Quantum Computing

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(Dated: April 9, 2024)

The aim of this project is study the fundamentals concepts as well as analytical and numerical tools for understanding of the quantum computing. Herein we use Qiskit package like a useful tool, to learn building circuits, use optimizers and other modules. Besides we introduce in Lipkin model to perform calculations for $N = 2$ and $N = 4$ particles using standard eigenvalue solver and VQE method for both cases. **Each section is dedicated to a project part**.

To introduce us to this quantum world, a first step is to know a very important definition: **Qubit**. Qubit is a basic unit of the quantum information, we can say colloquially that "It is the quantum version of a bit".

Unlike bits, which it can have two different values: 0 and 1; qubits can have this values, but also can have mixed values between the both above values. Mixed values means the qubits can be in superposition. In quantum mechanics this values are states, represented as $|0\rangle$ or $|1\rangle$, they are known as **computational basis**. The superposition state is given by:

$$|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle \tag{1}$$

where α and β are the complex amplitudes. This idea is much better to see in the Bloch sphere.

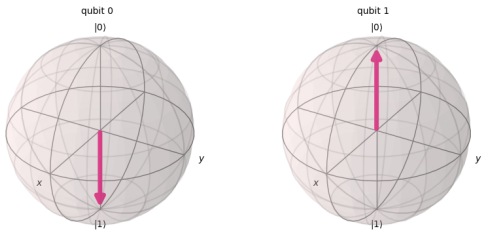

FIG. 1: Bloch sphere for the computational basis

On this representative scheme (FIG. 1), we can see that our two basis $|0\rangle$ and $|1\rangle$ are in the poles of the sphere. Using the concept of the Bloch sphere, we can use the state vector to describe the points on the surface, which represent pure qubit states. Mathematically the computational basis are:

$$|0\rangle = \begin{bmatrix}1\\0\end{bmatrix}, \qquad |1\rangle = \begin{bmatrix}0\\1\end{bmatrix} \tag{2}$$

I. THEORETICAL FRAMEWORK

This section is dedicated to the **Part a)**. Information about codes and procedures in detail, you can find in Github - Part a.

Following through this quantum world and seeking to understand quantum computers from the scratch. It is important to know how build this kind of computers using the quantum mechanics and circuits.

Quantum gates play a very important role in building circuits, since they allow us to manipulate and control the qubits. Unlike classical gates, the quantum gates are powerful since are reversibles.

Within this set of quantum gates, we can find the Pauli matrices, Hardamard, Phase, T, CNOT and any more [1]. Now we will discuss about this quantum gates and how they are applied mathematically on the computational basis.

A. Pauli X matrix

Pauli matrices σ_x, σ_y and σ_z are usually denoted by X, Y and Z, so we follow this last notation. Pauli X matrix is given by:

$$X = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \tag{3}$$

If we apply on the wave function from the equation (1), we get:

$$X|\Psi\rangle = \beta|0\rangle + \alpha|1\rangle \tag{4}$$

this gate switch the amplitudes. When $\alpha = \beta$, we can say that wave function is same.

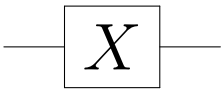

FIG. 2: Schematic representation of Pauli X gate

B. Pauli Y matrix

$$Y = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix} \tag{5}$$

Applying:

$$Y|\Psi\rangle = -i(\beta|0\rangle - \alpha|1\rangle) \tag{6}$$

this gate introduce the imaginary unit and switch the amplitudes.

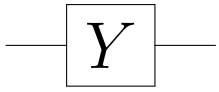

FIG. 3: Schematic representation of Pauli Y gate

C. Pauli Z matrix

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{7}$$

Applying:

$$Z|\Psi\rangle = \alpha|0\rangle - \beta|1\rangle \tag{8}$$

Only swicht this sign.

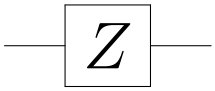

FIG. 4: Schematic representation of Pauli Z gate

D. Hardamard

Hadamard gate is a single qubit operation

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{9}$$

Applying:

$$H|\Psi\rangle = \alpha\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) + \beta\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{10}$$

this gate create a superposition with the above states, which are define by:

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{11}$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{12}$$

FIG. 5: Schematic representation of Hardamard gate

E. Z90 or S

Represent a 90-degree rotation around the z axis

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \tag{13}$$

Applying:

$$S|\Psi\rangle = \alpha|0\rangle + i\beta|1\rangle \tag{14}$$

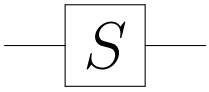

FIG. 6: Schematic representation of S gate

F. T($\pi/8$)

It is a single qubit operation given by:

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{bmatrix} \tag{15}$$

Applying:

$$T|\Psi\rangle = \alpha|0\rangle + \frac{1}{\sqrt{2}}\beta(1-i)|1\rangle \tag{16}$$

