Problem 2:

Objective: Derive Optimal Value Function and Optimal Policy for Linear Percentage Temporary (LPT) Price Impact Model

Serial Correlated Variable

$$P_{t+1} = P_t e^{Z_t}$$

Z_t is random variable w/ mean μ_z, variance σ_z^2

$$X_{t+1} = \rho X_t + \eta_t$$

i.i.d random w/ mean 0

$$Q_t = P_t(1 - \beta \cdot N_t - \theta \cdot X_t)$$

$$U(\cdot) = \text{Identity}$$

Effective Sale Price

Value Function: $V_t^{\pi}((P_t, R_t)) = \mathbb{E}_{\pi}\left[\sum_{i=1}^{T} N_i(P_i - \beta \cdot N_i - \theta X_t) \,\middle|\, (P_t, R_t)\right]$

- **Practice Approach 1:** Neglect X_t term

$$\therefore V_t^{\pi}((P_t, R_t)) = \mathbb{E}_{\pi}\left[\sum_{i=1}^{T} N_i(P_i - \beta N_i) \,\middle|\, (P_t, R_t)\right]$$

Optimal Value Function: $V_t^{*}((P_t, R_t)) = \max_{\pi} V_t^{\pi}((P_t, R_t))$

→ Satisfies Bellman Equation:

$$V_t^{*} = \max_{N_t}\left\{N_t \cdot (P_t - \beta N_t) + \mathbb{E}\left[V_{t+1}^{*}((P_{t+1}, R_{t+1}))\right]\right\}$$

Boundary Condition: Sell all remaining shares at $t = T-1$

$$\therefore V_{T-1}^{*} = N_{T-1}(P_{T-1} - \beta N_{T-1}) = R_{T-1}(P_{T-1} - \beta R_{T-1})$$

$$\therefore V_{T-2}^{*} = \max_{N_{T-2}}\left\{N_{T-2}(P_{T-2} - \beta N_{T-2}) + \mathbb{E}\left[R_{T-1}(P_{T-1} - \beta R_{T-1})\right]\right\}$$

$P_{T-1} = P_{T-2}\, e^{N(\mu, \sigma^2)}$

$R_{T-1} = R_{T-2} - N_{T-2}$

Let $\lambda = \mu + \frac{1}{2}\sigma^2$

$$= \max_{N_{T-2}} \left\{ N_{T-2}\left(P_{T-2} - \beta N_{T-2}\right) + \left(R_{T-2} - N_{T-2}\right)\left(P_{T-2}\, e^{\mu + \frac{1}{2}\sigma^2} - \beta\left(R_{T-2} - N_{T-2}\right)\right) \right\}$$

↪ Satisfied where partial derivative wrt N_{T-2} is equal to zero.

$$\therefore\ 0 = \frac{\partial}{\partial N_{T-2}} \left\{ N_{T-2}\left(P_{T-2} - \beta N_{T-2}\right) + \left(R_{T-2} - N_{T-2}\right)\left(P_{T-2}\, e^{\lambda} - \beta\left(R_{T-2} - N_{T-2}\right)\right) \right\}$$

$$\therefore\ 0 = \frac{\partial}{\partial N_{T-2}} \left\{ N_{T-2} P_{T-2} - \beta N_{T-2}^2 + R_{T-2} P_{T-2}\, e^{\lambda} - N_{T-2} P_{T-2}\, e^{\lambda} - \beta R_{T-2}^2 + \beta R_{T-2} N_{T-2} + \beta R_{T-2} N_{T-2} - \beta N_{T-2}^2 \right\}$$

$$\therefore\ 0 = P_{T-2} - 2\beta N_{T-2} - P_{T-2}\, e^{\lambda} + 2\beta R_{T-2} - 2\beta N_{T-2}$$

$$\therefore\ 0 = P_{T-2}\left(1 - e^{\lambda}\right) - 4\beta N_{T-2} + 2\beta R_{T-2}$$

$$\therefore\ N_{T-2}^{*} = \underbrace{\frac{P_{T-2}\left(1 - e^{\lambda}\right)}{4\beta}}_{C_{T-2}^{(1)}} + \frac{1}{2} R_{T-2}$$

$$\therefore\ V_{T-2}^{*} = C_{T-2}^{(1)} P_{T-2}\left(1 - e^{\lambda}\right) + \frac{1}{2} R_{T-2} P_{T-2}\left(1 - e^{\lambda}\right) - 2\beta\left({C_{T-2}^{(1)}}^2 + R_{T-2}\, C_{T-2}^{(1)} + \frac{1}{4} R_{T-2}\right) + R_{T-2} P_{T-2}\, e^{\lambda} - \beta R_{T-2}^2 + 2\beta\left(C_{T-2}^{(1)} + \frac{1}{2} R_{T-2}\right) R_{T-2}$$

- Algebra from here!

Problem 2 Continued:

- **Practice Approach 2:** All 6.

