000	0.000000	0.007554
1	39.0	304.0	13.045140	0.155009	0.000000	3.045140	0.024280
2	79.0	1248.0	13.105203	0.060063	1.367801	3.105203	0.223245
3	159.0	5056.0	13.116548	0.011344	2.404500	3.116548	5.080357
4	319.0	20352.0	13.119674	0.003126	1.859626	3.119674	79.097900

Remark : We can notice from the above plot that the solution is more stable compared to the one of the explicit scheme.

- **Propose a variant of the previous scheme**

$$\frac{U_j^{n+1}-U_j^n}{\Delta t}+\frac{1}{2}\left(AU^{n+1}+q(t_{n+1})\right)+\frac{1}{2}(AU^n+q(t_n))=0$$
$$\implies \left(Id+\frac{1}{2}\Delta tA\right)U^{n+1}=(Id-\frac{1}{2}\Delta tA)U^n-\frac{1}{2}\Delta t\left(q(t_n)+q(t_{n+1})\right)$$
$$\implies U^{n+1}=max(\left(Id+\frac{1}{2}\Delta tA\right)^{-1}(Id-\frac{1}{2}\Delta tA)U^n-\frac{1}{2}\Delta t\left(q(t_n)+q(t_{n+1})\right)$$

In [ ]:
```
option = AmericanOptionEuler(r=0.1, sigma=0.3, K=100, T=1, Smin=50, Smax=
option.run()
```



- **Convergence table**

In [ ]:
```
option.compute_error_table(90)
```

Out[ ]:

	I	N	U(s)	error	alpha	errex	tcpu
0	19.0	72.0	12.920051	0.000000	0.000000	0.000000	0.003629
1	39.0	304.0	13.054909	0.134858	0.000000	3.054909	0.030367
2	79.0	1248.0	13.107405	0.052496	1.361163	3.107405	0.227199
3	159.0	5056.0	13.117173	0.009768	2.426061	3.117173	3.209356
4	319.0	20352.0	13.119830	0.002657	1.878188	3.119830	94.037989

3. Implicit Euler Scheme

$$min(\frac{U^{n+1} - U^n}{\Delta t} + AU^{n+1} + q(t_{n+1}),\ U^{n+1} - g) = 0$$
$$U^0 = g$$
$$B := I_d + \Delta t\ A$$
$$b := U^n - \Delta t\ q(t_{n+1})$$
$$min(Bx - b, x - g) = 0\, x \in \mathbb{R}^I$$
$$U^{n+1} = x$$

3.1 PSOR Algorithm (PSOR = "Projected Successive Over Relaxation")

- **Check that the solution $x = x^{k+1}$ of $min(Lx - (b - Ux^k), x - g) = 0$ can be programmed in the following pseudo-code :**

In [ ]:
```
######################
'''
for i in range(1, n):
  x[i] = (b[i] - sum_{j = 1, ..., I, j != i} (B[i, j] * x[j])) / B[i, i]
  x[i] = max(x[i], g[i])
'''
######################
```

Proof: First of all, we can notice that the two functions are increasing functions of x so we can check for value of x for which the two functions are equal to zero and then, set x to be the maximum of the two.

$$Lx - (b - U\ x^k) = 0$$
$$(Lx)_i = (b - U\ x^k)_i$$

The structure of L (lower triangular) and U (upper triangular) and the fact that we start by the first line of matrix L makes the pseudo-code correct.

- **Complete the iterative method in a function PSOR**

In [ ]:
```
def PSOR(B, b, g, x0, eta, kmax):
    n = len(b)
    x = x0.copy()
    k = 1

    while k < kmax:
        x_old = x.copy()
        for i in range(n):
            sum_except_i = np.sum(B[i, :] * x) - B[i, i] * x[i]

            x[i] = (b[i] - sum_except_i) / B[i, i]
```

