- Lecture 8 Notes:

  - Simplified asset-Allocation Example on slide 18.

    ↦ Discrete time steps 0, 1, ..., T-1

    ↦ Asset allocate wealth W_t

    ↦ 1 risky asset, unconstrained allocation, no transaction costs

    ↦ Risky asset returns $\sim N(\mu, \sigma^2)$

    ↦ Use CARA utility function

    ↦ Solution not covered, but see slides 18-20.

  * Start with Bellman Optimality Equation - finite horizon

    $$V^*(t) = f(V^*(t+1))$$

    ↦ Closed form solution

    ↦ Maximum over actions of an expectation

    ↦ Maximize over $\pi \in [-\infty, \infty]$ of an expression which takes the form of an exponential

** Slides 18-20 will be very similar to midterm / final

- Wrap-up on asset allocation and consumptions:

  ↦ Analytical tractability because:

  1. N-dist. of asset returns
  2. Constant Relative Risk Aversion
  3. Frictionless, Continuous trading

- CRRA works well with Normal distribution

- Real world is a lot more complicated (see slide 25 for examples)

- Need to use approximate dynamic programming to solve this
- RL Algorithms learn from historical data and simulators.
  - ↪ In real world, building simulators to train RL algorithms is common.
  - ↪ In real world, you cannot sweep over all actions for a given state.
    - ↪ Numerical optimization over actions

- <u>Derivatives pricing and hedging:</u>
  - ↪ Not tested on theory! Math 238 covers this topic in great detail.
  - ↪ Derivative is something priced based on the price of an underlying stock (e.g. call, put contract).
    - ↪ <u>Lock Type contract:</u> Pricing structure based on future type of stock.
      - ↪ Must be executed, locked, not an option
    - ↪ <u>Options:</u> Owner has the option to execute the contract.
  - ↪ Derivatives are an artificial way to do leverage
  - ↪ Hedging and Replications are opposites
    - ↪ Hedging cancels out portfolio payoff
    - ↪ Replicating amplifies portfolio payoff
  - ↪ Stopping time τ is a set of conditions when reached prompts some action.

- Optimal Stopping Process for Stochastic process X_t:

$$W(x) = \max_{\tau} \mathbb{E}\left[H(X_\tau) \mid X_0 = x \right]$$

Reward function

↪ If more than one τ maximize expectation, take the first time.

↪ General Stopping time problem as Markov Decision Process:

- State X_t
- Action is Boolean: stop or continue
- Reward is 0, except upon stopping
- State transition governed by stochastic process X_t
- For discrete time step, the Bellman Optimality Equation is:

$$V^*(X_t) = \max\left(H(X_t),\ \mathbb{E}\left[V^*(X_{t+1} \mid X_t) \right] \right)$$

- You can prove that some derivatives are priced optimally when executed at their expiration date.

  ∴ Price them as European Options.

- If state space is not too large, use backward induction.
- Alternative, standard approach is Longstaff-Schwartz Algorithm

- Binomial Tree for Backward Induction:

  ↪ Multiplicative Random Walk

  ↪ Tree recombining: ↑↓ Move = ↓↑ Move

↦ After N steps, there are N+1 states

↦ V(s) (payoff) known at step N when American stock option gets executed.

↦ Backward induction tells when to execute stock option to maximize payoff, $V^*(s)$

- Slide 9: Optimal policy plotted as an optimal exercise boundary

- Arbitrage: Opportunity to make money without taking risk.

  ↦ Economic theory says this is impossible because any opportunity would be exploited until it goes away.

- Assumptions of arbitrage-free and completeness lead to (dynamic, exact, unique) replication of derivatives with underlying securities.

- Black-Scholes Formula for replicating in a complete market.

- In reality, we have an incomplete market

  ↦ Choices of risk-neutral measures (hence price) done in an ad hoc way.

  ↦ Alternatively, treat the problem as a portfolio optimization

- Utility Maximization = Maximizing risk-adjusted return.

- Consider Portfolio optimization as a MDP

  ↦ Deep Reinforcement Learning Model

    ↦ Dynamic Programming not suitable

    ↦ Optimal policy gives best hedging strategy.

- **Problem Setup:**
    - Assume a position (portfolio) D in m derivatives
        - → How do you hedge these derivatives?
    - Assume all derivatives will expire in time $\leq T$
    - Portfolio = Aggregated Contingent Cashflows at time t denoted $X_t \in \mathbb{R}$
    - Hedge positions (units held) at time t denoted $\alpha_t \in \mathbb{R}^m$