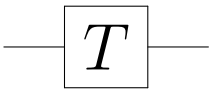

FIG. 7: Schematic representation of T gate

G. CNOT

It is two qubit operation. The first qubit is a control qubit and the second qubit is a target qubit.

$$CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \tag{17}$$

This gate work like this:

$$\begin{aligned} |0\rangle \otimes |0\rangle &\longrightarrow |00\rangle \\ |0\rangle \otimes |1\rangle &\longrightarrow |01\rangle \\ |1\rangle \otimes |0\rangle &\longrightarrow |11\rangle \\ |1\rangle \otimes |1\rangle &\longrightarrow |10\rangle \end{aligned}$$

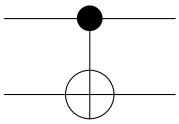

FIG. 8: Schematic representation of CNOT gate

H. Rotational $R_x(\theta)$

$$R_x(\theta) = \cos(\frac{\theta}{2})I - i\sin(\frac{\theta}{2})X \tag{18}$$

$$= \begin{bmatrix} \cos(\frac{\theta}{2}) & -i\sin(\frac{\theta}{2}) \\ -i\sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{bmatrix} \tag{19}$$

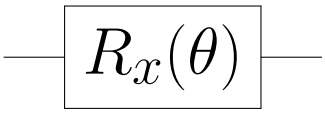

FIG. 9: Schematic representation of $R_x(\theta)$ gate

I. Rotational $R_y(\theta)$

$$R_y(\theta) = \cos(\frac{\theta}{2})I - i\sin(\frac{\theta}{2})Y \tag{20}$$

$$= \begin{bmatrix} \cos(\frac{\theta}{2}) & -\sin(\frac{\theta}{2}) \\ \sin(\frac{\theta}{2}) & \cos(\frac{\theta}{2}) \end{bmatrix} \tag{21}$$

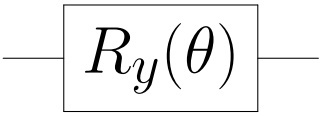

FIG. 10: Schematic representation of $R_y(\theta)$ gate

J. Rotational $R_z(\theta)$

$$R_z(\theta) = \cos(\frac{\theta}{2})I - i\sin(\frac{\theta}{2})Z \tag{22}$$

$$= \begin{bmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{bmatrix} \tag{23}$$

II. BELL STATES

This section is dedicated to the **Part a)**. Information about codes and procedures in detail, you can find in Github - Part a.

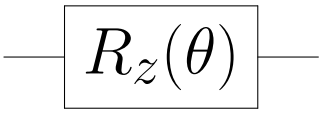

FIG. 11: Schematic representation of $R_z(\theta)$ gate

The Bell basis is a set of four two qubit wave functions, they are examples of quantum entanglement. Representated by:

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\left[|10\rangle + |01\rangle\right] \tag{24}$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left[|10\rangle - |01\rangle\right] \tag{25}$$

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left[|00\rangle + |11\rangle\right] \tag{26}$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}\left[|00\rangle - |11\rangle\right] \tag{27}$$

A. Measurement on $|\Phi^+\rangle$

Setting up the circuit (FIG. 12)

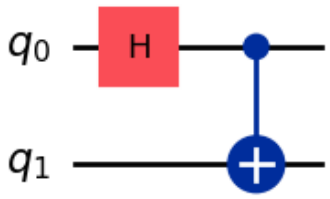

FIG. 12: Circuit for Bell state $|\Phi^+\rangle$

where $q_0 = |0\rangle_A$ and $q_1 = |0\rangle_B$
Steps:

- First step: Hardamard gate

$$H|0\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \tag{28}$$

- Second step: CNOT gate

$$(H|0\rangle)\otimes|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\otimes|0\rangle = \frac{1}{\sqrt{2}}(|00\rangle+|11\rangle) = |\Phi^+\rangle \tag{29}$$

We can see that the circuit is the correct for this state. Now we can measure (FIG. 13), one important thing is about the classical registers, they play a important role because collect the information of circuit. That means, the quantum computations depends on a classical computer.

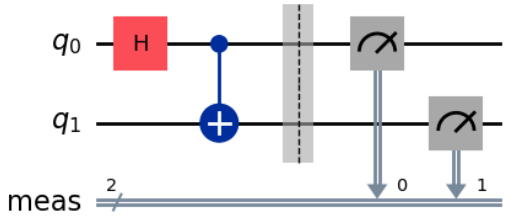

FIG. 13: Circuit for Bell state $|\Phi^+\rangle$ prepared for the measure

For make a correct measurement, we must performed several times the calculations, this will ensure accurate and stability in the results.