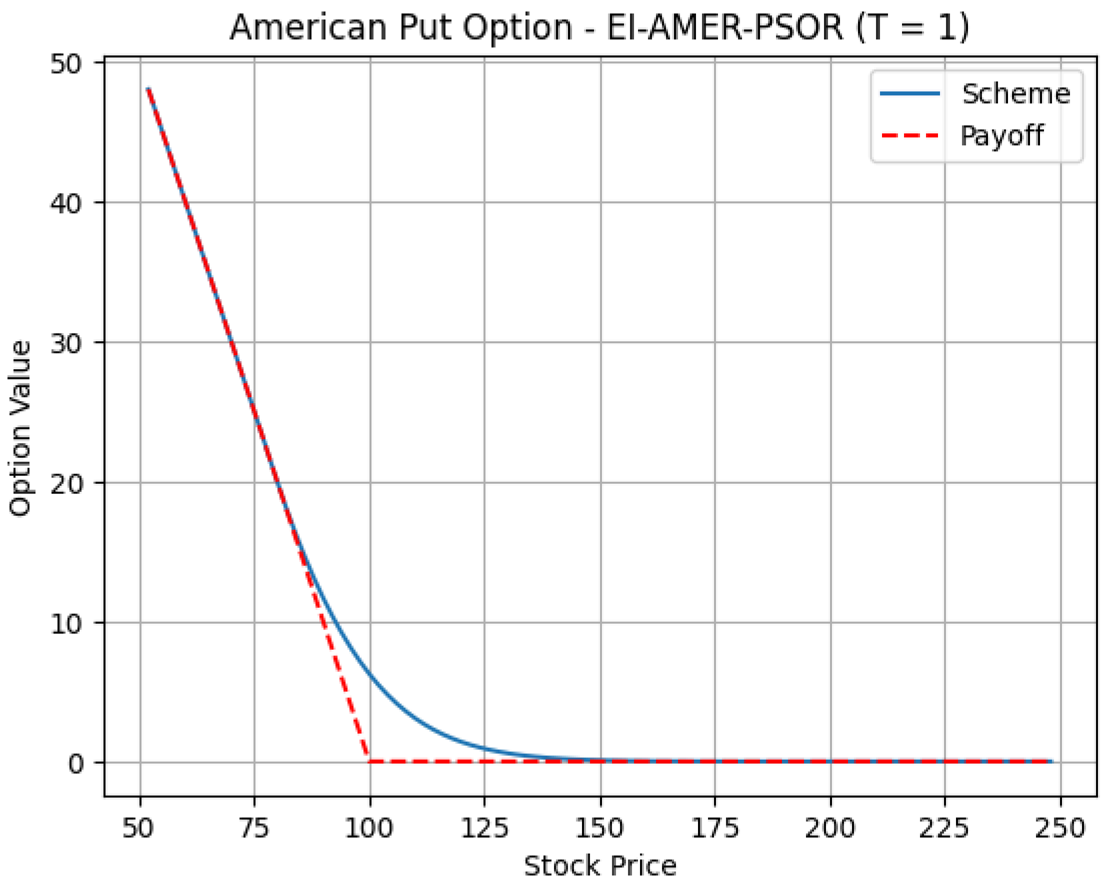
```
            x[i] = max(x[i], g[i])

        err1 = lng.norm(x - x_old)
        err2 = lng.norm(np.minimum(B @ x - b, x - g))

        print("k=%3i, |x-xold|=%10.6f, |min(Bx-b,x-g)|=%10.6f" % (k, err1

        if err1 <= eta:
            print("Converged in %d iterations." % k)
            return x
        k += 1
    print("WARNING: Maximum iterations reached. Solution may not have con
    return x
```

In [ ]:
```
def PSOR(B, b, g, x0, eta=1e-1, kmax=1000):
    x = np.array(x0, copy=True, dtype=float)
    n = len(x)
    k = 1

    while k < kmax :
        x_old = x.copy()

        for i in range(n):
            sum_offdiag = 0.0
            for j in range(n):
                if j != i:
                    sum_offdiag += B[i, j] * x[j]

            x[i] = (b[i] - sum_offdiag) / B[i, i]

            if x[i] < g[i]:
                x[i] = g[i]

        err1 = np.linalg.norm(x - x_old)
        err2 = np.linalg.norm(np.minimum(B @ x - b, x - g))

        print("k=%3d, |x - x_old|=%.6e, |min(Bx-b, x-g)|=%.6e" % (k, err1

        if err1 <= eta:
            print(f"Convergence reached at iteration k={k}.")
            return x
        k += 1

    print(f"WARNING: maximum number of iterations (kmax={kmax}) reached."
    return x
```