Value Function: $V_t^{\pi}((P_t, R_t)) = \mathbb{E}_{\pi}\left[\sum_{i=1}^{T} N_i (P_i - \beta \cdot N_i - \theta X_t) \;\Big|\; (P_t, R_t) \right]$

Optimal Value Function: $V_t^{*}((P_t, R_t)) = \max_{\pi} V_t^{\pi}((P_t, R_t))$

$\mapsto$ Satisfies Bellman Equation:

$$V_t^{*} = \max_{N_t} \left\{ N_t \cdot (P_t - \beta N_t - \theta X_t) + \mathbb{E}\left[V_{t+1}^{*}((P_{t+1}, R_{t+1})) \right] \right\}$$

Boundary Condition: Sell all at remaining at $t = T-1$

$$V_{T-1}^{*}((P_{T-1}, R_{T-1})) = N_{T-1}(P_{T-1} - \beta N_{T-1} - \theta X_{T-1}) = R_{T-1}(P_{T-1} - \beta R_{T-1} - \theta X_{T-1})$$

$$\therefore V_{T-1}^{*} = P_{T-1} R_{T-1} - \beta R_{T-1}^2 - \theta X_{T-1} R_{T-1}$$

$X_{T-1} = \rho X_{T-2} + \eta_{T-2}$

$$\therefore V_{T-2}^{*}((P_{T-2}, R_{T-2})) = \max_{N_{T-2}} \left\{ N_{T-2}(P_{T-2} - \beta N_{T-2} - \theta X_{T-2}) + \mathbb{E}[P_{T-1} R_{T-1}] - \beta R_{T-1}^2 - \theta X_{T-1} R_{T-1} \right\}$$

$R_{T-1} = R_{T-2} - N_{T-2}$

$P_{T-1} = P_{T-2}\, e^{Z_{T-2}}$

$\mu_2 + \frac{1}{2}\sigma_2^2$

$$= \max_{N_{T-2}} \left\{ N_{T-2}(P_{T-2} - \beta N_{T-2} - \theta X_{T-2}) + (P_{T-2} \exp(\mu_2 + \tfrac{1}{2}\sigma_2^2))(R_{T-2} - N_{T-2}) - \beta (R_{T-2} - N_{T-2})^2 - \theta(\rho X_{T-2} + \eta_{T-2})(R_{T-2} - N_{T-2}) \right\}$$

$$= \max_{N_{T-2}} \Big\{ N_{T-2} P_{T-2} - \beta N_{T-2}^2 - \theta N_{T-2} X_{T-2} + P_{T-2} R_{T-2} e^{\mu_2 + \frac{1}{2}\sigma_2^2} - P_{T-2} N_{T-2} e^{\mu_2 + \frac{1}{2}\sigma_2^2} - \beta R_{T-2}^2 - \beta N_{T-2}^2$$

$$+ 2\beta R_{T-2} N_{T-2} - \theta \rho X_{T-2} R_{T-2} + \theta \rho X_{T-2} N_{T-2} - \theta \eta_{T-2} R_{T-2} + \theta \eta_{T-2} N_{T-2} \Big\}$$

↪ Find where partial derivative wrt $N_{T-2} = 0$

let $\lambda = \mu + \frac{1}{2}\sigma^2$

$$\therefore \; 0 = P_{T-2} - 2\beta N_{T-2} - \theta X_{T-2} - P_{T-2} e^{\lambda} - 2\beta N_{T-2} + 2\beta R_{T-2} + \theta \rho X_{T-2} + \theta \eta_{T-2}$$

$$\therefore \; 4\beta N^*_{T-2} = P_{T-2}\left(1 - e^{\lambda}\right) + 2\beta R_{T-2} + \theta(\rho - 1) X_{T-2} + \theta \eta_{T-2}$$

$$\therefore \; N^*_{T-2} = \frac{\left(1 - e^{\lambda}\right)}{4\beta} P_{T-2} + \frac{1}{2}\beta R_{T-2} + \frac{\theta(\rho-1)}{4\beta} X_{T-2} + \frac{\theta \eta_{T-2}}{4\beta}$$

$\mathbb{E}[P_t] = P_0 e^{\lambda t}$

↪ Continue Traceback:

$$N_t^* = \underbrace{\frac{P_0\left(1 - e^{\lambda}\right)e^{\lambda t}}{4\beta} + \frac{\theta \eta_{T-2}}{4\beta}}_{c_t^{(1)}} + \underbrace{\frac{1}{2}}_{c_t^{(2)}} \beta R_{T-2} + \underbrace{\frac{\theta(\rho-1)}{4\beta}}_{c_t^{(3)}} X_{T-2}$$

$$\therefore \; N_t^* = c_t^{(1)} + c_t^{(2)} R_t + c_t^{(3)} X_t$$

$$\therefore \; V_t^*\left((P_t, R_t)\right) =$$

$$N_t^* P_t - \beta N_t^{*2} - \theta N_t^* X_t + P_t R_t e^{\lambda} - P_t N_t^* e^{\lambda} - \beta R_t^2 - \beta N_t^{*2}$$
$$+ 2\beta R_t N_t^* - \theta \rho X_t R_t + \theta \rho X_t N_t^* - \theta \eta_t R_t + \theta \eta_t N_t^*$$

$$= -2\beta\left(c_t^{(1)} + c_t^{(2)} R_t + c_t^{(3)} X_t\right)^2 + \left(P_t\left(1 - e^{\lambda}\right) - \theta X_t + 2\beta R_T + \theta\rho X_t + \theta \eta_t\right)$$
$$\cdot \left(c_t^{(1)} + c_t^{(2)} R_t + c_t^{(3)} X_t\right) + P_t R_t e^{\lambda} - \beta R_t^2 - \theta \rho X_t R_t - \theta \eta_t R_t$$

↪ Algebra from here!