EA PROJECT : Study of path-dependent CIR process

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Abstract

This project explores the boundary behavior of the Cox-Ingersoll-Ross (CIR) process both theoretically and numerically, extending to its path-dependent variant which incorporates memory effects via an integral term. By comparing explicit and implicit approximation schemes, we assess their convergence and applicability. While the non-Markovian nature of the path-dependent CIR process challenges traditional analysis; leveraging a Volterra framework, we demonstrate the existence and uniqueness of solutions, and further investigate its boundary behavior under the Feller conditions. Our study contributes to the understanding of CIR processes in financial modeling, especially in the context of rough volatility models, highlighting potential avenues for future research.

Contents

1 Introduction

The first stochastic volatility models generally had Markovian structures, and the best known of these is Heston's, which correctly reproduces the term structure of money skew for long maturities, but fails to explain certain stylized facts such as the convexity of implied volatility surfaces, observed for very short maturities. Rough processes, on the other hand, are neither Markovian nor semi-martingale and seem remarkably well suited to reproducing time series from financial data. Empirical studies indicate that volatility is rougher than Brownian Motion in Gatheral et al.(2018); Bennedsen et al.(2016). Subsequent development of stochastic models with this feature has been done in El Euch and Rosenbaum (2016, 2017). Hence, Rough volatility models are able to match roughness of time series data, fit implied volatility smiles remarkably well and admit in some cases microstructural justification. For this reason, the path dependent CIR processes could be useful to understand rough volatility models like the Rough Heston Model.

In this project, we first understand the boundary behaviour of the CIR process through both theoretical analysis and numerical simulations. We begin with a detailed exploration of the squared Bessel process and its connection with the CIR process. We further implement and evaluate the performance of explicit and implicit approximation schemes, comparing their convergence rates to discern their efficacy and applicability. Expanding our scope, we delve into the path-dependent CIR process, distinguished by its integration term $\int_0^t f(t-s)X_s ds$, which endows the process with memory, thereby enhancing its capacity to model real-world phenomena. However, this non-Markovian characteristic complicates the analysis, as traditional methodologies applied to the CIR process are predicated on its Markov property. By reinterpreting the process as a Volterra type, we leverage the seminal work of Eduardo Abi Jaber, Martin Larsson, and Sergio Pulido (2019) to establish the existence and uniqueness of the solution. Then, we demonstrate its boundary behavior under the Feller condition. In cases where the Feller condition violates, our numerical investigations shows that the process tends to touch the 0 as the classical CIR process. Additionally, we implement its characteristic function to better understand the property of the process.

This project not only advances our understanding of the CIR process and its path-dependent extensions but also opens avenues for future research into their boundary behaviors and practical applications in financial modeling and beyond.

Setting. Thoughout this report, we consider the filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in[0,\infty]}, P)$, where $(\mathscr{F}_t)_{t\in[0,\infty]}$ is completed filtration generated by the Brownian Motion $(W_t)_{t\in[0,\infty)}$.

2 Existing results of the CIR process

2.1 Theoretical analysis

In this section we present the well-known results of the CIR process, which is:

$$dr_t = k\left(\theta - r_t\right)dt + \sigma\sqrt{|r_t|}dW_t,\ r_0 = x. \tag{2.1}$$

We always assume $x, k\theta > 0$, $\sigma \in \mathbb{R}$. We first introduce the Yamada-Watanabe's criteria, which ensures the existence and uniqueness of the equation. We also introduce the comparison theorem, which helps justify that r_t is always non-negative.

Theorem 2.1. ***(Yamada-Watanabe)*** *Considering the process:*

$$X_t = X_0 + \int_0^t b\left(s, X_s\right)ds + \int_0^t \sigma\left(s, X_s\right)dW_s. \tag{2.2}$$

Suppose $\varphi : (0,\infty) \to (0,\infty)$ is a Borel function such that $\int_{0^+} \frac{1}{\varphi(a)}da = +\infty$. Under any of the following conditions:

1. *the Borel function b is bounded, the function σ does not depend on the time variable and satisfies*
$$|\sigma(x) - \sigma(y)|^2 \le \varphi(|x-y|)$$
and $|\sigma| \ge \epsilon > 0$
2. *$|\sigma(s,x) - \sigma(s,y)|^2 \le \varphi(|x-y|)$ and b is Lipschitz continuous,*

3. the function σ does not depend on the time variable and satisfies

$$|\sigma(x)-\sigma(y)|^2 \leq |f(x)-f(y)|$$

where f is a bounded increasing function, $\sigma \geq \epsilon > 0$ and b is bounded,

the equation (2.2) admits a unique solution which is strong, and the solution X is a Markov process.

Proof. See [1], Chapter IV, Section 3. □

Theorem 2.2. ***(Comparison Theorem)*** *Let*

$$dX_t^i = b_i\left(t, X_t^i\right)dt + \sigma\left(t, X_t^i\right)dW_t, i = 1, 2$$

where b_i $(i = 1, 2)$ are bounded Borel functions and at least one of them is Lipschitz and σ satisfies assumption 1 or 2 of theorem 2.1. Suppose also that $X_t^1 \geq X_t^2$ and $b_1(x) \geq b_2(x)$. Then

$$X_t^1 \geq X_t^2 \text{ for } \forall t, \text{ a.s.}$$

Proof. See [1], Chapter IX, Section 3. □

Proposition 2.3. *Equation (2.1) admits a unique strong solution. In particular, $r_t \geq 0$ for every t a.s. Thus equation (2.1) is reduced to*

$$dr_t = k\left(\theta - r_t\right)dt + \sigma\sqrt{r_t}dW_t,\ r_0 = x. \tag{2.3}$$

Proof. As $|\sqrt{x}-\sqrt{y}| \leq \sqrt{|x-y|}$, for $x, y \geq 0$, we applying the theorem 2.1 and know that (2.1) admits a strong solution. Then we apply the comparison theorem 2.2 and derive $r_t \geq 0$.