In [ ]:
```
option = AmericanOptionEuler(r=0.1, sigma=0.3, K=100, T=1, Smin=50, Smax=
option.run()
```

```
k=  1, |x - x_old|=1.101387e+00, |min(Bx-b, x-g)|=3.824036e+01
k=  2, |x - x_old|=7.573657e-01, |min(Bx-b, x-g)|=2.642782e+01
k=  3, |x - x_old|=5.988323e-01, |min(Bx-b, x-g)|=2.169534e+01
k=  4, |x - x_old|=5.075527e-01, |min(Bx-b, x-g)|=1.899141e+01
k=  5, |x - x_old|=4.541934e-01, |min(Bx-b, x-g)|=1.682820e+01
k=  6, |x - x_old|=4.054264e-01, |min(Bx-b, x-g)|=1.529618e+01
k=  7, |x - x_old|=3.736798e-01, |min(Bx-b, x-g)|=1.404325e+01
k=  8, |x - x_old|=3.429566e-01, |min(Bx-b, x-g)|=1.291749e+01
k=  9, |x - x_old|=3.146415e-01, |min(Bx-b, x-g)|=1.198724e+01
k= 10, |x - x_old|=2.946085e-01, |min(Bx-b, x-g)|=1.122660e+01
k= 11, |x - x_old|=2.759148e-01, |min(Bx-b, x-g)|=1.052898e+01
k= 12, |x - x_old|=2.582222e-01, |min(Bx-b, x-g)|=9.886692e+00
k= 13, |x - x_old|=2.417335e-01, |min(Bx-b, x-g)|=9.295809e+00
k= 14, |x - x_old|=2.264740e-01, |min(Bx-b, x-g)|=8.761044e+00
k= 15, |x - x_old|=2.135871e-01, |min(Bx-b, x-g)|=8.308568e+00
k= 16, |x - x_old|=2.027122e-01, |min(Bx-b, x-g)|=7.892787e+00
k= 17, |x - x_old|=1.922213e-01, |min(Bx-b, x-g)|=7.503341e+00
k= 18, |x - x_old|=1.822608e-01, |min(Bx-b, x-g)|=7.138240e+00
k= 19, |x - x_old|=1.728588e-01, |min(Bx-b, x-g)|=6.795898e+00
k= 20, |x - x_old|=1.640102e-01, |min(Bx-b, x-g)|=6.474808e+00
k= 21, |x - x_old|=1.556954e-01, |min(Bx-b, x-g)|=6.173517e+00
k= 22, |x - x_old|=1.478883e-01, |min(Bx-b, x-g)|=5.890637e+00
k= 23, |x - x_old|=1.405598e-01, |min(Bx-b, x-g)|=5.624856e+00
k= 24, |x - x_old|=1.336802e-01, |min(Bx-b, x-g)|=5.376833e+00
k= 25, |x - x_old|=1.275607e-01, |min(Bx-b, x-g)|=5.155079e+00
k= 26, |x - x_old|=1.222084e-01, |min(Bx-b, x-g)|=4.947370e+00
k= 27, |x - x_old|=1.170254e-01, |min(Bx-b, x-g)|=4.750090e+00
k= 28, |x - x_old|=1.120578e-01, |min(Bx-b, x-g)|=4.562562e+00
k= 29, |x - x_old|=1.073141e-01, |min(Bx-b, x-g)|=4.384250e+00
k= 30, |x - x_old|=1.027926e-01, |min(Bx-b, x-g)|=4.214654e+00
k= 31, |x - x_old|=9.848718e-02, |min(Bx-b, x-g)|=4.053293e+00
Convergence reached at iteration k=31.
k=  1, |x - x_old|=2.793846e-01, |min(Bx-b, x-g)|=1.108110e+01
k=  2, |x - x_old|=2.653352e-01, |min(Bx-b, x-g)|=1.052867e+01
k=  3, |x - x_old|=2.526289e-01, |min(Bx-b, x-g)|=1.003346e+01
k=  4, |x - x_old|=2.409135e-01, |min(Bx-b, x-g)|=9.580514e+00
k=  5, |x - x_old|=2.299840e-01, |min(Bx-b, x-g)|=9.161107e+00
k=  6, |x - x_old|=2.197075e-01, |min(Bx-b, x-g)|=8.769531e+00
k=  7, |x - x_old|=2.099931e-01, |min(Bx-b, x-g)|=8.401797e+00
k=  8, |x - x_old|=2.007763e-01, |min(Bx-b, x-g)|=8.054992e+00
k=  9, |x - x_old|=1.920107e-01, |min(Bx-b, x-g)|=7.727028e+00
k= 10, |x - x_old|=1.836676e-01, |min(Bx-b, x-g)|=7.420633e+00
k= 11, |x - x_old|=1.762666e-01, |min(Bx-b, x-g)|=7.137638e+00
k= 12, |x - x_old|=1.692837e-01, |min(Bx-b, x-g)|=6.869379e+00
k= 13, |x - x_old|=1.625755e-01, |min(Bx-b, x-g)|=6.614118e+00
k= 14, |x - x_old|=1.561460e-01, |min(Bx-b, x-g)|=6.370866e+00
k= 15, |x - x_old|=1.499894e-01, |min(Bx-b, x-g)|=6.138844e+00
k= 16, |x - x_old|=1.440977e-01, |min(Bx-b, x-g)|=5.917389e+00
k= 17, |x - x_old|=1.384622e-01, |min(Bx-b, x-g)|=5.705909e+00
k= 18, |x - x_old|=1.330740e-01, |min(Bx-b, x-g)|=5.503864e+00
k= 19, |x - x_old|=1.279234e-01, |min(Bx-b, x-g)|=5.310754e+00
k= 20, |x - x_old|=1.230009e-01, |min(Bx-b, x-g)|=5.126113e+00
k= 21, |x - x_old|=1.182969e-01, |min(Bx-b, x-g)|=4.949501e+00
k= 22, |x - x_old|=1.138018e-01, |min(Bx-b, x-g)|=4.780507e+00
k= 23, |x - x_old|=1.095061e-01, |min(Bx-b, x-g)|=4.618742e+00
k= 24, |x - x_old|=1.054007e-01, |min(Bx-b, x-g)|=4.463838e+00
k= 25, |x - x_old|=1.014766e-01, |min(Bx-b, x-g)|=4.315449e+00
k= 26, |x - x_old|=9.772518e-02, |min(Bx-b, x-g)|=4.173249e+00
Convergence reached at iteration k=26.
k=  1, |x - x_old|=1.976253e-01, |min(Bx-b, x-g)|=8.153662e+00
```

```
k=  2, |x - x_old|=1.900457e-01, |min(Bx-b, x-g)|=7.862365e+00
k=  3, |x - x_old|=1.827678e-01, |min(Bx-b, x-g)|=7.584520e+00
k=  4, |x - x_old|=1.757921e-01, |min(Bx-b, x-g)|=7.319381e+00
k=  5, |x - x_old|=1.691834e-01, |min(Bx-b, x-g)|=7.070063e+00
k=  6, |x - x_old|=1.631632e-01, |min(Bx-b, x-g)|=6.833902e+00
k=  7, |x - x_old|=1.573578e-01, |min(Bx-b, x-g)|=6.607954e+00
k=  8, |x - x_old|=1.517690e-01, |min(Bx-b, x-g)|=6.391451e+00
k=  9, |x - x_old|=1.463929e-01, |min(Bx-b, x-g)|=6.183819e+00
k= 10, |x - x_old|=1.412236e-01, |min(Bx-b, x-g)|=5.984571e+00
k= 11, |x - x_old|=1.362544e-01, |min(Bx-b, x-g)|=5.793270e+00
k= 12, |x - x_old|=1.314784e-01, |min(Bx-b, x-g)|=5.609524e+00
k= 13, |x - x_old|=1.268890e-01, |min(Bx-b, x-g)|=5.432969e+00
k= 14, |x - x_old|=1.224793e-01, |min(Bx-b, x-g)|=5.263266e+00
k= 15, |x - x_old|=1.182426e-01, |min(Bx-b, x-g)|=5.100099e+00
k= 16, |x - x_old|=1.141724e-01, |min(Bx-b, x-g)|=4.943168e+00
k= 17, |x - x_old|=1.102621e-01, |min(Bx-b, x-g)|=4.792193e+00
k= 18, |x - x_old|=1.065053e-01, |min(Bx-b, x-g)|=4.646906e+00
k= 19, |x - x_old|=1.028960e-01, |min(Bx-b, x-g)|=4.507053e+00
k= 20, |x - x_old|=9.942794e-02, |min(Bx-b, x-g)|=4.372396e+00
Convergence reached at iteration k=20.
```



