Problem 1:

Refresher: Avellaneda - Stoikov Continuous Time Formulation: * Based on Course Slides *

$W_t \in \mathbb{R}$ :: Market Maker's trading cash

$I_t \in \mathbb{Z}$ :: Market Maker's inventory of stock at t

$S_t \in \mathbb{R}$ :: Midprice of Stock

$P_t^{(b)} \in \mathbb{R}^+$, $N_t^{(b)} \in \mathbb{Z}^+$ :: Bid price and # of shares

$P_t^{(a)} \in \mathbb{R}^+$, $N_t^{(a)} \in \mathbb{Z}^+$ :: Ask price and # of shares

$\delta_t^{(b)} = S_t - P_t^{(b)}$ is Bid Spread

$\delta_t^{(a)} = P_t^{(a)} - S_t$ is ask Spread

$X_t^{(b)}$ = Accumulated bids hit up to t

$X_t^{(a)}$ = Accumulated asks lifted up to t

$$\therefore W_{t+1} = W_t + P_t^{(a)} \cdot \left(X_{t+1}^{(a)} - X_t^{(a)} \right) - P_t^{(b)} \left(X_{t+1}^{(b)} - X_t^{(b)} \right)$$

$$I_t = X_t^{(b)} - X_t^{(a)}$$

In Continuous Case:

$dX_t^{(b)} \sim \text{Poisson}\left(\lambda_t^{(b)} \cdot dt \right)$ :: Hit rate

$dX_t^{(a)} \sim \text{Poisson}\left(\lambda_t^{(a)} \cdot dt \right)$ :: lift rate

↪ Assume $\lambda_t^{(b)} = f^{(b)}(\delta_t^{(b)})$, $\lambda_t^{(a)} = f^{(a)}(\delta_t^{(a)})$

↪ $f^{(b)}$, $f^{(a)}$ are decreasing functions

$$\therefore dW_t = P_t^{(a)} \, dX_t^{(a)} - P_t^{(b)} \, dX_t^{(b)}$$

$$I_t = X_t^{(b)} - X_t^{(a)}$$

↪ In infinitesimal time, assume $N_t^{(b)}, N_t^{(a)} = 1$ and $dX_t^{(b)}, dX_t^{(a)}$ are Bernoulli distributed (1 or 0)

↳ Action is pricing ∴ $a_t = (\delta_t^{(b)}, \delta_t^{(a)})$

Assume Stochastic midprice : $dS_t = \sigma \cdot dz_t$

Apply Utility Function : $U(x) = -e^{-ax}$ where $a > 0$ is coeff. of risk aversion

• Optimal Value Function given by HJB Equation :

$$V^*(t, S_t, W_t, I_t) = \max_{\delta_u^{(b)}, \delta_u^{(a)} \, : \, t \le u < t_1} \mathbb{E}\left[V^*(t_1, S_{t_1}, W_{t_1}, I_{t_1})\right]$$

$$\hookrightarrow V^*(t, S_t, W_t, I_t) = \max_{\delta_u^{(b)}, \delta_u^{(a)} \, : \, t \le u < T} \mathbb{E}\left[-e^{-a(W_T + I_T S_T)}\right]$$

Money evaluated at time T

$$\begin{cases} \max_{\delta_t^{(a)}, \delta_t^{(b)}} \mathbb{E}\left[dV^*(t, S_t, W_t, I_t)\right] = 0 \quad \text{for } t < T \\ V^*(T, S_T, W_T, I_T) = -e^{-a(W_T + I_T S_T)} \end{cases}$$

$V^*(t, S_t, W_t, I_t)$ affected by :

1. Changes in t
2. Stochasticity in Midprice S_t of order book
3. Hit / Lift randomness

W is deterministic

∴ Taking the derivative of the HJB Eq yields :

$$dV^*(t, S_t, W_t, I_t) =$$

$$\max_{\delta_t^{(a)}, \delta_t^{(b)}} \left\{ \frac{\partial V^*}{\partial t} dt + \mathbb{E}\left[\overbrace{\sigma \frac{\partial V^*}{\partial S_t} dz_t}^{dS_t} + \overbrace{\frac{\sigma^2}{2} \frac{\partial^2 V^*}{\partial S_t^2} (dz)^2}^{(dS_t)^2} \right] \right.$$

Think of this in differential space

HIT $\longrightarrow$ $+ \lambda_t^{(b)} dt \, V^*(t, S_t, W_t - S_t + \delta_t^{(b)}, I_t + 1)$

LIFT $\longrightarrow$ $+ \lambda_t^{(b)} dt \, V^*(t, S_t, W_t + S_t + \delta_t^{(a)}, I_t - 1)$

NEITHER $\longrightarrow$ $+ (1 - \lambda_t^{(b)} dt - \lambda_t^{(a)} dt) \, V^*(t, S_t, W_t, I_t)$

$$- V^*(t, S_t, W_t, I_t) = 0$$

Taking differential w.r.t. 1 discrete time step

$\Rightarrow$ Take a difference!