□

Definition 2.4. *If (2.3) further satisfies*

$$2k\theta > \sigma^2,$$

*we say it satisfies the **Feller condition**.*

The following 2 propositions clarify that the Feller condition fully describe the boundary behavior of the CIR process.

Theorem 2.5. *If (2.3) satisfies the Feller condition, then*

$$\mathbb{P}_x\left(r_t > 0, \forall t > 0\right) = 1.$$

For the proof, we use the Gronwall's inequality on its integral form, which is stated in the Lemma below:

Lemma 2.6. *Let $I \subset [a, +\infty)$ for some $a \in \mathbb{R}$ and α, β and u be real-valued functions defined on I. Assume that β and u are continuous and that the negative part of α is integrable on every closed and bounded subinterval of I.*

- *If β is non-negative and if u satisfies the integral inequality:*

$$\forall t \in I, \quad u(t) \leq \alpha(t) + \int_a^t \beta(s)u(s)ds$$

Then:

$$\forall t \in I, \quad u(t) \leq \alpha(t) + \int_a^t \alpha(s)\beta(s)\exp\left(\int_s^t \beta(r)dr\right)ds$$

- *If, in addition, the function α is non-decreasing, then:*

$$\forall t \in I, \quad u(t) \leq \alpha(t)\exp\left(\int_a^t \beta(s)ds\right)$$

Using the Lemma above, we now prove the theorem 2.5.

Proof. For $\varepsilon > 0$, we define $\tau_\varepsilon := \min\{t : r_t \leq \varepsilon\}$. We want to show that, when $\varepsilon \to 0$, we have:

$$\mathbb{P}(\tau_\varepsilon < t) \to 0$$

Where $\tau_\varepsilon \wedge t := \min\{\tau_\varepsilon, t\}$. Define $m = \frac{2k\theta - \sigma^2}{\sigma^2}$. Applying the Ito formula for the function $f(x) = x^{-m}$, we obtain:

$$d\left(r_t^{-m}\right) = -mr_t^{-(m+1)}\left[k(\theta - r_t dt + \sigma\sqrt{r_t}dW_t\right] + \frac{1}{2}m(m+1)r_t^{-(m+2)}\sigma^2 r_t dt$$

Reorganizing the terms:

$$d\left(r_t^{-m}\right) = mkr_t^{-m}dt - m\sigma r_t^{-(m+\frac{1}{2})}dW_t + m\left(\frac{(m+1)\sigma^2}{2} - k\theta\right)r_t^{-(m+1)}dt$$

Where the last term vanishes, since $\left(\frac{(m+1)\sigma^2}{2} - k\theta\right) = 0$. Writing the expression above in the integral form and evaluating it on $\tau_\varepsilon \wedge t$:

$$r_{\tau_\varepsilon \wedge t}^{-m} = r_0^{-m} + mk\int_0^{\tau_\varepsilon \wedge t} r_{\tau_\varepsilon \wedge s}^{-m} ds - m\sigma \int_0^{\tau_\varepsilon \wedge t} r_{\tau_\varepsilon \wedge s}^{-(m+\frac{1}{2})} dW_s$$

Taking the expectations and using the property that the Itô's integral is a local martingale, we obtain:

$$\mathbb{E}\left[r_{\tau_\varepsilon \wedge t}^{-m}\right] = r_0^{-m} + mk\int_0^{\tau_\varepsilon \wedge t} \mathbb{E}\left[r_{\tau_\varepsilon \wedge t}^{-m}\right] ds$$

Applying the Gronwall's inequality, we obtain:

$$\mathbb{E}\left[r_{\tau_\varepsilon \wedge t}^{-m}\right] \leq r_0^{-m} e^{mkt}$$

Finally, we apply the Markov inequality to bound the desired probability:

$$\mathbb{P}(\tau_\varepsilon \leq t) = \mathbb{P}(\tau_\varepsilon \wedge t = \tau_\epsilon) = \mathbb{P}(r_{\tau_\varepsilon \wedge t}^{m} \leq \varepsilon^m) = \mathbb{P}(r_{\tau_\varepsilon \wedge t}^{-m} \geq \varepsilon^{-m}) \leq \varepsilon^m r_0^{-m} e^{mkt}$$

Hence $\mathbb{P}(\tau_\varepsilon < t) \to 0$ when $\varepsilon \to 0$.

□

Remark 2.1. Theorem 2.5 still holds when $2k\theta = \sigma^2$. The proof of this case is similar to the proof above, we basically need to change $f(x)$ into $\int_1^x y^{-1}e^{\theta y}dy$.

Theorem 2.7. *If $2k\theta < \sigma^2$, then for every $s > 0$,*

$$\mathbb{P}\left(r_t > 0, \forall t > s\right) = 0.$$

The proof of theorem 2.7 is more tricky. To achieve this, We first reduce it to a squared Bessel process.

Definition 2.8. ***(Squared Bessel process)*** *Given $\delta > 0$ and $x > 0$, the unique strong solution to the equation*

$$u_t = x + \delta t + 2\int_0^t \sqrt{u_s}dW_s$$

is called squared Bessel process with dimension δ, starting at x. We have equivalently:

$$du_t = \sigma dt + 2\sqrt{u_t}dW_t,\ u_0 = x$$

Proposition 2.9. *The conditional characteristic function of the squared Bessel process can be computed as:*

$$\mathbb{E}\left[e^{-\lambda u_t} \mid \mathscr{F}_s\right] = (1 + 2\lambda(t-s))^{-\frac{n}{2}} \exp\left(\frac{-\lambda u_s}{1 + 2\lambda(t-s)}\right). \tag{2.4}$$