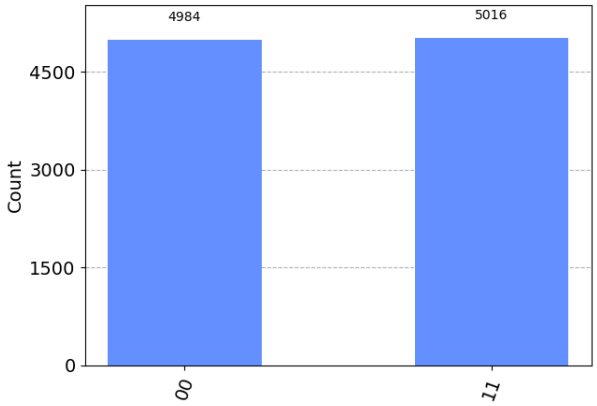

FIG. 14: Results for Bell state $|\Phi^+\rangle$

This histogram (FIG. 14) indicate that the probability to find either $|00\rangle$ or $|11\rangle$ is equal to 50%.

We also have created other code using the probabilities to find each qubit. Following this approach, we can generate random number in some interval, there the range is from 0 to 1, since we consider the probabilities of finding the qubits. You can check the created code in Github - Report.

III. STUDYING HAMILTONIANS

This section is dedicated to the **Part b)**. Information about codes and procedures in detail, you can find in Github - Part b.

Hamiltonians are a good way to implement more realistic cases, because we can use the perturbational theory. For some system we can define the hamiltonian by $H = H_0 + H_I$. Where:

$$H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}, \tag{30}$$

$$H_0 = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}, \tag{31}$$

$$H_I = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}, \tag{32}$$

where V_{ij} represent various interaction matrix elements. We can view H_0 as the non-interacting solution

$$H_0|0\rangle = E_1|0\rangle \tag{33}$$

$$H_0|1\rangle = E_2|1\rangle \tag{34}$$

where we have defined the orthogonal computational one-qubit basis states $|0\rangle$ and $|1\rangle$.

We rewrite H via Pauli matrices

$$H_0 = \mathcal{E}I + \Omega Z, \mathcal{E} = \frac{E_1 + E_2}{2},\ \Omega = \frac{E_1 - E_2}{2} \tag{35}$$

therefore

$$H_I = cI + \omega_z Z + \omega_x X, \tag{36}$$

with $c = (V_{11} + V_{22})/2$, $\omega_z = (V_{11} - V_{22})/2$ and $\omega_x = V_{12} = V_{21}$. We let our Hamiltonian depend linearly on a strength parameter λ

$$H = H_0 + \lambda H_{\mathrm{I}}, \tag{37}$$

with $\lambda \in [0, 1]$.

For study this problem, we must to use standard eigenvalue solvers. That means, Use the hamiltonian for diagonalize it and calculate the eigenvalues

$$|H - IE| = 0 \tag{38}$$

$$\begin{vmatrix} E_1 + \lambda V_{11} - E & \lambda V_{12} \\ \lambda V_{21} & E_2 + \lambda V_{22} - E \end{vmatrix} = 0 \tag{39}$$

solving and using some conditions $V_{12} = V_{21} = V$ and $V_{11} = -V_{22} = \delta$

$$\begin{vmatrix} E_1 + \lambda\delta - E & \lambda V \\ \lambda V & E_2 - \lambda\delta - E \end{vmatrix} = 0 \tag{40}$$

follow the next conditions: $\delta = 2$, $V = 0.2$, $E_1 = 0$ and $E_2 = 4$. We can write the solutions as function of λ

$$E(\lambda) = 2 \pm \sqrt{9.04\lambda^2 - 12\lambda + 4} \tag{41}$$

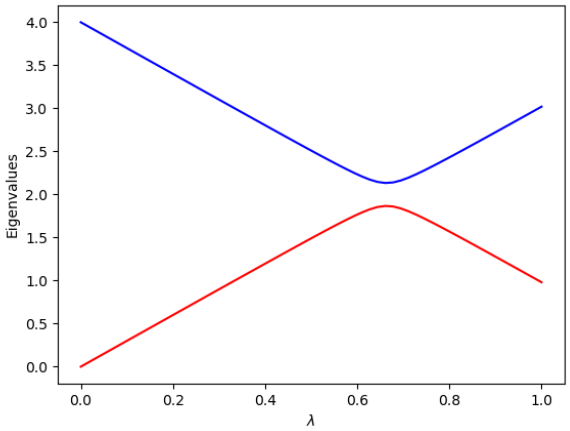

FIG. 15: Eigenvalues as function of the interaction strength λ

From FIG. 15, When we increase λ, the eigenvalues are influenced by interaction (perturbed). In the case for the qubit $|0\rangle$ the eigenvalues also increase, but for the qubit $|1\rangle$ decrease. The breaking point is when λ is equal to 0.67, there the interaction reach the maximal point, however the states don't entanglement.
Herein the minimal expectation value is $E = 0.9802$, when $\lambda = 1$.

IV. VARIATIONAL QUANTUM EIGENSOLVER (VQE)

This section is dedicated to the **Part c)**. Information about codes and procedures in detail, you can find in Github - Part c.