3.2 Semi-smooth Newton's method

$$F(x) = 0 \quad ; F(x) := min(Bx - b, x - g)$$

Iteration : $x^{k+1} = x^k - F'(x^k)^{-1} F(x^k)$

Stopping conditions : $F(x^k) = 0$ or $x^{k+1} = x^k$

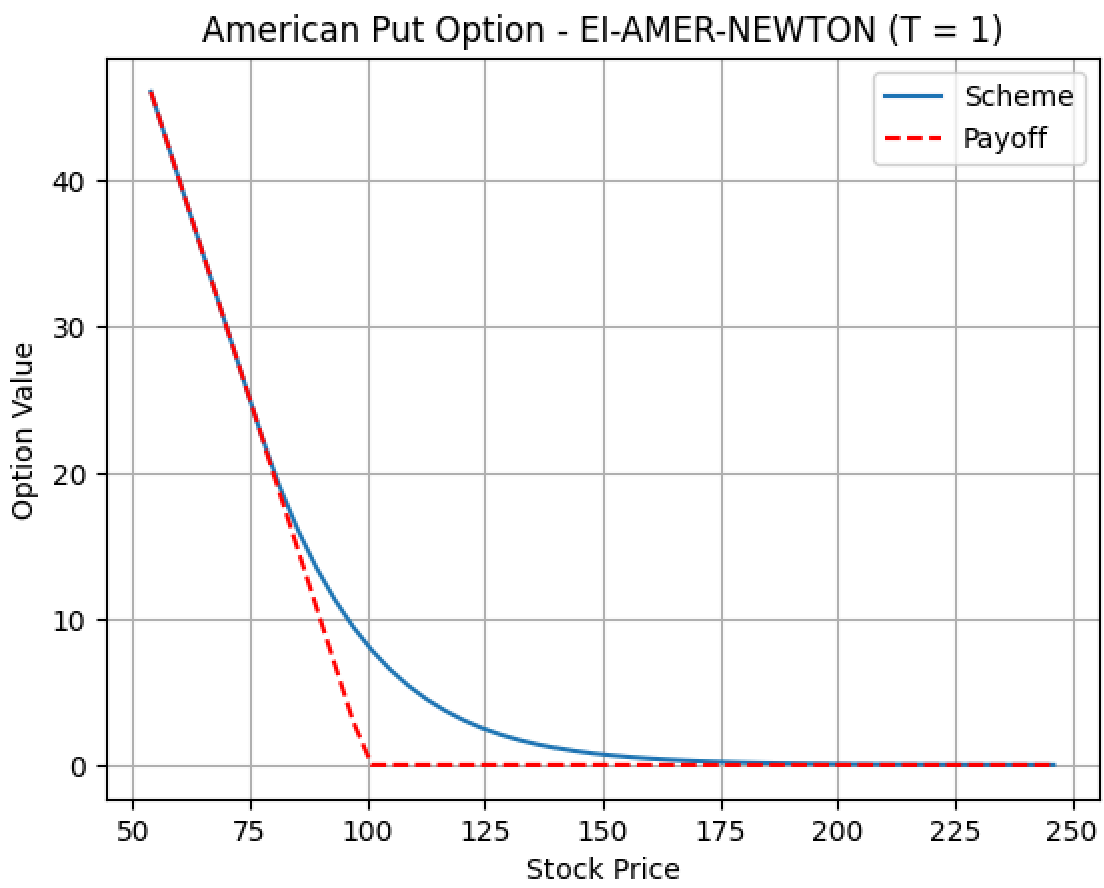
$$F'(x^k)_{i,j} = \begin{cases} B_{i,j} & \text{if } (Bx^k - b)_i \leq (x^k - g)_i \\ \delta_{i,j} & \text{otherwise} \end{cases}$$