Proof. Notice that u_t has quadratic variation $4\int_0^t u_s ds$, and $M_t = u_t - \delta t$ is a continuous local martingale, Ito's lemma gives

$$e^{-\lambda u_t} = e^{-\lambda u_s} + \int_s^t e^{-\lambda u_r}\left(\frac{1}{2}\lambda^2 4u_r - \lambda\delta\right)dr - \lambda\int_s^t e^{-\lambda u_r}dM_r.$$

for constant $\lambda \geq 0$ and times $s \leq t$. The final term is a local martingale and is bounded on finite time intervals (as all the other terms are). So, it is a proper martingale. Fixed a bounded $\mathscr{F}_s$-measurable random variable Z, Multiplying by it and taking expectations, we get

$$\mathbb{E}\left[Ze^{-\lambda X_t}\right] = \mathbb{E}\left[Ze^{-\lambda X_s}\right] + \lambda\int_s^t \mathbb{E}\left[Ze^{-\lambda X_r}\left(2\lambda X_r - \delta\right)\right]dr.$$

We now introduce the function $\phi(t,\lambda) = \mathbb{E}\left[Ze^{-\lambda X_t}\right]$ and, noting that this has partial derivative $\partial\phi/\partial\lambda = -\mathbb{E}\left[Ze^{-\lambda X_t}X_t\right]$,

$$\phi(t,\lambda) = \phi(s,\lambda) - \lambda\int_s^t\left(2\lambda\frac{\partial\phi(r,\lambda)}{\partial\lambda} + \delta\phi(r,\lambda)\right)dr.$$

So, ϕ is continuously differentiable over $t \geq s$ and, by differentiating the above equation, it satisfies the following partial differential equation

$$\frac{\partial\phi}{\partial t} + 2\lambda^2\frac{\partial\phi}{\partial\lambda} + \delta\lambda\phi = 0.$$

We can solve this PDE and get

$$\phi\left(t,\lambda_t\right) = e^{-n\int_s^t\lambda_u du}\phi\left(s,\lambda_s\right). \tag{2.5}$$

with λ_t the unique solution of the ODE $\frac{d\lambda_t}{dt} = 2\lambda_t^2$, which is $\lambda_t = \frac{\lambda_s}{1-2\lambda_s(t-s)}$. Substituting back into (2.5) we derive equation (2.4), and thus completing the proof. □

The lemma below illustrates the link between the CIR process and the squared Bessel process:

Lemma 2.10. *Given* $\delta = \frac{4k\theta}{\sigma^2}$, *then* $r_t = e^{-kt}u_{\frac{\sigma^2}{4k}(e^{kt}-1)}$ *is the unique solution of* (2.3).

Proof. By the uniqueness of the CIR process, we simply apply Ito formula and finish the proof. □

Notice that $r_t = 0$ equals to $u_{\frac{\sigma^2}{4k}(e^{kt}-1)} = 0$ and $2k\theta < \sigma^2$ equals to $\delta \in (0,2)$. Therefore it suffices to consider the boundary behavior the u_t with $\delta \in (0,2)$. We then define $\tau = \inf\{t \in [0,\infty) : u_t = 0\} \cup \{\infty\}$, it is clear that τ is a stopping time.

Lemma 2.11. *Let* u_t *be a squared Bessel process with dimension* σ. *Then there exists a function* $f : \mathbb{R}_+ \to \mathbb{R}_+$, *such that:*

1. $\lim_{x\to+\infty} f(x) = +\infty$,

2. $Y_t = f(u_{\tau\wedge t})$ *is a continuous local martingale.*

Particularly, we can let $f(x) = \frac{x^{1-\frac{\delta}{2}}}{(2-\delta)^{2-\delta}}$ *and thus have*

$$dY_t = \mathbb{1}_{\{Y_t>0\}}Y_t^{\frac{1-n}{2-n}}dW_t.$$

Proof. The proof for $\sigma \neq 2$ consists on applying the Ito's Lemma to the function $f(x) = (ax)^b$, while the case $\sigma = 2$ considers the function $f(x) = \frac{1}{2}\log(x)$. In our work, we consider only the first case. We consider the interval $[0,\, \tau_n]$, where $\tau_n = \inf\{t : u_t \leq \frac{1}{b}\}$, in such a way that the stopped process $u_{\tau_n\wedge t}$ is always non-negative. Applying the Ito's lemma to the function $f(x) = (ax)^b$ and defining $Y_t = f(u_t)$, we have:

$$dY_t = a^b b u_t^{b-1}(\sigma dt + 2\sqrt{u_t}dW_t) + \frac{1}{2}a^b b(b-1)x^{b-2}\sigma^2 dt$$

$$dY_t = 2a^b b u_t^{b-\frac{1}{2}} dW_t + 2a^b b \left(\frac{\sigma}{2} + b - 1\right) u_t^{b-1} dt$$

Since $\sigma \neq 2$, we define $b = 1 - \frac{\sigma}{2} \neq 0$, so the last term vanishes. Defining $c = 1 - \frac{1}{2b}$ and writing $u_t = \frac{Y^{\frac{1}{b}}}{a}$, we have:

$$dY_t = 2a^b b \left(\frac{Y^{\frac{1}{b}}}{a}\right)^{b-\frac{1}{2}} dW_t = 2a^{\frac{1}{2}} b Y^c dW$$

Using $a^{\frac{1}{2}} = |2b|^{-1}$, we have $2a^{\frac{1}{2}}b = \pm 1$, according to whether b is positive ($\sigma < 2$) or b is negative ($\sigma > 2$). Hence, on the interval $[0, \tau_n]$, we have:

$$dY_t = \pm Y^c dW_t$$

Letting $n \to \infty$, the equation above holds also for τ.

