Problem 1:

$$U(x) = x - \frac{\alpha x^2}{2}. \quad \text{Assume } x \sim N(\mu, \sigma^2)$$

Denote $\mathbb{E}[x] = \bar{x}$

① $$\mathbb{E}[U(x)] \approx U(\bar{x}) + \frac{1}{2} U''(\bar{x}) \sigma_x^2$$

$$= U(\mu) + \frac{1}{2} U''(\mu) \sigma_x^2$$

$$U'(x) = 1 - \alpha x$$

$$U''(x) = -\alpha$$

$$\therefore \mathbb{E}[U(x)] \approx \mu - \frac{\alpha \mu^2}{2} - \frac{\alpha \sigma^2}{2} = \mu - \frac{\alpha}{2}(\mu^2 + \sigma^2)$$

② Certainty Equivalent Value

$$x_{CE} = U^{-1}\left(\mathbb{E}[U(x)]\right)$$

↪ The value of X which yields the expected utility

$$\therefore X_{CE} - \frac{\alpha X_{CE}^2}{2} = \mu - \frac{\alpha}{2}(\mu^2 + \sigma^2)$$

$$\therefore -\frac{\alpha X_{CE}^2}{2} + X_{CE} - \left(\mu - \frac{\alpha}{2}(\mu^2 + \sigma^2)\right) = 0$$

$$\therefore X_{CE}^2 - \frac{2}{\alpha} X_{CE} + \left(\frac{2\mu}{\alpha} - (\mu^2 + \sigma^2)\right) = 0$$

↪ Complete the square:

$$X_{CE}^2 - \frac{2}{\alpha} X_{CE} + \frac{1}{\alpha^2} + \frac{2\mu}{\alpha} - (\mu^2 + \sigma^2) - \frac{1}{\alpha^2} = 0$$

$$\therefore \left(X_{CE} - \frac{1}{\alpha}\right)^2 + \frac{2\mu}{\alpha} - (\mu^2 + \sigma^2) - \frac{1}{\alpha^2} = 0$$

$$\therefore \quad x_{CE} - \frac{1}{\alpha} = \pm\sqrt{\frac{-2\mu}{\alpha} + (\mu^2+\sigma^2) + \frac{1}{\alpha^2}}$$

$$\therefore \quad \boxed{x_{CE} = \frac{1}{\alpha} \pm \sqrt{\frac{-2\mu}{\alpha} + (\mu^2+\sigma^2) + \frac{1}{\alpha^2}}}$$

<u>Checking Answer:</u>

$$\frac{1}{\alpha} + \sqrt{-\frac{2\mu}{\alpha} + (\mu^2+\sigma^2) + \frac{1}{\alpha^2}}$$

$$- \frac{\alpha}{2}\left(\frac{1}{\alpha} + \sqrt{\frac{-2\mu}{\alpha} + (\mu^2+\sigma^2) + \frac{1}{\alpha^2}}\right)^2$$

$$= \frac{1}{\alpha} + \sqrt{\frac{-2\mu}{\alpha} + (\mu^2+\sigma^2) + \frac{1}{\alpha^2}} - \frac{\alpha}{2}\left(\frac{1}{\alpha^2} - \frac{2\mu}{\alpha} + (\mu^2+\sigma^2) + \frac{1}{\alpha^2}\right.$$

$$\left. + \frac{2}{\alpha}\sqrt{\frac{-2\mu}{\alpha} + (\mu^2+\sigma^2) + \frac{1}{\alpha^2}}\right)$$

$$= \cancel{\frac{1}{\alpha}} + \cancel{\sqrt{\frac{-2\mu}{\alpha} + (\mu^2+\sigma^2) + \frac{1}{\alpha^2}}} - \cancel{\frac{1}{2\alpha}} + \mu - \frac{\alpha(\mu^2+\sigma^2)}{2} - \cancel{\frac{1}{2\alpha}}$$

$$- \cancel{\sqrt{\frac{-2\mu}{\alpha} + (\mu^2+\sigma^2) + \frac{1}{\alpha^2}}}$$

$$= \mu - \frac{\alpha}{2}(\mu^2+\sigma^2) \quad \checkmark$$

$\rightarrow$ By inspection $x_{CE} = \frac{1}{\alpha} - \sqrt{\frac{-2\mu}{\alpha} + (\mu^2+\sigma^2) + \frac{1}{\alpha^2}}$ also satisfies expected value.

Absolute Risk Premium $= \Pi_A = \mathbb{E}[x] - x_{CE}$

$$= \mu - \left\{\frac{1}{\alpha} \pm \sqrt{\frac{-2\mu}{\alpha} + (\mu^2+\sigma^2) + \frac{1}{\alpha^2}}\right\}$$

$$\therefore \quad \boxed{\Pi_A = \mu - \frac{1}{\alpha} \pm \sqrt{\frac{-2\mu}{\alpha} + (\mu^2+\sigma^2) + \frac{1}{\alpha^2}}}$$

- \$1,000,000 to invest
- \$ z in a risky asset w/ returns $N(\mu, \sigma^2)$
- \$1000000 - \$ z in riskless asset with fixed annual return r
- Static Asset Allocation Problem
- Risk Aversion based on : $U(W) = W - \frac{\alpha W^2}{2}$
- ? optimal value of z for fixed α ?

$$dW_t = (r + \pi \underbrace{(\mu - r)}_{\text{Excess return}}) W_t \cdot dt + \pi \sigma W_t \cdot dz_t$$

$$\frac{dW_t}{W_t} = (r + \pi(\mu - r))\, dt + \pi \sigma\, dz_t$$

$\hookrightarrow$ Let $Y_t = \log W_t$

$$\therefore dY_t = \frac{\partial \log W_t}{\partial W_t} dW_t + \frac{\partial \log W_t}{\partial t} dt + \frac{1}{2} \frac{\partial^2 \log W_t}{\partial W_t^2} dW_t^2$$

