- February 3, 2021 Notes: Utility Theory
  - Slide 13

↳ Consider 1 riskless asset and 1 risky asset

Riskless Asset: $dR_t = r \cdot R_t \cdot dt$

Fractional growth

Risky Asset: $dS_t = \mu \cdot S_t \cdot dt + \sigma \cdot S_t \cdot dz_t$

Geometric Brownian
$N(0, dt)$

Variance proportional to time traveled

Wealth

↳ Determine constant fraction π of W_t to allocate to risky asset

↳ Frictionless setting: instantaneous, free money transfer

$$dW_t = \Big(r + \pi \underbrace{(\mu - r)}_{\text{Risky asset excess return}}\Big) \cdot W_t \cdot dt + \pi \cdot \sigma \cdot W_t \cdot dz_t$$

Assume CRRA Utility $U(W) = \dfrac{W^{1-\gamma} - 1}{1 - \gamma}$, $0 < \gamma \neq 1$

See appendix on Stochastic Calculus.

↳ No testing on Stochastic Calculus, thank God.

↳ Always look at certain part and uncertain part separately

W is log-normally distributed

$\therefore$ maximize $\mu + \sigma^2/2$

$\Rightarrow$ Optimal investment fraction in risky asset:

$$\pi^* = \frac{\mu - r}{\gamma \sigma^2}$$

- Chapter 6 Notes: Dynamic Asset Allocation and Consumption
  - ↪ Broad topic is Investment Management
    - ↪ Applies to corporations and individuals
      - ① How to allocate money across assets in one's investment portfolio
      - ② How much to consume for one's needs / operations / pleasures
    - ↪ Allocation and Consumption decisions at each time step
      - ↪ Spend money now or later
      - ↪ Objective: Horizon - Aggregated Expected Utility of Consumption

- Personal Finance Example:
  - Recieve Money: Salary, Bonus, Rental income, Asset liquidation, etc.
  - Consume Money: food, cloths, rent/mortgage, cars, vacations, etc.
  - Investing Money: Saving Account, Stocks, Real Estate, Gold, etc.

- Framed as a MDP
- State: Age, Asset Holdings, Asset Valuation, Career situation, etc.
- Action: Changes in Asset Holdings, Optional Consumption
- Reward: Utility of Consumption of Money

- Merton's Frictionless Continuous - Time Formulation:

  Assume: Current wealth $W_0 > 0$, you'll live for T more years

  ↪ You can invest in a risky assets and a riskless asset

- ↪ Each risky asset has known normal distribution of returns
- ↪ Allow long or short any fractional quantities of assets
- ↪ Trading in continuous time $0 \leq t < T$ with no transaction costs.
- ↪ Consume any fractional amount of wealth

Riskless Asset : $dR_t = r \cdot R_t \cdot dt$

Risky Asset : $dS_t = \mu S_t \, dt + \sigma \cdot S_t \, dz_t$

$\mu > r > 0$

$\sigma > 0$

Wealth at time t : $W_t > 0$

Fraction in risky asset : $\pi(t, W_t)$

$\therefore$ Riskless asset fraction $1 - \pi(t, W_t)$

Consumption : $c(t, W_t) \geq 0$

Utility of consumption : $U(x) = \dfrac{x^{1-\gamma}}{1-\gamma}$ for $0 < \gamma \neq 1$

$U(x) = \log(x)$ for $\gamma = 1$

$$dW_t = ((\pi_t \cdot (\mu - r) + r) \cdot W_t - c_t) \cdot dt + \pi_t \cdot \sigma \cdot W_t \cdot dz_t$$

State W

State transition : $W + dW$

- At any time t, determine optimal $[\pi(t, W_t), c(t, W_t)]$ to maximize utility of reward:

inflation / consumption - associated discount factor

$$\mathbb{E}\left\{ \frac{e^{-\rho(s-t)} \cdot C_s^{1-\gamma}}{1-\gamma} \, ds + \frac{e^{-\rho(T-t)} \cdot B(t) \cdot W_T^{1-\gamma}}{1-\gamma} \;\middle|\; W_t \right\}$$

Utility function

Utility of money when you die

↦ Assume $B(t) = 0$

- This is a continuous - time stochastic control problem
  - State at time t is (t, W_t)
  - The action at t is $[\pi_t, c_t]$
  - Continuous - time $\Rightarrow$ Calculus.
  - Reward:

$$\int_t^T e^{-\rho(s-t)} \cdot \frac{C_s^{1-\gamma}}{1-\gamma} \cdot ds$$

  - If $(1-\pi) < 0$, you are borrowing money

$$V^*(t, W_t) = \max_{\pi, c} \mathbb{E}_t \left\{ \int_t^T \frac{e^{-\rho(s-t)} \, C_s^{1-\gamma}}{1-\gamma} \, ds \right\}$$

Max over policies

→ Set this up as a recursive function

→ Continuous time Bellman Optimality Equation

HJB Formulation $\quad V^*(t, W_t) = \max_{\pi, c} \mathbb{E}_t \left\{ \int_t^{t_1} \cdots \right.$ See slide 8

$$\Rightarrow \max_{\pi_t, c_t} \mathbb{E} \left\{ d\left(e^{-\rho t} \cdot V^*(t, W_t)\right) + \frac{e^{-\rho t} \cdot c_t^{1-\gamma}}{1-\gamma} dt \right\} = 0$$

Discounted reward

- Not tested on Ito's lemma
- Higher order differentials go away except $dz_t^2 = dt$, associated with dW^2
- Ito's lemma gives partial DE form of HJB Equation
- Stochastic Calculus, Continuous-time stochastic Control useful (Math 238)

Consider function Φ given by Ito's lemma applied to HJB

$$\max_{\pi_t, c_t} \Phi(\underbrace{t, W_t}_{\text{State}}; \underbrace{\pi_t, c_t}_{\text{actions}}) = \rho \cdot V^*(t, W_t)$$

Expectation of $dz_t = 0$

Take partial derivatives w.r.t. actions

$$\pi_t^* = f\left(\frac{\partial V^*}{\partial W_t}, \frac{\partial^2 V^*}{\partial W_t^2}\right)$$

We will solve for V^*, find partial derivatives and substitute

...

$$\frac{\partial V^*}{\partial t} - \frac{(\mu - r)^2}{2\sigma^2} \cdot \left(\frac{\partial V^*}{\partial W_t}\right)^2 \cdots$$ See slide 11

- When Solving Differential Equation, Start by guessing form of Solution

$$V^*(t, W_t) = f(t)^\gamma \cdot \frac{W_t^{1-\gamma}}{1-\gamma}$$

↪ Evaluate partial DE's and substitute into equation on slide 11.

$$\Rightarrow \quad f'(t) = \nu \cdot f(t) - 1$$

Constant

Solution is an exponential

$f(t)$ is an exponential wrt time

↳ Substitute $f(t)$ into V^* ansatz

Separation of time, wealth state dependencies

Take derivatives of V^* and substitute into equation on slide 11

$$\pi^*(t, W_t) = \frac{\mu - r}{\sigma^2 \gamma}$$

CONSTANT!! allocation into risky asset

Independent of time and wealth

- Consumption rate is not constant, but almost constant:

$$c^*(t, W_t) = \frac{W_t}{f(t)}$$

Fractional consumption rate dependent only on time.

- The approach to solving this is a template for similar continuous-time stochastic control problems.

$$\frac{dW_t}{W_t} = f(c^*, \pi^*)$$ on slide 15

Growth rate of wealth has deterministic and stochastic part

Expected portfolio returns

Uncertainty from risky asset