- In particular, if $\sigma < 2$, b is positive and $Y = (au_{\tau \wedge t})^b$ is equal to zero over the interval $[\tau, \infty)$. Hence, defining $Y_t = f(u_{\tau \wedge t})$ with $f(x) = \frac{x^{1-\frac{\sigma}{2}}}{(2-\sigma)^{2-\sigma}}$ (which is obtained considering the values of a, b and c used in the proof):

$$dY_t = \mathbb{1}_{\{Y_t > 0\}} Y_t^{\frac{1-n}{2-n}} dW_t.$$

- For the case $\sigma > 2$, and b negative, we conclude that, since $Y_{\tau_n \wedge t}$ is a non-negative local martingale, that $\tau = \infty$ and Y is almost surely bounded, since it cannot explode in a finite time, for which case we have that $Y = (au_t)^b$ satisfies:

$$dY_t = -Y^c dW_t.$$

□

Lemma 2.12. *For every $\delta > 0$,* $\limsup_{t \to +\infty} u_t = +\infty$

Proof. By 2.9 we know that $\frac{X_t}{t}$ has the $\chi_n^2\left(\frac{x}{t}\right)$ distribution, which can be written as the sum of independent random variables $U \sim \chi_n^2(0)$ and $V \sim \chi_0^2\left(\frac{x}{t}\right)$. So, for any constant $K \geq 0$,

$$\mathbb{P}\left(X_t > K\right) = \mathbb{P}\left(U + V > \frac{K}{t}\right) \geq \mathbb{P}\left(U > \frac{K}{t}\right).$$

Notice that $\lim_{t \to +\infty} \mathbb{P}(U > \frac{K}{t}) = \mathbb{P}(U > 0) = 1$, we prove that $\limsup_{t \to \infty} X_t = +\infty$. □

Proposition 2.13. *Let M_t be a continuous local martingale, then*

$$\left\{\omega \in \Omega : \liminf_{t \to \infty} M_t \neq -\infty\right\} \subset \left\{\omega \in \Omega : \exists c_\omega \in \mathbb{R} \ s.t. \lim_{t \to +\infty} M_t = c_\omega\right\}.$$

Proof. The proof is a direct result of theorem 5 in the blog of George Lowther. □

With these properties related to the squared Bessel process and the local martingales, we give now the proof of the theorem 2.7:

Proof of theorem 2.7. Since $f(x)$ is always non-negative, $\liminf_{t \to \infty} Y_t \neq -\infty$ a.s.. While Y_t is a continuous local martingale, we apply proposition 2.13 and know that $\lim_{t \to \infty} Y_t$ exists a.s.. On the other hand, Combined with lemma 2.12 with the property of f, we know that $\limsup_{t \to \infty} f(u_t) = +\infty$ a.s.. This shows that $\tau = +\infty$ a.s., and thus completing the proof.

2.2 Numerical experiments

To simulate the CIR process, we've tried two different implicit schemes mentioned in On the discretization schemes for the CIR processes, Aurélien Alfonsi (2005) and an explicit scheme obtained by applying Ito's formula to $f : (t,x) \mapsto e^{kt}x$. Let's consider the classic CIR process :

$$\begin{cases} X_t = x_0 + \int_0^t (a - kX_s)ds + \sigma \int_0^t \sqrt{X_s} dW_s \\ x_0, \ a \geq 0, \ k \in \mathbb{R} \end{cases}$$

Values of parameters for implementation :

$$\begin{cases} x_0 = 0.9 \\ a = 2 \\ k = 2 \\ \sigma = 1.5 \end{cases}$$

Remark 2.2. For this set of parameters, the Feller condition is well satisfied as $2a > \sigma^2$ and $x_0 > 0$.

A natural way to simulate the CIR process is the explicit Euler-Maruyama scheme :

$$\widehat{X}^n_{t_{i+1}} = \widehat{X}^n_{t_i} + \frac{T}{n}(a - k\widehat{X}^n_{t_i}) + \sigma\sqrt{\widehat{X}^n_{t_i}}(W_{t_i+1} - W_{t_i})$$

But this scheme can lead to negative values since the Gaussian increment is not bounded from below. Thus, this scheme is not well defined even in the Feller condition. However, Brigo and Alfonsi proposed a discretization to obtain positivity using an implicit scheme by rewriting the CIR process with the posticipated stochastic integral, since, $d < \sqrt{X}, W >_s = \frac{\sigma}{2}ds$, we get the result below.

2.2.1 First Implicit scheme

The first discretization scheme is taken by considering the CIR process as the square root of a second-degree polynomial deduced from the dynamic of the process X_T. So we have :

$$\widehat{X}^n_{t_{i+1}} = \left(\frac{\sigma(W_{t_{i+1}} - W_{t_i}) + \sqrt{\sigma^2(W_{t_{i+1}} - W_{t_i})^2 + 4(\widehat{X}^n_{t_i} + (a - \frac{\sigma^2}{2})\frac{T}{n})(1 + k\frac{T}{n})}}{2(1 + k\frac{T}{n})} \right)^2$$

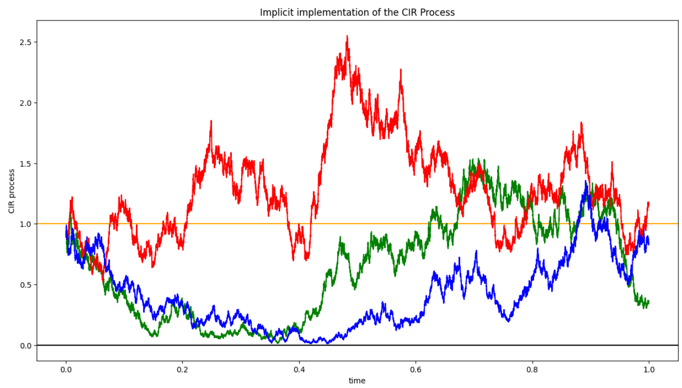

Figure 1: First implicit scheme of CIR process

Remark 2.3. The different curves obtained above are three independent simulations of the CIR process with the same value of parameters previously mentioned.