VQE is a hybrid algorithm that variationally calculate the ground state energy of a hamiltonian [2]. Hybrid because use classical optimization techniques. We can use Qiskit for plot the quantum circuit with the gates. For this case we need to select some ansatz (parameterized quantum circuit), which get information global and give us a good expectation value of the hamiltonian. The idea is to go through all the points of the Bloch sphere, optimizate values and minimize the expectation value.

Now we're going to solve the hamiltonian from equation (37) using the VQE. For this case a good ansatz must include rotationals gates:

$$|\Psi\rangle = R_y(\phi)R_x(\theta)|0\rangle \tag{42}$$

To work with VQE, we must to modify the Hamiltonian. Modify or rewrite via Pauli matrices and identity matrix. Energy, using the ansatz

$$E = \langle\Psi|H|\Psi\rangle = \langle 0|R_x(\theta)R_y(\phi)HR_y(\phi)R_x(\theta)|0\rangle \tag{43}$$

Writing the hamiltonian via Pauli matrices

$$H = 2I + Z - 0.2X \tag{44}$$

Setting up the circuit (FIG. 16), we have used the module TwoLocal from qiskit with COBYLA optimizer.

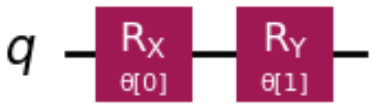

FIG. 16: Circuit for one qubit

with this calculation we found the expectation value $E = 0.9802$. If we compare with the above value, the value for the energy is similar when $\lambda = 1$, so this method provides exact information about the ground state energy. It is also possible get information about the angles. We can use this angles (optimal parameters) to measurement the circuit (FIG. 17), set up the circuit with the classical registers.

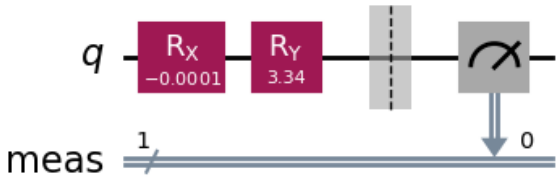

FIG. 17: Measurement for the circuit with one qubit

Keep in mind that each parameters are not unique, every time that run the code we will get a new parameters, but the same expectation value.

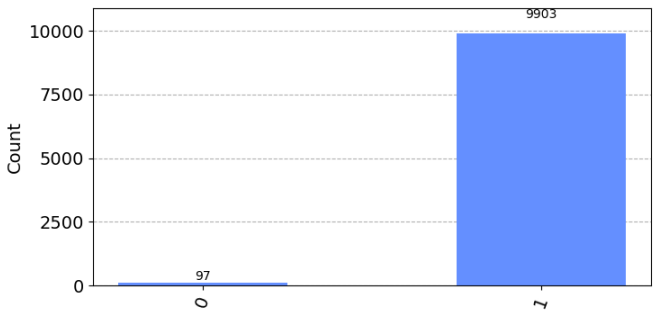

FIG. 18: Histogram

From FIG. 18, We can conclude that the probability to find the state $|1\rangle$ is much greater than $|0\rangle$, close to the

100 %.
We also have created other code using qiskit but define the wave function (ansatz) and the explicit calculation for the expectation value. You can check the created code in Github - Report.

V. TWO-QUBIT SYSTEM

This section is dedicated to the **Part d)**. Information about codes and procedures in detail, you can find in Github - Part d.

This system can be thought of as composed of two subsystems A and B. Each subsystem has computational basis states define by equation (2). The subsystems could represent single particles or composite many-particle systems of a given symmetry.

$$|00\rangle = |0\rangle_{\mathrm{A}} \otimes |0\rangle_{\mathrm{B}} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \tag{45}$$

$$|01\rangle = |0\rangle_{\mathrm{A}} \otimes |1\rangle_{\mathrm{B}} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \tag{46}$$

$$|10\rangle = |1\rangle_{\mathrm{A}} \otimes |0\rangle_{\mathrm{B}} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \tag{47}$$

$$|11\rangle = |1\rangle_{\mathrm{A}} \otimes |1\rangle_{\mathrm{B}} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \tag{48}$$

These computational basis states define also the eigenstates of the non-interacting Hamiltonian

$$H_0|00\rangle = \epsilon_{00}|00\rangle \tag{49}$$

$$H_0|10\rangle = \epsilon_{10}|10\rangle \tag{50}$$

$$H_0|01\rangle = \epsilon_{01}|01\rangle \tag{51}$$

$$H_0|11\rangle = \epsilon_{11}|11\rangle \tag{52}$$

The interacting part of the Hamiltonian H_{I} is given by the tensor product of two X and Z matrices, respectively, that is

$$H_{\mathrm{I}} = H_x X \otimes X + H_z Z \otimes Z \tag{53}$$

where H_x and H_z are interaction strength parameters. Our final Hamiltonian matrix is given by

$$H = \begin{bmatrix} \epsilon_{00} + H_z & 0 & 0 & H_x \\ 0 & \epsilon_{10} - H_z & H_x & 0 \\ 0 & H_x & \epsilon_{01} - H_z & 0 \\ H_x & 0 & 0 & \epsilon_{11} + H_z \end{bmatrix} \tag{54}$$

The values for the parameters are $H_z = 3.0$, $H_x = 2.0$, $\epsilon_{00} = 0.0$, $\epsilon_{10} = 2.5$, $\epsilon_{01} = 6.5$ and $\epsilon_{11} = 7.0$.

