Useful information

$$[A, BC] = [A, B]C + B[A, C]$$

$$[AB, C] = A[B, C] + [A, C]B$$

$$e^{\lambda A} B e^{-\lambda A} = B + \lambda[A, B] + \frac{\lambda^2}{2}[A, [A, B]] + \dots$$

$$e^X e^Y = e^{X+Y+Z}$$

$$Z = \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y[X, Y]] + \dots$$

when we have a function of an operator A, we have to expand it as

$$f(A) = \sum_n f(a_n)|n\rangle\langle n|$$

where a_n is the eigenvalue and $|n\rangle$ is the eigenvectors. Trace is defined as

$$Tr(A) = \sum_k \langle k|A|k\rangle$$

where $|k\rangle$ is a complete orthonormal basis. Further the trace is invariant of basis, and can therefore be thought of as the sum of the eigenvalues

$$d\Omega = d\phi dcos\theta, \phi \in \{0, 2\pi\}, cos\theta \in \{-1, 1\}$$

Relations

$$\sigma_z|+\rangle = |+\rangle$$
$$\sigma_z|-\rangle = -|-\rangle$$
$$\sigma_x|+\rangle = |-\rangle$$
$$\sigma_x|-\rangle = |+\rangle$$
$$\sigma_y|+\rangle = i|-\rangle$$
$$\sigma_y|-\rangle = -i|+\rangle$$
$$\sigma_\pm = \sigma_x \pm i\sigma_y$$

Harmonic Oscillator

For a HO we have that

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a)$$

$$p = i\sqrt{\frac{\hbar}{2m\omega}}(a^\dagger - a)$$

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, a|n\rangle = \sqrt{n}|n-1\rangle$$

HO an coherent states

$$D(z) = e^{za^\dagger - z^* a}$$

Displacement operator in phase space

$$D(z)^\dagger a D(z) = a + z$$
$$D(z)^\dagger a^\dagger D(z) = a^\dagger + z^*$$
$$|z\rangle = D(z)|0\rangle$$
$$D(\beta)D(\alpha) = e^{\frac{\beta\alpha^* - \beta^*\alpha}{2}} D(\alpha + \beta)$$
$$= e^{\beta\alpha^* - \beta^*\alpha} D(\alpha)D(\beta)$$
$$D(z)^\dagger x D(z) = x + x_c$$
$$D(z)^\dagger p D(z) = p + p_c$$

$$\psi_z(x) = \langle x|z\rangle \longrightarrow \text{coherent state}$$

Schrodinger eq. and Heisenberg

$$i\hbar\frac{d|\psi(t)\rangle}{dt} = H|\psi(t)\rangle$$

$$\frac{d}{dt}A_H(t) = \frac{i}{\hbar}[H(t), A_H(t)] + \left(\frac{\partial A_S}{\partial t}\right)_H$$

$$A_H(t) = U^\dagger(t) A_S U(t)$$

Pauli matrices

$$\sigma_j = \begin{bmatrix} \delta_{3j} & \delta_{1j} - i\delta_{2j} \\ \delta_{1j} + i\delta_{2j} & -\delta_{3j} \end{bmatrix}$$

conmutator and anti

$$[\sigma_i, \sigma_j] = 2i\sum_k \epsilon_{ijk}\sigma_k, \{\sigma_j, \sigma_k\} = 2\delta_{jk} I$$