2.2.2 Second Implicit scheme

The second implementation considers now the dynamic of the square root of the CIR process X_t.

$$d\sqrt{X_t} = \frac{a - \frac{\sigma^2}{4}}{2\sqrt{X_t}} dt \ - \ \frac{k}{2}\sqrt{X_t} dt \ + \ \frac{\sigma}{2} dW_t$$

This dynamic allows us to derive the discretization of the process X_t by considering $\sqrt{X_t}$ as the square root of a second degree polynomial as before and we have the following result.

$$\widehat{X}^n_{t_{i+1}} = \left(\frac{\frac{\sigma}{2}(W_{t_{i+1}} - W_{t_i}) \ + \ \sqrt{\widehat{X}^n_{t_i}} \ + \ \sqrt{(\frac{\sigma}{2}(W_{t_{i+1}} - W_{t_i}) + \sqrt{\widehat{X}^n_{t_i}})^2 \ + \ 4(1 + \frac{kT}{2n})\frac{a - \frac{\sigma^2}{4}}{2}\frac{T}{n}}}{2(1 + \frac{kT}{2n})} \right)^2$$

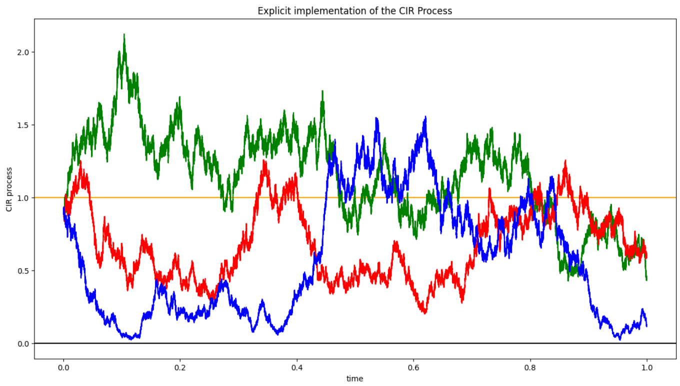

Figure 2: Explicit scheme of CIR process

2.2.3 Explicit scheme

By appling Itô's formula to the function

$$f : (t, x) \mapsto e^{kt} x$$

we obtain for $f(t, X_t)$:

$$X_{t+h} = X_t e^{-kh} + \int_t^{t+h} a \ e^{-k(t+h-u)} du + \int_t^{t+h} \sigma \ e^{-k(t+h-u)} \ \sqrt{X_u} \ dW_u$$

By discretization, we have then with $h = t_{i+1} - t_i$:
$X_{t_{i+1}} = X_{t_i} e^{-k(t_{i+1} - t_i)} + a(t_{i+1} - t_i) e^{-k(t_{i+1} - t_i)} + \sigma \ e^{-k(t_{i+1} - t_i)} \sqrt{X_{t_i}} (W_{t_{i+1}} - W_{t_i})$

$$X_{t_{i+1}} = e^{-k(t_{i+1} - t_i)} \left(X_{t_i} + a(t_{i+1} - t_i) + \sigma \sqrt{X_{t_i}} (W_{t_{i+1}} - W_{t_i}) \right)$$

2.2.4 Violation of Feller condition

An unsurprising observation is that when we simulate the CIR process in the case where the condition $2a > \sigma^2$ is no more satisfied and we observe the following fact : Some paths disappear after a certain time, because it is impossible in this case to have a negative number under the square root which confirms that this type of phenomenon can be observed when the Feller condition is violated.

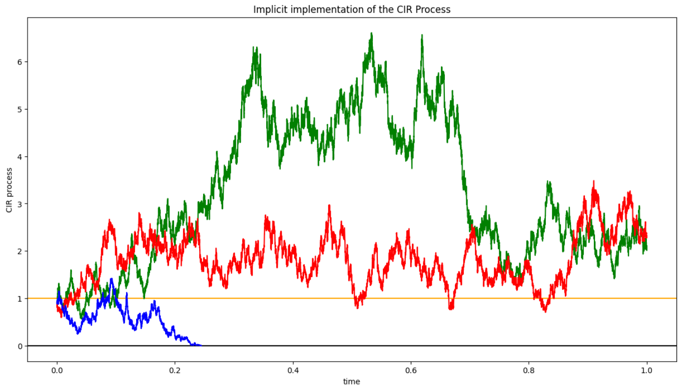

Figure 3: CIR process with Feller condition violated

2.2.5 Conclusion of the discretization schemes

- The explicit implementation using the Itô's formula is faster than the implicit implementations used on the paper of Aurélien Alfonsi
- The implementation using the square root of the CIR process is the slowest among the three discretization schemes that have been mentioned above.
- At the end of the day, we have good convergence of both implementations to the mean of the CIR process (1 in this case).

The table below summarizes the convergence to the mean for both implementation schemes for $n = 10^6$ by applying the Law of Large Number (LLN). The running time mentioned in this table has been normalized according to the fastest discretization scheme which is the explicit one.

Implementation	Mean	Running time
First Scheme	0.93	1.12
Second Scheme	1.08	1.93
Explicit Scheme	1.00	1

Remark 2.4. Recall that the mean reversion for the values of parameters in this implementation is given by : $\frac{a}{k} = \frac{2}{2} = 1$.

3 Our results of the path dependent CIR process

3.1 Theoretical analysis

In this section we discuss the following path dependent CIR process:

$$dX_t = \left(a - kX_t + \int_0^t f(t-s)X_s ds\right) dt + \sigma\sqrt{X_t^+} dW_t, \ X_0 = x > 0, \tag{3.1}$$

where $a, k > 0$, $f \in L^2([0,T])$, and $f(s) > 0$ for every $s \in [0,T]$. The function x^+ means $\max\{x, 0\}$. This is a path dependent extension of 2.3.

Proposition 3.1. *If* (3.1) *has a continuous solution* X_t*, then* $X_t \geq 0$ *a.s.*

Proof. It suffices to consider the stopping time $\tau = \inf\{t \in [0,T] : X_t < 0\}$, and then do pathwise analysis. □

Thus (3.1) is reduced to:

$$dX_t = \left(a - kX_t + \int_0^t f(t-s)X_s ds\right) dt + \sigma\sqrt{X_t} dW_t, \ X_0 = x > 0. \tag{3.2}$$

Remark 3.1. We can also get equation (3.2) from the equation:

$$dX_t = \left(a - kX_t + \int_0^t f(t-s)X_s ds\right) dt + \sigma\sqrt{|X_t|}dW_t,\ X_0 = x > 0,$$

But it should be noted that, this process is a non-markovian process, so we could not apply the comparison theorem as what we did for the CIR process.

We first show that equation (3.2) admits a unique continuous strong solution by transforming it into a Volterra equation and then apply the results in [2].