Calculating the eigenvalues using standard eigenvalue solvers as functions of the interaction strength λ.

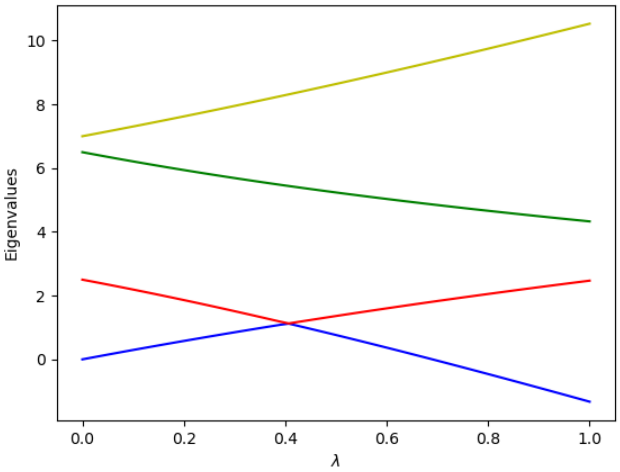

FIG. 19: Eigenvalues as function of the interaction strength λ for two qubits

From Fig. 19, we can see that for this hamiltonian is possible to reach the entanglement, which it is due to the increase in λ. From the FIG. 9, we can say that the entanglement occurs in all the states. If we change the values for ϵ_{00}, ϵ_{10}, ϵ_{01} and ϵ_{11}, by decreasing these values the entanglement is stronger. Herein the minimal expectation value is $E = -1.3284$, when $\lambda = 1$.
The four eigenstates of the above Hamiltonian matrix can in turn be used to define density matrices. As an example, the density matrix of the first eigenstate (lowest energy E_0) Ψ_0 is

$$\rho_0 = (\alpha_{00}|00\rangle\langle 00| + \alpha_{10}|10\rangle\langle 10| + \alpha_{01}|01\rangle\langle 01| + \alpha_{11}|11\rangle\langle 11|) \tag{55}$$

where using the eigenvector coefficients resulting from the solution of the above $\alpha_{00} = 0.2567$, $\alpha_{10} = 0.3827$, $\alpha_{01} = 0.9239$ and $\alpha_{11} = 0.9665$.
We can then in turn define the density matrix for the subsets A or B as

$$\rho_A = \mathrm{Tr}_B(\rho_0) = \langle 0|\rho_0|0\rangle_B + \langle 1|\rho_0|1\rangle_B \tag{56}$$

$$\rho_B = \mathrm{Tr}_A(\rho_0) = \langle 0|\rho_0|0\rangle_A + \langle 1|\rho_0|1\rangle_A \tag{57}$$

The density matrices for these subsets can be used to compute the so-called von Neumann entropy, which is one of the possible measures of entanglement. A pure state has entropy equal zero while entangled state have an entropy larger than zero. The von-Neumann entropy is defined as

$$S(A,B) = -\mathrm{Tr}\left(\rho_{A,B}\log_2(\rho_{A,B})\right) \tag{58}$$

In this system the Von-Neumann entropy for subsystem A is

$$S(A) = 0.5511 \tag{59}$$

VI. VQE FOR TWO-QUBIT SYSTEM

This section is dedicated to the **Part e)**. Information about codes and procedures in detail, you can find in Github - Part e.

In this section we are going to development the hamiltonian from the equation (53) using the VQE method. Herein we again to use the rotationals for define the ansatz. we rewriting the hamiltonian via Pauli matrices

$$H = \alpha I\otimes I+\beta I\otimes Z+\gamma Z\otimes I+(\delta+H_z)Z\otimes Z+H_xX\otimes X \tag{60}$$

we have used the same steps that the section III. Setting up the quantum circuit (Fig. 20).

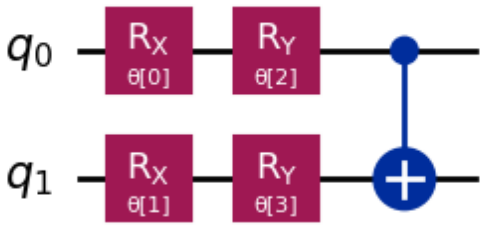

FIG. 20: Quantum circuit for a two-qubit system

We found the energy $E = -1.3284$. If we compare with the previous result, are same. We again use the optimal parameters for check what state is most probable to find, so add the classical registers (FIG. 21).