- **Write the code of Newton's method in a function *newton***

In [ ]:
```
def F_example(B, b, g, x):
    return np.minimum(B @ x - b, x - g)

def F_prime_example(B, b, g, x):
    n = len(x)
    F_prime = np.zeros((n, n))
    for i in range(n):
        if (B @ x - b)[i] <= (x - g)[i]:
            F_prime[i, :] = B[i, :]
        else:
            F_prime[i, i] = 1
    return F_prime

def newton(F, F_prime, x0, eta, kmax, B, b, g):
    x = x0.copy()
    for k in range(kmax):
        Fx = F(B, b, g, x)
        F_prime_x = F_prime(B, b, g, x)

        dx = lng.solve(F_prime_x, -Fx)

        x = x + dx

        err = lng.norm(Fx)
        #print(f"k={k}, |F(x)|={err:.6f}")

        if err <= eta:
            #print("Converged in %d iterations." % k)
            return x
    return x
```

- Program the implicit Euler scheme using Newton's method : *Done above*
- Test the method with $N = 20, I = 50$ and the classical payoff function φ_1

In [ ]:
```
option = AmericanOptionEuler(r=0.1, sigma=0.3, K=100, T=1, Smin=50, Smax=
option.run()
```



- Draw error tables : $N = I$

In [ ]:
```
option = AmericanOptionEuler(r=0.1, sigma=0.3, K=100, T=1, Smin=50, Smax=
option.compute_error_table(90)
```

Out[ ]:

	I	N	U(s)	error	alpha	errex	tcpu
0	20.0	20.0	13.137800	0.000000	0.000000	0.000000	0.013871
1	40.0	40.0	13.084578	0.053221	0.000000	3.084578	0.055344
2	80.0	80.0	13.106557	0.021979	1.298869	3.106557	0.192780
3	160.0	160.0	13.111972	0.005414	2.039480	3.111972	1.111530
4	320.0	320.0	13.116142	0.004170	0.378387	3.116142	11.279583

- Draw error tables : $N = I/10$

In [ ]:
```
option = AmericanOptionEuler(r=0.1, sigma=0.3, K=100, T=1, Smin=50, Smax=
option.compute_error_table(90)
```

Out[ ]:

	I	N	U(s)	error	alpha	errex	tcpu
0	20.0	2.0	12.719620	0.000000	0.000000	0.000000	0.003668
1	40.0	4.0	12.835853	0.116233	0.000000	2.835853	0.006236
2	80.0	8.0	12.965750	0.129897	-0.163235	2.965750	0.024464
3	160.0	16.0	13.038234	0.072484	0.849222	3.038234	0.163176
4	320.0	32.0	13.077756	0.039522	0.878945	3.077756	1.437687

3.3. Brennan and Schwartz Algorithm

$$
\begin{aligned}
min\ (Bx - b, x - g) = 0 \\
B = UL\ (U_{i,i} = 1\ \forall i) \\
\implies min(Lx - U^{-1}b, x - g) = 0
\end{aligned}
$$