Rotated Pauli matrix

$$n \cdot \sigma = \begin{bmatrix} cos\theta & e^{-i\phi}sin\theta \\ e^{i\phi}sin\theta & -cos\theta \end{bmatrix}$$

eigenvalue ± 1, with eigenvectors

$$\psi_n = \begin{bmatrix} cos\frac{\theta}{2} \\ e^{i\phi}sin\frac{\theta}{2} \end{bmatrix}, \psi_{-n} = \begin{bmatrix} sin\frac{\theta}{2} \\ -e^{i\phi}cos\frac{\theta}{2} \end{bmatrix}$$

$$\langle\psi_n|\sigma_n|\psi_n\rangle = n = (cos\phi sin\theta, sin\phi sin\theta, cos\theta)$$

$$e^{-\frac{i}{2}\alpha\sigma_n} = I cos\frac{\alpha}{2} - i\sigma_n sin\frac{\alpha}{2}$$

$$e^{-\frac{i}{2}\alpha\sigma_z}\sigma_x e^{\frac{i}{2}\alpha\sigma_z} = cos\alpha\sigma_x - isin\alpha\sigma_n$$

Density matrix

Density operator for a mixed state is

$$\rho = \sum_{k=1}^{n} p_k|\psi_k\rangle\langle\psi_k|, 0 < p_k < 1$$

$$\rho = \sum_{kl} p_{kl}\rho_k^A \otimes \rho_l^B$$

when $p_{kl} = 1$ are separable states (classical correlations).

$$\rho = \rho_A \otimes \rho_B \longrightarrow \langle AB\rangle = \langle A\rangle\langle B\rangle$$

where p_k are classical probabilities and $|\psi_k\rangle$ are the corresponding states, if it is a pure state the density matrix is

$$\rho = |\psi\rangle\langle\psi|$$

if $\rho^2 = \rho$ we have a pure state, if not it is mixed. The eigenvalues of a density matrix can be interpreted as the probabilities associated with the eigenvectors. Expectation values are found by

$$\langle A\rangle = Tr(\rho A)$$

If you diagonalize the density matrix you get the probabilities for the eigenvalues, NB these may not be the original states. The density matrix evolves in time as

$$i\hbar\frac{\partial}{\partial t}\rho = [H, \rho]$$

We can further find the reduced density operator for a subsystem by taking the partial trace of the other subsystems, e.g.

$$\rho_A = Tr_B(\rho)$$

$$\rho_B = Tr_A(\rho)$$

Reduced density matrices often correspond to mixed state of the subsystem, even if the original system was pure.
Any density matrix for a 2D system can be written as

$$\rho = \frac{1}{2}(I + n \cdot \sigma) = \frac{1}{2}\begin{bmatrix} 1 + r_z & r_x - ir_y \\ r_x + ir_y & 1 - r_z \end{bmatrix}$$

where $\vec{n} = (r_x, r_y, r_y)$ is the Bloch vector. If $\vec{n}^2 = 1$ then the state is pure, if $\vec{n}^2 < 1$ then it is a mixed state. Eigenvalue are

$$p_\pm = \frac{1 \pm |n|}{2}$$

The density matrix is hermitian $\rho_\dagger = \rho$, has positive semi-definite eigenvalues and has

$$\sum_k p_k = 1 \rightarrow Tr(\rho) = 1$$

Entanglement and entropy

The mixing entropy and entanglement entropy are both found in the same way, the mixing entropy is how pure the system is, a pure state has mixing entropy 0. The maximum amount of entropy for a n dimensional Hilbert space is

$$S_{max} = log(n)$$

$$S = -Tr(\rho log\rho)$$

$$S = -\sum_k p_k log(p_k)$$

where p_k are the eigenvalues of the density matrix, the probabilities for different states.

Entanglement entropy

Entanglement entropy is defined by the eigenvalues of the partial trace, for two entangled systems $A \otimes B$

$$S_A = S_B = -\sum_n |d_n|^2 log(|d_n|^2)$$

where $|d_n|^2$ are the eigenstates of one of the reduced density operators. Non-entangled systems have entanglement entropy 0. Maximal entanglement

$$S_{max} = ln(min\{n_A, n_B\})$$

Entropy

$$S = -Tr(\rho log\rho)$$

Von Neumann entropy measure the freedom of entanglement of a system.
Terms of eigenvalues

$$S = -\sum_k p_k log p_k, \text{monotonic function}$$

*For mixed states the entropy measure how far the state is from being pure.
*Entropy increases when the probabilities get distributed over many states.
*Maximal entropy state exist where all states are equally probable.