Analyzing the histogram (FIG. 22), we will find only two states $|01\rangle$ and $|10\rangle$, which $|10\rangle$ is most probably to find, close to 86 %.

We also have created other code using qiskit but define the wave function (ansatz) and the explicit calculation for the expectation value. You can check the created code in Github - Report.

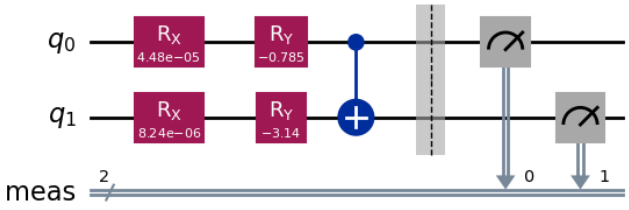

FIG. 21: Measurement for the quantum circuit for a two-qubit system

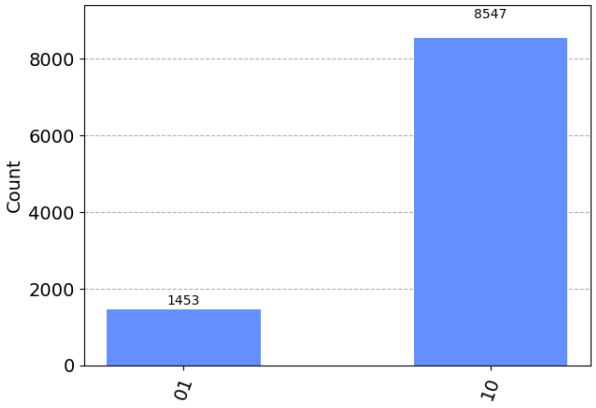

FIG. 22: Histogram for a two-qubit system

VII. LIPKIN MODEL

This section is dedicated to the **Part f)** and **Part g)**. Information about codes and procedures in detail, you can find in Github - Part f and Part g.

The main aim of this project is implement to Lipkin model [3], this model is versatil and good example to introduce us in this field. The Hamiltonian in second quantization is given by

$$H = H_0 + H_1 + H_2 \tag{61}$$

with

$$H_0 = \frac{1}{2}\varepsilon\sum_{p\sigma}\sigma a^\dagger_{p\sigma}a_{p\sigma} \tag{62}$$

$$H_1 = \frac{1}{2}V\sum_{p,p',\sigma}a^\dagger_{p\sigma}a^\dagger_{p'\sigma}a_{p'-\sigma}a_{p-\sigma} \tag{63}$$

$$H_2 = \frac{1}{2}W\sum_{p,p',\sigma}a^\dagger_{p\sigma}a^\dagger_{p'-\sigma}a_{p'\sigma}a_{p-\sigma} \tag{64}$$

We can rewrite this Hamiltonian in terms of the so-called quasispin operators leading to

$$H_0 = \varepsilon J_z \tag{65}$$

$$H_1 = \frac{1}{2}V\left(J_+^2 + J_-^2\right) \tag{66}$$

$$H_2 = \frac{1}{2}W\left(-N + J_+J_- + J_-J_+\right) \tag{67}$$

A. Case $J = 1$

We start here with a simpler case, namely the $J = 1$ case and we set $W = 0$. Show that Hamiltonian matrix is then given by

$$H_{J=1} = \begin{pmatrix} -\epsilon & 0 & -V \\ 0 & 0 & 0 \\ -V & 0 & \epsilon \end{pmatrix} \tag{68}$$

To solve the above hamiltonian problem on a quantum computer we need to rewrite the Hamiltonian in terms of the Pauli spin matrices, we also consider $N = 2$. Diagonalize the hamiltonian we can study the relation between eigenvalues and the strength of the interaction V while keeping the single-particle energies fixed. we fixed ϵ and vary V with random numbers.

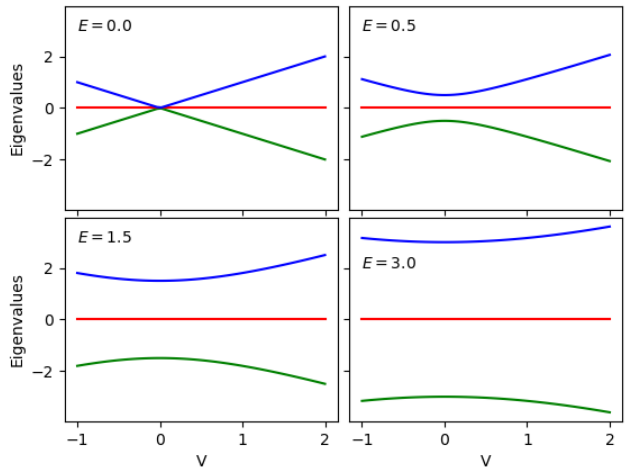

FIG. 23: Quantum circuit for a two-qubit system

In some papers report V/ϵ, but here we decided to work with ϵ independent because when change their values the entanglement appears. We report for four differents ϵ values.