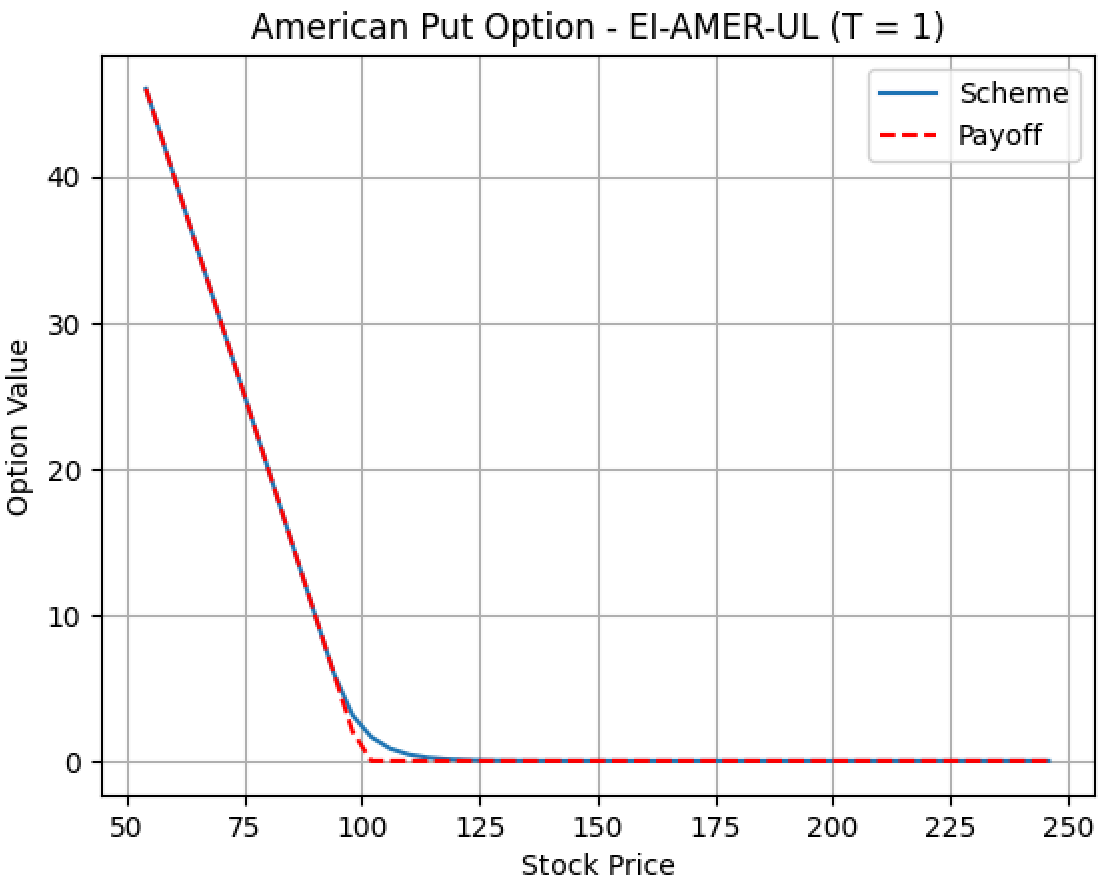
In [5]:

```python
def ul_decomposition(B):
    n = B.shape[0]
    L = np.zeros_like(B)
    U = np.eye(n)
    for i in range(n):
        L[i, i] = B[i, i]

    for i in range(n-1):
        U[i, i+1] = B[i, i+1] / B[i+1, i+1]
        L[i+1, i] = B[i+1, i]


    #print('norme de B-UL:',lng.norm(B-U@L, np.inf))
    return L, U

def solve_upwind(U, b):
    n = len(b)
    c = np.zeros_like(b)

    for i in range(n - 1, -1, -1):
        c[i] = b[i] - np.dot(U[i, i+1:], c[i+1:])
    return c

def descente_p(L, c, g):
    n = len(c)
    x = np.zeros_like(c)

    for i in range(n):
        x[i] = max((c[i] - np.dot(L[i, :i], x[:i])) / L[i, i], g[i])

    print("||min(Bx-b,x-g)||:", lng.norm(np.minimum(L@x - g, x - g)))
    return x
```

- **Test the method for** $N = 20, I = 49$

In [10]:
```python
option = AmericanOptionEuler(r=0.1, sigma=0.3, K=100, T=1, Smin=50, Smax=
option.run()
```

```
||min(Bx−b,x−g)||: 2.647301057881577e-23
||min(Bx−b,x−g)||: 1.469181269471013
||min(Bx−b,x−g)||: 1.9419532002858342
||min(Bx−b,x−g)||: 2.077394015210682
||min(Bx−b,x−g)||: 2.1385091259998203
||min(Bx−b,x−g)||: 2.169549073658953
||min(Bx−b,x−g)||: 2.1855511992083936
||min(Bx−b,x−g)||: 2.193876128473527
||min(Bx−b,x−g)||: 2.1982381925221137
||min(Bx−b,x−g)||: 2.20053696708473
||min(Bx−b,x−g)||: 2.2017540384086334
||min(Bx−b,x−g)||: 2.2024008415216816
||min(Bx−b,x−g)||: 2.202745636126504
||min(Bx−b,x−g)||: 2.2029298980386622
||min(Bx−b,x−g)||: 2.2030285711090363
||min(Bx−b,x−g)||: 2.2030814994682806
||min(Bx−b,x−g)||: 2.203109929279275
||min(Bx−b,x−g)||: 2.2031252172103137
||min(Bx−b,x−g)||: 2.203133445807879
||min(Bx−b,x−g)||: 2.203137878161421
```