$$\rho_{max} = \frac{1}{n}\mathbb{1} \longrightarrow S_{max} = logn$$

Temperature dependent density operator
Thermal state $\longrightarrow$ mixed state

$$\rho = Ne^{-\beta H} = N\sum_k e^{-\beta E_k}|\psi_k\rangle\langle\psi_k|$$

$$N^{-1} = Tre^{-\beta H} = \sum_k e^{-\beta E_k}$$

Schmidt decomposition
For a composite Hilbert space of two other spaces there exist a representation known as the Schmidt decomposition, it can be written as

$$|\psi\rangle = \sum_n d_n|n\rangle_A \otimes |n\rangle_B$$

where d_n is the root of the eigenvalues of one of the reduced density matrix, $|n\rangle_A$ are the eigenstates of the reduced density operator ρ_A and likewise $|n\rangle_B$ are the eigenstates of the reduced density operator ρ_B. The method of finding the Schmidt decomposition is finding d_n and either ρ_A or the other basis and then massaging the original state into this form. As the eigenvalues cant tell us about the relative phase.
If $n = 1$, then the state ψ is not entangled.

Quantum computing

All quantum gates must be reversable.
Hadamard gate

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}, |0\rangle = \begin{bmatrix}1\\0\end{bmatrix}, |1\rangle = \begin{bmatrix}0\\1\end{bmatrix}$$

$$H^2 = I$$

CNOT gate
If the first bit is the control bit and the second is the target bit then the CNOT gate acts as

$$|00\rangle \overset{CNOT}{\longrightarrow} |00\rangle$$
$$|01\rangle \longrightarrow |01\rangle$$
$$|10\rangle \longrightarrow |11\rangle$$
$$|11\rangle \longrightarrow |10\rangle$$

Tensor product and red.den. op.

The tensor product between two vectors

$$|0\rangle = \begin{bmatrix}\alpha\\\beta\end{bmatrix}, |1\rangle = \begin{bmatrix}a\\b\end{bmatrix}$$

is

$$|0\rangle \otimes |1\rangle = \begin{bmatrix}\alpha|1\rangle\\\beta|1\rangle\end{bmatrix} = \begin{bmatrix}\alpha a\\\alpha b\\\beta a\\\beta b\end{bmatrix}$$

The tensor product between two matrices is just the matrix of each matrix element of the first matrix multiplied with the whole second matrix.
For a system of $|0A\rangle \otimes |B\rangle$ where A and B are 2D Hilbert space we have that the two reduced density matrix are

$$\rho_A = Tr_B\begin{bmatrix}a & b & c & d\\e & f & g & h\\i & j & k & l\\m & n & o & p\end{bmatrix} = \begin{bmatrix}a+f & c+h\\i+n & k+p\end{bmatrix}$$

$$\rho_B = Tr_A\begin{bmatrix}a & b & c & d\\e & f & g & h\\i & j & k & l\\m & n & o & p\end{bmatrix} = \begin{bmatrix}a+k & b+l\\e+o & f+p\end{bmatrix}$$

Linblad equation

The Linblad equation describes open quantum systems, where a system interacts with a large heat reservoir, it is

$$\frac{d\rho}{dt} = -\frac{i}{\hbar}[H,\rho] - \frac{1}{2}\sum_i \gamma_i(L_i^\dagger L_i\rho + \rho L_i^\dagger L_i - 2L_i\rho L_i^\dagger)$$

*L_i are the Lindblad operators, taking us between states
*γ_i is the transition rate
* $H = H_S + H_{LS}$ so it is the sum of the system and the larger system.