Let $K(t) = 1 - \frac{1}{k}\int_0^t f(s)ds$ and $g(t) = x + at - a\int_0^t K(s)ds$, then the equation (3.2) can be transformed into

$$X_t = g(t) + \int_0^t K(t-s)(a - kX_s)ds + \sigma \int_0^t \sqrt{X_s}dW_s, \tag{3.3}$$

which is a **Volterra process**. If we apply the convolution notation in [2], we can write equation (3.3) as

$$X_t = g_t + K * (a - kX)_t + \sigma 1 * \sqrt{X_t}dW_t. \tag{3.4}$$

However, here the two kernels are different, so we need to transform it to the standard Volterra equation. Since $K \in L^2([0,T])$, there exists a unique function $R \in L^2([0,T])$, such that

$$R = -kK - kR * K.$$

Actually we can write R explicitly by Fourier transformation. Then by lemma 2.1 in [2], we know that

$$\begin{aligned} R * (K * -kX) &= (R * -kK) * X \\ &= (R + kK) * X \\ &= R * X + kK * X. \end{aligned}$$

On the other hand, we simply do convolution on (3.4) with R and get

$$R * X = R * g + R * (K * (a - kX)) + R * \left(1 * \left(\sigma\sqrt{X}dW\right)\right).$$

Combined the two equations above, we derive

$$\begin{aligned} -kK * X &= R * g + R * (K * a) + R * \left(1 * \left(\sigma\sqrt{X}dW\right)\right) \\ &= R * g + R * (K * a) + (R * 1) * \left(\sigma\sqrt{X}dW\right). \end{aligned}$$

Substitute it into equation (3.4) we get the standard Volterra equation:

$$X_t = \tilde{g}_t + \tilde{K} * \left(\sigma\sqrt{X_t}dW_t\right), \tag{3.5}$$

where $\tilde{g}_t = g_t + (K * a)_t + (R * g)_t + (R * (K * a))_t$, and $\tilde{K}(t) = (R * 1)_t + 1$. Thus, we can apply theorem 3.3 in [2] and know that (3.5) admits a continuous strong solution. As the previous section, We also want to understand the boundary behaviour of the process X_t. To do this, we need to introduce the following lemma:

Lemma 3.2. *(McKean's Argument.) Let $Z = (Z_s)_{s\in\mathbb{R}_+}$ be an adapted càdlàg $\mathbb{R}^+\backslash\{0\}$ valued stochastic process on a stochastic interval $[0, \tau_0)$ such that $Z_0 > 0$ a.s. and $\tau_0 = \inf\{0 < s \leq \tau_0 : Z_{s-} = 0\}$. Suppose $h : \mathbb{R}_+\backslash\{0\} \to \mathbb{R}$ is continuous and satisfies the following:*

1. *For all $t \in [0, \tau_0)$, we have $h(Z_t) = h(Z_0) + M_t + P_t$, where*
   (a) *P is an adapted càdlàg process on $[0, \tau_0)$ such that $\inf_{t\in[0,\tau_0\wedge T)} P_t > -\infty$ a.s. for each $T \in \mathbb{R}^+\backslash\{0\}$,*
   (b) *M is a continuous local martingale on $[0, \tau_0)$ with $M_0 = 0$,*

2. and $\lim_{z\downarrow 0} h(z) = -\infty$.

Then $\tau_0 = \infty$ *a.s.*

Proof. See [3, proposition 4.3.] □

We found that the feller condition could also describe the boundary behaviour of the path-dependent CIR process, as the following proposition shows.

Proposition 3.3. *If* $2a \geq \sigma^2$*, then* $X_t > 0 \ \forall t \in [0,T]$ *a.s.*

Proof. We first consider the case where $2a > \sigma^2$. Let $\tau = \inf\{t \in [0,T] : X_t = 0\}$. We aim at proving $P(\tau = \infty) = 1$. We apply lemma 3.2 by taking $h(x) = -x^{-\alpha}$ for $\alpha > 0$ small enough s.t. $a - \frac{\sigma^2(\alpha+1)}{2} > 0$. By Ito formula we derive:

$$h(X_t) = h(X_0) + M_t + P_t,$$

where

$$\begin{aligned} M_t &= \int_0^t \alpha\sigma X_s^{-\alpha-\frac{1}{2}} dW_s, \\ P_t &= \int_0^t \left(\alpha \left(a - \frac{\sigma^2(\alpha+1)}{2} \right) X_s^{-(\alpha+1)} + \alpha X_s^{-\alpha-1} \int_0^s f(s-r) X_r dr \right) ds. \end{aligned}$$

As $X_t > 0$ for $t \in [0,\tau)$, M_t is naturally a local martingale on $[0,\tau)$. On the other hand, by the choice of α, $P_t \geq 0$. Above all, we have checked all the condition of lemma 3.2 and thus finish the proof of the first part.
When $2a = \sigma^2$ we apply the same method with $h(x) = \int_1^x y^{-1} e^{\frac{k}{a}y} dy$. □

3.2 Numerical experiments

3.2.1 Discretization scheme

Let's recall the path dependent SDE :

$$dX_t = \left(a - kX_t + \int_0^t f(t-s) X_s ds \right) dt + \sigma \sqrt{X_t} dW_t$$

Here, we'll approximate the integral term above $\int_0^t f(t-s)ds$ by:

$$\int_0^t f(t-s) X_s ds \approx \sum_{j=0}^{i-1} X_{t_j} f(t-t_j)(t_{j+1}-t_j)$$

By applying the Itô's formula as above with the new expression of a, we now have a path-dependent process given by the following expression :

$$X_{t+h} = X_t e^{-kh} + a \int_t^{t+h} e^{-k(t+h-s)} ds + \sigma \int_t^{t+h} e^{-k(t+h-s)} \sqrt{X_s} dW_s + \int_t^{t+h} e^{-k(t+h-u)} \left(\int_0^u f(u-s) X_s \, ds \right) du$$