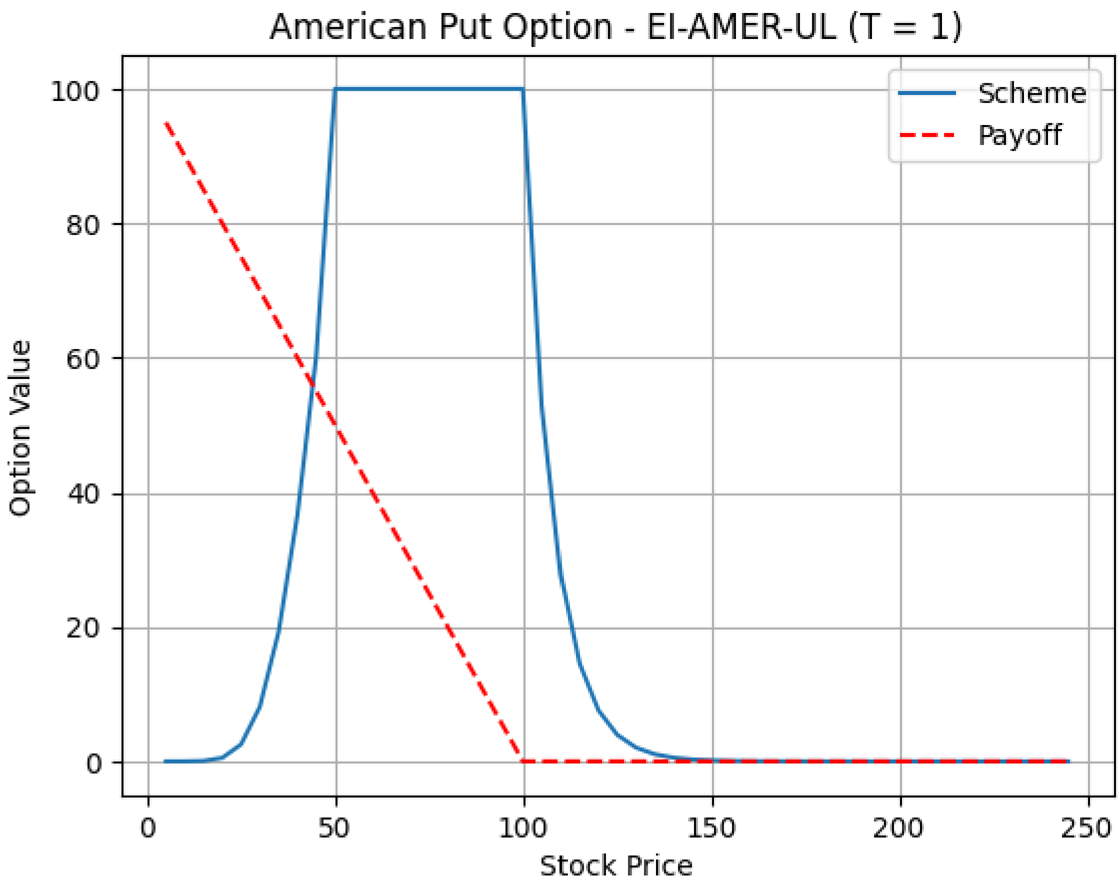


- Run the program again with the particular payoff φ_2 instead of φ_1. Check that in that case $min(Bx-b, x-g) \neq 0$ as soon as $n = 0$.

$$\varphi_2(s) = \begin{cases} K & \text{if } \frac{K}{2} \leq s \leq K \\ 0 & \text{otherwise} \end{cases}$$

In [8]:
```
option = AmericanOptionEuler(r=0.1, sigma=0.3, K=100, T=1, Smin=0, Smax=2
option.run()
```

```
||min(Bx-b,x-g)||: 12.59575359939473
||min(Bx-b,x-g)||: 43.42925854627938
||min(Bx-b,x-g)||: 60.89000591469507
||min(Bx-b,x-g)||: 67.86618374938274
||min(Bx-b,x-g)||: 72.3186475616518
||min(Bx-b,x-g)||: 75.71036229026303
||min(Bx-b,x-g)||: 78.46494397282096
||min(Bx-b,x-g)||: 80.79200831929052
||min(Bx-b,x-g)||: 82.81416059674221
||min(Bx-b,x-g)||: 84.60754262037001
||min(Bx-b,x-g)||: 86.22182333291043
||min(Bx-b,x-g)||: 87.6909276003538
||min(Bx-b,x-g)||: 89.03904367994113
||min(Bx-b,x-g)||: 90.2841082398639
||min(Bx-b,x-g)||: 91.43989387831178
||min(Bx-b,x-g)||: 92.51729829372258
||min(Bx-b,x-g)||: 93.52516474332138
||min(Bx-b,x-g)||: 94.4708201813502
||min(Bx-b,x-g)||: 95.36043905069108
||min(Bx-b,x-g)||: 96.19929668520822
```