Therefore from FIG. 23, the entanglement depends of how we define ϵ and the interaction strenght V. For example if we decrease the values for ϵ, the entanglement increase. When $\epsilon = 0$, the entenglament reach the maximal point. On the other hand, when we increase the values for ϵ, the entanglement between states disappears. Herein the minimal expectation value is $E = -1.1180$, when $\lambda = 1$.

Now we will work with the VQE for this hamiltonian. Writing the hamiltonian via Pauli matrices for $N = 2$ [4].

$$H = \frac{\epsilon}{2}\left(Z \otimes I + I \otimes Z\right) - \frac{V}{2}\left(X \otimes X - Y \otimes Y\right) \tag{69}$$

circuit for this case is same to the FIG. 10, since it is for two qubits, we again follow the same steps from section IV. Expectation value is $E = -1.1180$. We again use the optimal parameters to perform measurements, so setting up the circuit to perform measurements (FIG. 24).

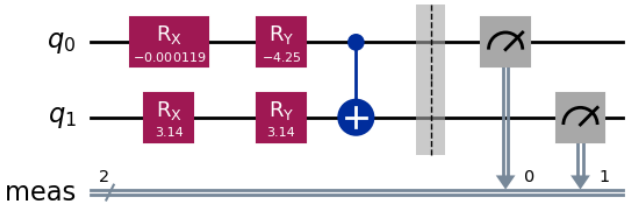

FIG. 24: Measurement for the quantum circuit for Lipkin Model, $N = 2$

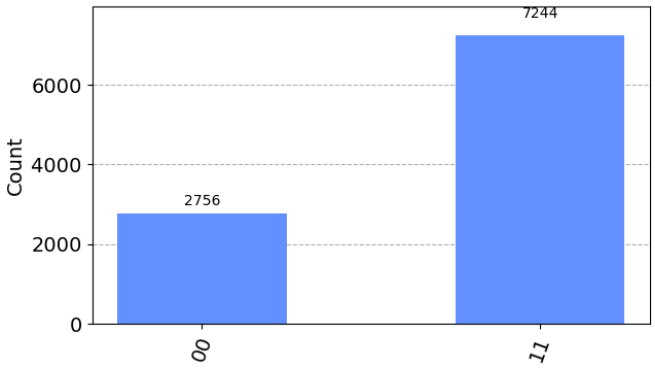

FIG. 25: Histogram for Lipkin Model, $N = 2$

Same to the previous calculation, we found only two states $|00\rangle$ and $|11\rangle$. We can see from FIG. 25 that the state $|11\rangle$ is most probably to find.

We also have created other code using qiskit but define the wave function (ansatz) and the explicit calculation for the expectation value. You can check the created code in Github - Report. We have used as inspiration [5].

B. Case $J = 2$

Matrix is given by

$$H_{J=2} = \begin{pmatrix} -2\epsilon & 0 & \sqrt{6}V & 0 & 0 \\ 0 & -\epsilon + 3W & 0 & 3V & 0 \\ \sqrt{6}V & 0 & 4W & 0 & \sqrt{6}V \\ 0 & 3V & 0 & \epsilon + 3W & 0 \\ 0 & 0 & \sqrt{6}V & 0 & 2\epsilon \end{pmatrix} \tag{70}$$

We again study the the relation between eigenvalues and the strength of the interaction V ($W = 0$) for $N = 4$.

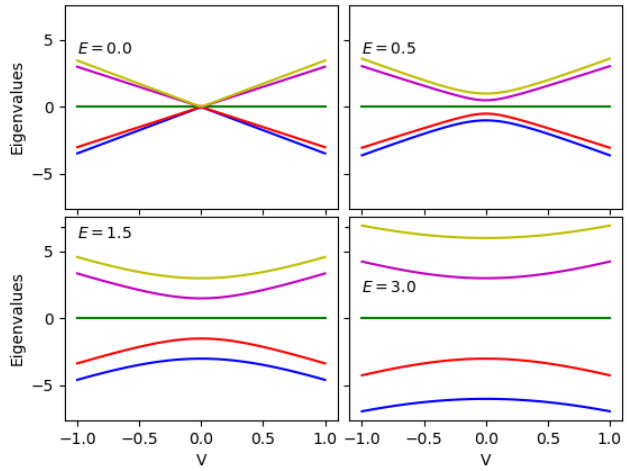

FIG. 26: Quantum circuit for a four-qubit system

From FIG. 26, Similarly to above case, the entanglement depends of how we define ϵ and the interaction strenght V. For example if we decrease the values for ϵ, the entanglement increase. When $\epsilon = 0$, the entenglament reach the maximal point. On the other hand, when we increase the values for ϵ, the entanglement between states disappears. Herein the minimal expectation value is $E = -3.6056$, when $\lambda = 1$.