... Continuous case continues on slide 27 of Ch.8 slide deck, but this does not answer this question...

- <u>Objective 1</u>: Evaluate $\mathbb{E}\left[-\exp(-\gamma(W + I S_T)) \mid (t, S_t) \right]$ to get a simple expression for $V(t, S_t, W, I)$.

$$\mathbb{E}\left[-\exp\left(-\gamma\left(W + I S_T\right)\right) \mid (t, S_t) \right]$$

Deterministic

Stochastic

Drop conditions to lighten notations, but remember them

$$= -\exp(-\gamma W)\, \mathbb{E}\left[\exp(-\gamma I S_T)\right]$$

$S_T \sim N(S_t, \sigma^2(T-t))$

Expectation of log normal

$$= -\exp(-\gamma W) \exp\left[-\gamma I S_t - \tfrac{1}{2}\gamma^2 I^2 \sigma^2 (T-t)\right]$$

$$= -\exp\left[-\gamma\left(W + I\left(S_t + \tfrac{1}{2}\gamma I \sigma^2 (T-t)\right)\right)\right]$$

$$\therefore \boxed{V(t, S_t, W, I) = -\exp\left[-\gamma\left(W + I\left(S_t + \tfrac{1}{2}\gamma I \sigma^2 (T-t)\right)\right)\right]}$$

- <u>Objective 2</u>: Calculate Indifference Bid and Ask Prices Satisfying:

$$V\left(t, S_t, W - Q^{(b)}(t, S_t, I), I + 1\right) = V(t, S_t, W, I)$$

$$V\left(t, S_t, W + Q^{(a)}(t, S_t, I), I - 1\right) = V(t, S_t, W, I)$$

$\hookrightarrow$ Start with $Q^{(b)}$:

$$-\cancel{e}xp\left[-\cancel{\gamma}\left(\cancel{W} + I\left(S_t + \frac{1}{2}\gamma I \sigma^2 (T-t)\right)\right)\right] =$$

$$- \cancel{e}xp\left[-\cancel{\gamma}\left(\cancel{W} - Q^{(b)}(t, S_t, I) + (I+1) S_t + \frac{1}{2}\gamma (I+1)^2 \sigma^2 (T-t)\right]\right.$$

$$\Rightarrow I\cancel{S_t} + \frac{1}{2}\gamma \cancel{I^2 \sigma^2}(T-t) = -Q^{(b)}(t, S_t, I) + I\cancel{S_t} + S_t + \frac{1}{2}\gamma I^2 \cancel{\sigma^2}(T-t)$$
$$+ (2I+1)\frac{1}{2}\gamma\sigma^2(T-t)$$

$$\Rightarrow \boxed{Q^{(b)}(t, S_t, I) = S_t + \left(I + \frac{1}{2}\right)\gamma\sigma^2(T-t)}$$

$\hookrightarrow$ Now do $Q^{(a)}$:

$$-\cancel{e}xp\left[-\cancel{\gamma}\left(\cancel{W} + I\left(S_t + \frac{1}{2}\gamma I \sigma^2 (T-t)\right)\right)\right] =$$

$$- \cancel{e}xp\left[-\cancel{\gamma}\left(\cancel{W} + Q^{(a)}(t, S_t, I) + (I-1) S_t + \frac{1}{2}\gamma (I-1)^2 \sigma^2 (T-t)\right]\right.$$

$$\Rightarrow I\cancel{S_t} + \frac{1}{2}\gamma \cancel{I^2 \sigma^2}(T-t) = Q^{(a)}(t, S_t, I) + I\cancel{S_t} - S_t + \frac{1}{2}\gamma I^2 \cancel{\sigma^2}(T-t)$$
$$+ (-2I+1)\frac{1}{2}\gamma\sigma^2(T-t)$$

$$\Rightarrow \boxed{Q^{(a)}(t, S_t, I) = S_t + \left(I - \frac{1}{2}\right)\gamma\sigma^2(T-t)}$$