4. Higher Order Schemes

i) Implicit Euler : Done in section 2

ii) Crank-Nikolson : Done in section 2

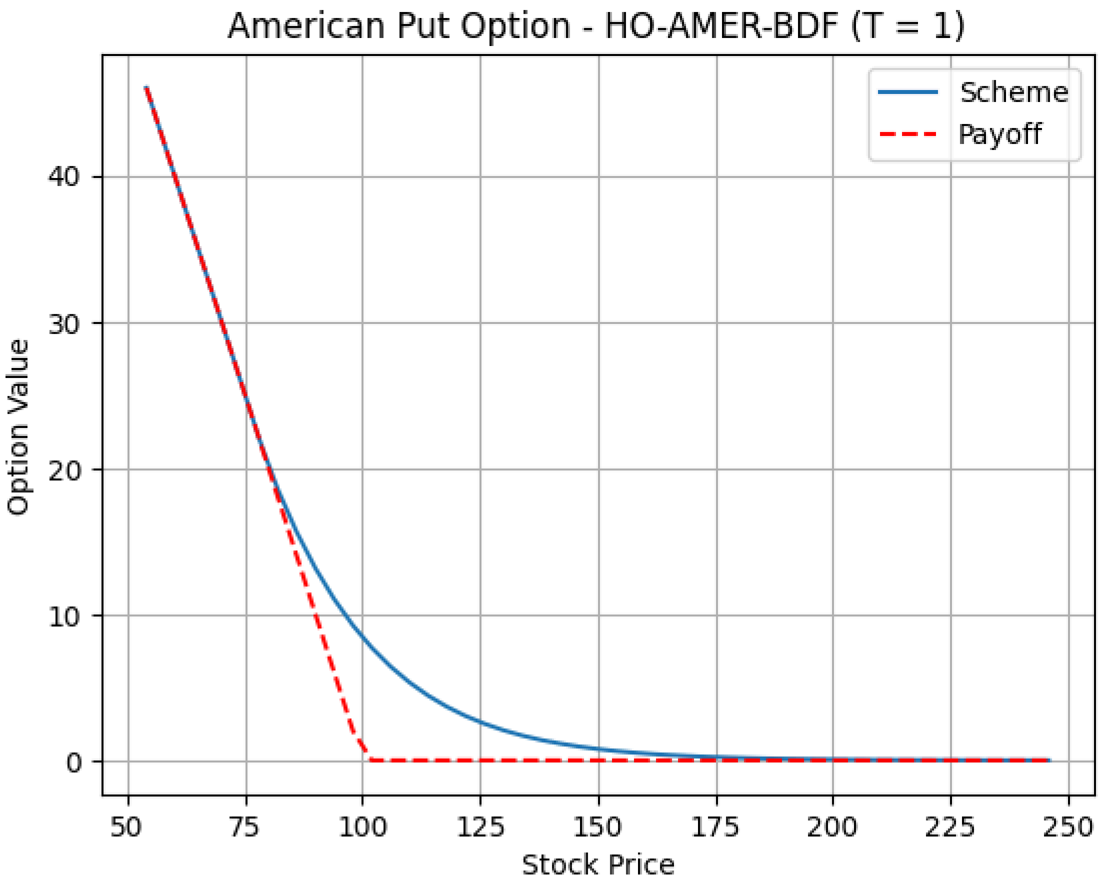
iii) BDF (Backward Difference Formula) Scheme

$$U^0 = g$$

$$min(\frac{3U^{n+1} - 4U^n + U^{n-1}}{2\Delta t} + AU^{n+1} + q(t_{n+1}),\ U^{n+1} - g) = 0$$

$$U^{n+1} = max(\left(\frac{3}{2}Id + \Delta t A\right)^{-1}(\frac{4U^n - U^{n-1}}{2} - \Delta t\, q(t_{n+1})),\ g\ )$$

In [ ]:
```
option = AmericanOptionEuler(r=0.1, sigma=0.3, K=100, T=1, Smin=50, Smax=
option.run()
```



- Table for $N = I/10$

In [ ]:
```
option.compute_error_table(90)
```

Out[ ]:

	I	N	U(s)	error	alpha	errex	tcpu
0	20.0	20.0	13.230914	0.000000	0.000000	0.000000	0.001306
1	40.0	40.0	13.135376	0.095537	0.000000	3.135376	0.004780
2	80.0	80.0	13.128265	0.007112	3.815358	3.128265	0.033784
3	160.0	160.0	13.122898	0.005367	0.409765	3.122898	0.475919
4	320.0	320.0	13.121262	0.001636	1.722048	3.121262	1.975531

- Table for $N = I$

In [ ]:
```
option.compute_error_table(90)
```

Out[ ]:

	I	N	U(s)	error	alpha	errex	tcpu
0	20.0	20.0	13.230914	0.000000	0.000000	0.000000	0.003254
1	40.0	40.0	13.135376	0.095537	0.000000	3.135376	0.007097
2	80.0	80.0	13.128265	0.007112	3.815358	3.128265	0.013460
3	160.0	160.0	13.122898	0.005367	0.409765	3.122898	0.099361
4	320.0	320.0	13.121262	0.001636	1.722048	3.121262	0.881788

Remark : The code in this case is faster than the previous one but more stable in term of values.