Nuclear and photon physics

For any vector $\vec{b}$ the following relation is true

$$\sum_a |\vec{\epsilon}_{\vec{k}a}\cdot\vec{b}|^2 = |\vec{b}|^2 - |\vec{b}\cdot\frac{\vec{k}}{|\vec{k}|}|^2$$

Complicated vector products can be found on page 62 in Rottmann. When faced with something of the form

$$(\vec{k}\times\vec{\epsilon}_{\vec{k}a})\cdot\vec{m}$$

a handy relation is

$$a\cdot(b\times c) = b\cdot(c\times a) = c\cdot(a\times b)$$

Also remember $\vec{\epsilon}_{\vec{k}1} \perp \vec{\epsilon}_{\vec{k}2} \perp \vec{k}$ where $\vec{\epsilon}_{\vec{k}a}$ are photon polarization and $\vec{k}$ is the propagation vector. The polarization vectors are unit vectors which satisfy the relation $\vec{\epsilon}_{\vec{k}a}\cdot\vec{\epsilon}_{\vec{k}a'} = \delta_{aa'}$.

The quantised electric field and vector fields satisfy the commutator relation

$$[E^\dagger_{\vec{k}a}, A_{\vec{k}'b}] = -\frac{1}{c}[\dot{A}_{\vec{k}'a}, A_{\vec{k}'b}] = i\frac{\hbar}{\epsilon_0}\delta_{\vec{k}\vec{k}'}\delta_{ab}$$

$$[a_{\vec{k}a}, a^\dagger_{\vec{k}'b}] = \delta_{\vec{k}\vec{k}'}\delta_{ab}$$

$$[a_{\vec{k}a}, a_{\vec{k}'b}] = 0$$

The electric and vector fields can be expressed as

$$A_{\vec{k}a} = \sqrt{\frac{\hbar}{2\omega_k\epsilon_0}}(a_{\vec{k}a} + a^\dagger_{\vec{k}a})$$

$$E_{\vec{k}a} = i\sqrt{\frac{\omega_k\hbar}{2\epsilon_0}}(a_{\vec{k}a} - a^\dagger_{\vec{k}a})$$

If given the Heisenberg picture vector potential, the electric and magnetic fields can be found by

$$E(r,t) = -\frac{\partial}{\partial t}A(r,t)$$

$$B(r,t) = \nabla\times A(r,t)$$

NI when working with the atomic and photonic states remember which tensor the operators acts on!. To calculate the probability for a trasition, we must take the sum of polarizations of the absolute squared matrix element in question.
Stimulated emission
Correlation function for spin flip radion

$$\langle B, 1_{k,a}|H_1|A,0\rangle = \frac{ie\hbar}{2m}\sqrt{\frac{\hbar}{2V\epsilon\omega}}(k\times\epsilon_{ka})\cdot\sigma_{BA}$$

where

$$\sigma_{BA} = \langle B, 1_{k,a}|H_1|A,0\rangle (= \langle\downarrow|\sigma|\uparrow\rangle)$$

Fermi's golden rule

$$p(\phi,\theta) = N|\langle B, 1_{ka}|H_1|A,0\rangle|^2$$

Normalise over solid angle to obtain N.
Bloch vector $\longrightarrow$ intial state $|\psi\rangle$ has Bloch vector $m^{(0)}$, the sphere of states for two-level system is the Bloch sphere

$$\rho_0 = |\psi\rangle\langle\psi| = \frac{1}{2}(\mathbb{1} + m_i^{(0)}\sigma_i)$$

For two-level system

$$\rho = \frac{1}{2}(\mathbb{1} + m_i^{(0)}\vec{r}\cdot\vec{\sigma}) \longrightarrow \text{mixed states}$$

$$\rho = \frac{1}{2}(\mathbb{1} + m_i^{(0)}\vec{n}\cdot\vec{\sigma}) \longrightarrow \text{pure states}$$

Ensemble $\longrightarrow$ mixed states

$$\langle A\rangle = \sum_{k=1}^n p_k\langle A\rangle_k = \sum_{k=1}^n p_k\langle\psi_k|A|\psi_k\rangle$$