Now we will work with the VQE for this hamiltonian. Writing the hamiltonian via Pauli matrices for $N = 4$.

$$H = \frac{\epsilon}{2}(Z\otimes I\otimes I\otimes I + I\otimes Z\otimes I\otimes I + I\otimes I\otimes Z\otimes I + I\otimes I\otimes I\otimes Z) - \frac{V}{2}(X\otimes X\otimes I\otimes I + X\otimes I\otimes X\otimes I + I\otimes X\otimes X\otimes I + I\otimes X\otimes I\otimes X + I\otimes I\otimes X\otimes X + X\otimes I\otimes I\otimes X) + \frac{V}{2}(Y\otimes Y\otimes I\otimes I + Y\otimes I\otimes Y\otimes I + I\otimes Y\otimes Y\otimes I + I\otimes Y\otimes I\otimes Y + I\otimes I\otimes Y\otimes Y + Y\otimes I\otimes I\otimes Y)$$

Now the circuit [6], we have 8 parameters to optimize (FIG. 27).

We again follow the same steps from section IV. Expectation value is $E = -3.2098$, there is a slight difference with the calculations using standard eigenvalue solver. Perform measurements (FIG. 28),

From the histogram (FIG. 29), we get several final states. The state $|1111\rangle$ is the most probably to find.

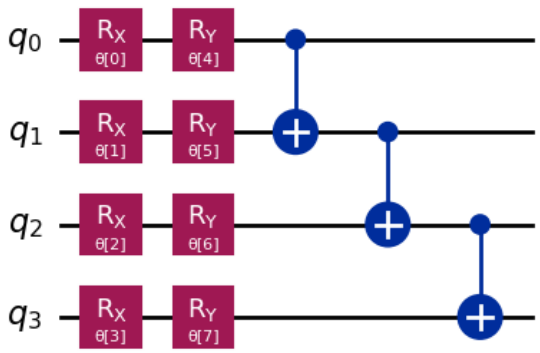

FIG. 27: Quantum circuit for Lipkin Model, $N = 4$

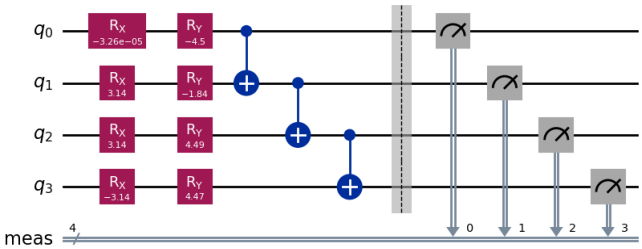

FIG. 28: Measurement for the quantum circuit for a four-qubit system

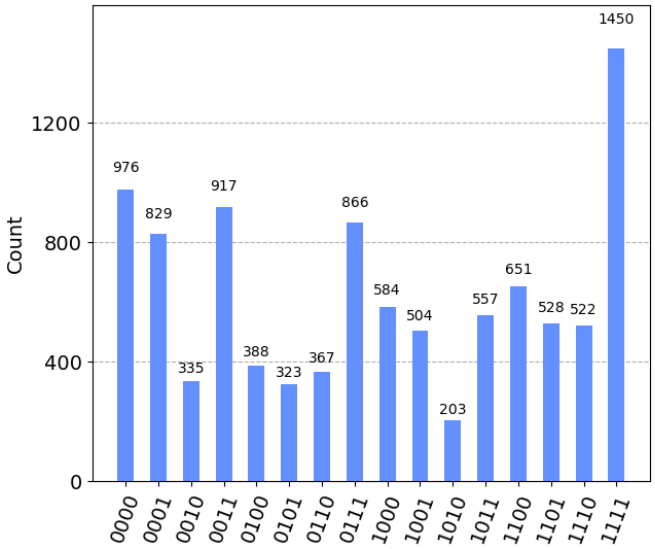

FIG. 29: Histogram for Lipkin Model, $N = 4$

VIII. CONCLUSION

VQE is a powerful tool for study quantum systems, in specific states like qubits in circuits. Mixed with the definition of Hamiltonian via Pauli matrices and a

good ansatz can find the lowest energy state either non-interaction or interacting systems. Besides qiskit provides a good support for the analysis. This work and its calculations can still be improved. A good idea will be add the strength of the interaction W, that mean $W \neq 0$, for get realistic results.

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[4] Cervia, M.J. et al.(2021). Lipkin model on a quantum computer. *Phys. Rev. C*, 104,024305. DOI: 10.1103/PhysRevC.104.024305.

[5] Hlatshwayo, M.Q. et al.(2022). Simulating excited states of the Lipkin model on a quantum computer. *Phys. Rev. C*, 106, 024319. DOI: 10.1103/PhysRevC.106.024319.

[6] Shee, Y. et al.(2022). Qubit-efficient encoding scheme for quantum simulations of electronic structure. *Phys. Rev. Research*, 4, 023154. DOI: 10.1103/PhysRevResearch.4.023154.