By discretization, we have :
$X_{t_{i+1}} = X_{t_i} e^{-k(t_{i+1}-t_i)} + a(t_{i+1}-t_i) e^{-k(t_{i+1}-t_i)} + \sigma \, e^{-k(t_{i+1}-t_i)} \sqrt{X_{t_i}} (W_{t_{i+1}} - W_{t_i}) + e^{-k(t_{i+1}-t_i)} (t_{i+1}-t_i) \sum_{j=0}^{i-1} X_{t_j} f(t_i - t_j)(t_{j+1}-t_j)$

We set $f(x) = e^{-\lambda x}$ with $\lambda > 0$. For this expression of the function f, we obtain the following curve for the CIR-Volterra process. Here the values of parameters are the same as before and $\lambda = 0.02$

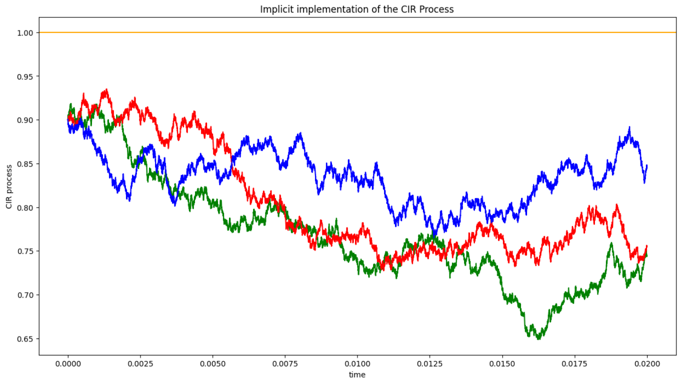

Figure 4: CIR-Volterra Process

3.2.2 Violation of Feller condition with the non-Markovian CIR Process

Values of parameters for implementation :

$$\begin{cases} x_0 = 0.9 \\ a = 2 \\ k = 2 \\ \sigma = 2.5 \\ \lambda = 0.02 \end{cases}$$

In the case of the classical CIR Process, it was observed that a simulation with a set of parameters violating the Feller condition leads to negative values. Consequently, the paths diminish to zero after a certain time. This phenomenon can also be observed here in the case of the path-dependent CIR process, as indicates the graph below.

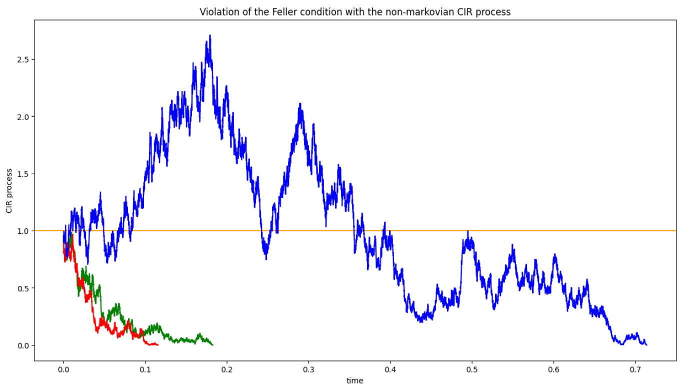

Figure 5: Violation of Feller condition with the non-Markovian CIR process

3.2.3 Riccati-Volterra equation

The Riccati-Volterra equation is given by :

$$\psi = \lambda K \ + \ K * F(\psi)$$

with

$$F(\psi) = \frac{c^2}{2}\psi^2 \ + \ a\psi$$

It's easy to show that the function ψ defined above is a solution of the following Riccati differential equation :

$$\begin{cases} \psi(0) = \lambda \\ \psi'(t) = \frac{c^2}{2}\psi^2(t) + a\psi(t) + b\int_0^t f(t-s)\psi(s)ds \end{cases}$$

By using the characteristic function stated in the previous section, we are going to verify by simulation that for ψ, function solution of the Riccati differential equation, we obtain approximately the same value for the two terms of the equality below. Our implementation aimed to test the two terms for $t = 0$.

$$\mathbb{E}[e^{uX_T}|\mathscr{F}_0] = e^{ug_0(T)+\frac{c^2}{2}\int_0^T \psi^2(T-s)g_0(s)ds}$$

By discretization of the Riccati differential equation, we have :

$$\begin{cases} \psi(0) & = \lambda, \\ \psi'(t) & = \frac{c^2}{2}\psi^2(t) + a\psi(t) + b\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} f(t-s)\psi(s)ds, \\ & = \frac{c^2}{2}\psi^2(t) + a\psi(t) + b\sum_{i=0}^{n-1} f(t-t_i)\psi(t_i)(t_{i+1}-t_i). \end{cases}$$

By substitution,

$$\implies \mathbb{E}[e^{uX_T}|\mathscr{F}_0] \ (*) = e^{ug_0(T)+\frac{c^2}{2}\sum_{i=0}^{n-1}\psi^2(T-t_i)g_0(t_i)(t_{i+1}-t_i)} \ (**)$$

We are now going to compare the result obtained by solving the differential equation and the one using Monte Carlo simulation.
We will also consider that $\mathbb{E}[e^{uX_T}|\mathscr{F}_0] = \mathbb{E}[e^{uX_T}]$ because of independence of the increments in the Brownian motion that appears in the discretization of the process. We have then considered that the filtration associated to the CIR process is the same as the one associated to the underlying brownian motion.
Values of parameters for implementation :

$$\begin{cases} x_0 = 0.9 \\ a = 2 \\ k = 1 \\ \sigma = 0.1 \\ u = -0.5 \end{cases}$$

By Monte Carlo simulation on $(*)$ and computation of $(**)$ we have a relative error of 0.177%.
We define in the following a confidence interval for $\mathbb{E}[e^{uX_T}]$.
A Monte Carlo estimator of $\mathbb{E}[e^{uX_T}]$ is $\hat{I}_n = \frac{1}{n}\sum_{k=1}^n e^{uX_T^{(k)}}$, where $X_T^{(k)},\ k = 1, \cdots, n$ are i.i.d copies of the random variable X_T. By the law of large numbers (LLN), we have:

$$\frac{1}{n}\sum_{k=1}^n e^{uX_T^{(k)}} \xrightarrow{a.s} \mathbb{E}[e^{uX_T}]$$

