- Derivatives Pricing and Hedging:

  - Incomplete market $\Rightarrow$ Not all positions can be hedged

  - Classical theory of hedging not in course

  - Assume we have a position D in m derivatives, each of which expires in time $\leq T$

  - State space includes: $\underbrace{\alpha_t}_{\text{hedge positions}}$, $\underbrace{P_t}_{\text{prices}}$, $\underbrace{\beta_t}_{\text{PnL}}$, $\underbrace{D}_{\text{Derivatives on book}}$

    a_t = # of hedge positions to buy or sell

    ① Observe States

    ② Perform actions (trades) a_t to produce trading PnL $= -a_t \cdot P_t$

    ③ Trading Transaction costs, example $= -\gamma P_t \cdot |a_t|$ for some $\gamma > 0$

    ... Continue slide 15

  - Pricing and Hedging:

    → Create incremental position D' with cash flow required to take on D' should not change optimal value

$$V_0^*\big((\alpha_0,\ P_0,\ \underbrace{\beta_0 - x}_{\text{Price to take on new positions}},\ \underbrace{D \cup D'}_{\text{Adding } D' \text{ to derivatives portfolio}})\big) = \underbrace{V_0^*\big((\alpha_0,\ P_0,\ \beta_0,\ D)\big)}_{\text{This MDP solved before trades}}$$

Now you have a new state and you want to solve for x such that V_0^* does not change

→ Want to maximize net present value of utility of accumulated future cash flow

→ Now for V_0^*, determine hedging strategy with π_t^*

- DRL Approach for practical Trading
- DRL = Deep Reinforcement Learning is breakthrough for market dynamics
  - ↪ **Advantages:**
    - ↪ Independent of Market Dynamics
    - ...
  - A lot of banks looking at DRL, e.g. JPM

- **Review:** Finance Examples
  1. Asset Allocation ⟵ Complete
  2. American Options Pricing
  3. Derivatives Pricing and Hedging

  $\Big\}$ Don't worry too much

  4. Optimal Execution of trades ⟵ Complete

Order Book Algorithms:

- People place limit orders on a stock
- Order book has limit sell orders and limit buy orders
- Trading book aggregates limit orders at each unique price
  - ↪ Represent books as two sorted lists:

Bids : $[(P_i^{(b)}, N_i^{(b)})]$ descending → Best is "bid"

Ask : $[P_i^{(a)}, N_i^{(a)})]$ ascending → Best is "ask"

$$\text{midprice} = \frac{\text{bid} - \text{ask}}{2}$$

$$\text{Spread} = \text{ask} - \text{bid}$$

- ↪ Retail asks for shares at best possible price = Market order

- ↳ Normally, you don't set sell limit order below highest bid order
  - ↳ If you do by mistake, you eat into the bid order book
  - ↳ Eat book block is the main function to consider

- How does a market order alter the order book?
  - ↳ Large sell or buy orders leave a gap in the order book
    - ↳ Order book dynamics = How bids / asks change with gap in order book
    - ↳ Change in bid/ask spread is Price Impact
      - ↳ Temporary and Permanent Price Impacts

- <u>Question:</u> How do you break up sell orders?
  - ↳ MDP
  - ↳ Too slow = Uncertainty ⟶ Poor utility
  - ↳ Too fast = low average price ⟶ Poor utility

- <u>Problem Annotation:</u>
  - T = time step indexed by $t = 0, 1, \ldots, T$
  - P_t = Bid price at start of time step t
  - N_t = number of shares sold at time step t

- See MDP Formulation on slide 11.

- <u>Similar Linear Impact Model w/ No Risk Aversion</u>

  $N, N_t, P_t \in \mathbb{R}$

  Price Dynamics: $P_{t+1} = P_t - \alpha N_t + \epsilon_t$, where $\alpha \in \mathbb{R}$

  ϵ_t is i.i.d. ...

$$V_t^{\pi}\left((P_t, R_t)\right) = \mathbb{E}_{\pi}\left[\sum_{i=t}^{T} N_i\left(P_i - \beta N_i\right) \,\middle|\, (P_t, R_t)\right]$$

- Boundary Condition: On last day, sell all remaining shares
  - ↳ Kind of like asset allocation problem
    - ... see slides 12 - 14
  - ↳ Take partial derivative wrt α for 2 cases: $\alpha \geq 2\beta$ or not
    - ↳ $\alpha \geq 2\beta \Rightarrow$ Sell everything on one day (work out the math)
      - ↳ Trivial case
    - ↳ If $\alpha < 2\beta$, $N_t^* = \dfrac{R_t}{T-t}$ ← Remaining # shares ← # of days remaining

- Price impact and market movement operate additively and don't interact in this basic example.

- Optimal Expected Total Sales Proceeds $= NP_0 - \underbrace{\frac{N^2}{2}\left(\alpha + \frac{2\beta - \alpha}{T}\right)}_{\text{"Slippage" = Implementation Shortfall}}$
  - ↳ There is always implementation shortfall unless price snaps back completely

- <u>Slide 17</u> gives a paper on Optimal Trade Order Execution with more complicated price impact model
  - ↳ We are asked to work out model on slide 18
  - ↳ Limitation: Did not assume a utility function. People are not interested just in maximizing revenue of sales. They want to balance expected returns and variance of returns.

- <u>Slide 19</u>: Almgren - Chriss breakthrough paper "Efficient Frontier"

- Real World Optimal Trade Order Execution
    - Lots of time building models of how order book will change (basis of algorithmic trading)
    - Incorporating various market factors in states bloats state space
    - Next lecture: How best to submit limit orders.