Also, by the Central Limit Theorem (CLT):

$$\frac{1}{n}\sum_{k=1}^n e^{uX_T^{(k)}} \xrightarrow{\mathcal{L}} \mathcal{N}\left(\mathbb{E}[e^{uX_T}], \frac{1}{n}(\mathbb{E}[e^{2uX_T}] - \mathbb{E}[e^{uX_T}]^2)\right)$$

Let's denote by:

$$\bar{X}_n = \frac{1}{n}\sum_{k=1}^n e^{uX_T^{(k)}}, \quad \mu = \mathbb{E}[e^{uX_T}] \quad \text{and} \quad V = \frac{1}{n}(\mathbb{E}[e^{2uX_T}] - \mathbb{E}[e^{uX_T}]^2)$$

We have then:

$$\frac{\bar{X}_n - \mu}{\sqrt{V}} \sim \mathcal{N}(0,1)$$

So a confidence interval of $\mu = \mathbb{E}[e^{uX_T}]$ is given by:

$$I_{95\%}(\mu) = \left[\bar{X}_n - 1.96\sqrt{V}, \bar{X}_n + 1.96\sqrt{V}\right]$$

In the following, the integer n corresponds to the number of points used in the discretization to solve the Riccati differential equation.

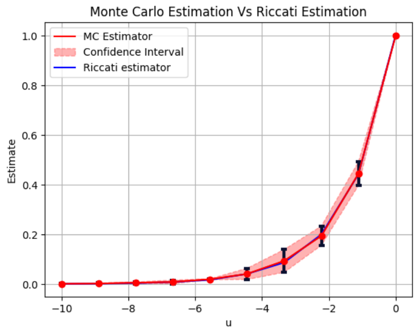

Figure 6: Comparison for n = 10 (Scale : X5)

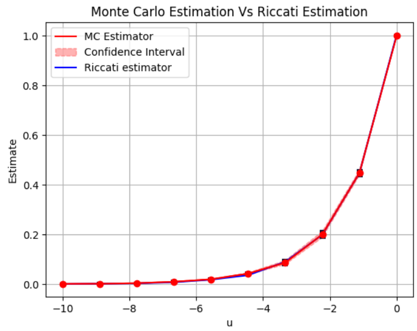

Figure 7: Comparison for n = 10 (Normal scale)

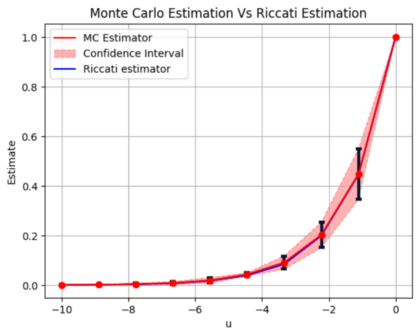

Figure 8: Comparison for n = 100

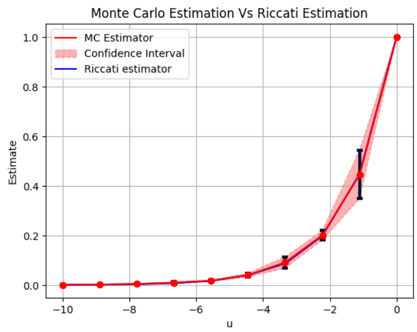

Figure 9: Comparison for n = 300

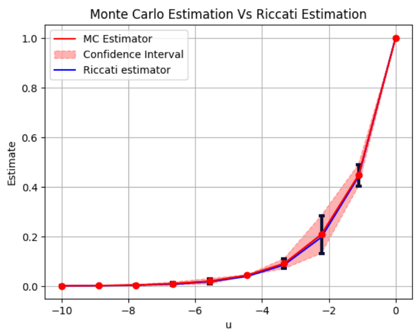

Figure 10: Comparison for n = 500

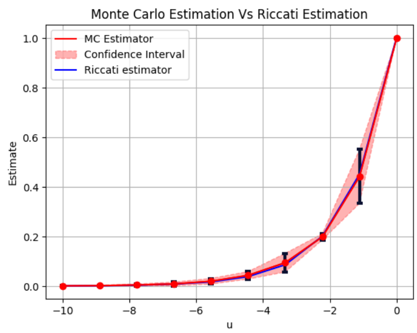

Figure 11: Comparison for n = 1000

Remark 3.2. The width of the confidence interval has been multiplied by a factor of 5 in order for them to appear in the graph and you can see the normal scale on the right plot.

Comments : We can see from those graphs that as n increases, the values of u where the confidence intervals where large decrease. In a nutschell, there is a strong match between the Monte Carlo estimator and the characteristic function that uses the solution of the Riccati differential equation.

3.3 Further discussion

In this project, we understand the boundary behaviour of the path dependent process when $2a \geq= X_t$, still it remains interesting problems to be solved, for example :

1. What will happen if the Feller condition is violated? Do we still have: for $2a < \sigma^2$ and $\forall r \geq 0, P\left(\cap_{t>r}\{X_t > 0\}\right) = 0$?

2. What are $\limsup_{n\to\infty} X_t$ and $\liminf_{n\to\infty} X_t$?

3. Can we show that the boundary 0 is instantaneously reflecting, as the classical CIR process? (Which means $L_t^0(X) = 0$ $a.s.$ and here L^0 represents the local time at 0.)

For the first question, our numerical experiments support it, but we have not found a theoretical proof. An approach could involve mirroring the proof strategy employed for the squared Bessel process as delineated in theorem 2.7. In that proof we need to find a function f such that $f(u_t)$ is a continuous local martingale. However, as the path dependent CIR process is non-markovian, we can prove that, in general, there does not exist a deterministic function $F(t,x)$, such that $F(t,X_t)$ is a continuous local martingale. It might possible to find a function $F : \mathbb{R}_+ \times C(\mathbb{R}_+,\mathbb{R}) \to \mathbb{R}$ such that $F(t, X_{s:0\leq s\leq t})$ is a continuous martingale, but still it will involve more profound study.

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