$dW_t^2 = \pi^2 \sigma^2 W_t^2\, dz_t^2$, $dz_t^2 \updownarrow dt$

$$= \frac{\partial W_t}{W_t} - \frac{1}{2} \frac{1}{W_t^2} \partial W_t^2$$

$$= (r + \pi(\mu - r))\, dt + \pi \sigma\, dz_t - \frac{1}{2} \pi^2 \sigma^2\, dt$$

$$\therefore dY_t = \left(r + \pi(\mu - r) - \frac{\pi^2 \sigma^2}{2}\right) dt + \pi \sigma\, dz_t$$

$$\therefore \log W_t = \int_0^t \left(r + \pi(\mu - r) - \frac{\pi^2 \sigma^2}{2}\right) dt' + \int_0^t \pi \sigma\, dz_{t'}$$

$$\Rightarrow \log W \sim N\left(r + \pi(\mu - r) - \frac{\pi^2 \sigma^2}{2},\ \pi^2 \sigma^2\right)$$

Maximize $\mathbb{E}[U(W)] \Leftrightarrow \mathbb{E}[U(\log W)]$

$$\therefore \pi^* = \underset{\pi}{\operatorname{argmax}} \left[\mu - \frac{\alpha}{2}(\mu^2 + \sigma^2)\right]$$

$$= \underset{\pi}{\operatorname{argmax}} \left[r + \pi(\mu - r) - \frac{\pi^2 \sigma^2}{2} - \frac{\alpha}{2}\left\{\left(r + \pi(\mu - r) - \frac{\pi^2 \sigma^2}{2}\right)^2 + \pi^4 \sigma^4\right)\right]$$

$$= \underset{\pi}{\operatorname{argmax}} \Big[r + \pi(\mu - r) - \frac{\pi^2\sigma^2}{2} - \frac{r^2\alpha}{2} - \frac{\pi^2(\mu-r)^2\alpha}{2} - \frac{3\pi^4\sigma^4\alpha}{4}$$

$$- \frac{\cancel{2}\alpha r \pi(\mu - r)}{\cancel{2}} + \frac{\cancel{2}\alpha r \pi^2\sigma^2}{\cancel{2}} + \frac{\cancel{2}\alpha\pi^3\sigma^2(\mu-r)}{\cancel{4}\,2}$$

$$\cancel{- \frac{\alpha\pi^4\sigma^4}{2}} \Big]$$

$$\Rightarrow 0 = \frac{\partial}{\partial \pi} \Big[r + \pi(\mu-r) - \frac{\pi^2\sigma^2}{2} - \frac{r^2\alpha}{2} - \frac{\pi^2(\mu-r)^2\alpha}{2}$$

$$- \frac{3}{4}\pi^4\sigma^4\alpha - \alpha r \pi(\mu-r) + \alpha r \pi^2 \sigma^2$$

$$+ \frac{\alpha\pi^3\sigma^2(\mu-r)}{2} \Big]$$

$$\Rightarrow 0 = \mu - r - \pi\sigma^2 - \pi(\mu-r)^2\alpha - 3\pi^3\sigma^4\alpha - \alpha r(\mu - r)$$

$$+ 2\alpha r \pi \sigma^2 + \frac{3}{2}\alpha\sigma^2(\mu-r)\pi^2$$

$$\therefore 0 = \pi^3(-3\sigma^4\alpha) + \pi^2\left(\frac{3}{2}\alpha\sigma^2(\mu-r)\right)$$

$$+ \pi\left(-\sigma^2 - (\mu-r)^2\alpha + 2\alpha r\sigma^2\right)$$

$$+ \left((\mu-r)(1-\alpha r)\right)$$

- Apply numerical root finding method to find π^* that satisfies the above equation and note:

$$z = \$1{,}000{,}000\,\pi$$

- TAKEAWAYS:
  1. Find an expression for expected utility of X, $E[U(X)]$
  2. Find the distribution of W_t
  3. Find the value of π that maximizes $E[U(W_t)]$.

Problem 2:

Utility Function : $U(x) = \log(x)$

Setup :

- Single - Period, static problem
- 1 risky asset, 1 riskless asset

$\therefore$ Like Problem 1, model equation for wealth is:

$$dW_t = (r + \pi(\mu - r)) \cdot W_t \cdot dt + \pi\sigma W_t \, dz_t$$

Let $Y(t, W_t) = \log W_t$

$$\therefore \quad dY = \cancel{\frac{\partial Y}{\partial t} dt} + \frac{\partial Y}{\partial W_t} dW_t + \frac{1}{2} \frac{\partial^2 Y}{\partial W_t^2} dW_t^2$$

$$= \frac{1}{W_t} dW_t - \frac{1}{2} \frac{1}{W_t^2} \pi^2 \sigma^2 W_t^2 \, dz_t^2$$

$$= (r + \pi(\mu - r)) \, dt + \pi\sigma \, dz_t + \frac{\pi^2\sigma^2}{2} dt$$

$$\therefore \quad dY = \left(r + \pi(\mu - r) + \frac{\pi^2\sigma^2}{2} \right) dt + \pi\sigma \, dz_t$$

$$\therefore \quad \log W_t = \int_0^t \left\{ r + \pi(\mu - r) + \frac{\pi^2\sigma^2}{2} \right\} dt' + \int_0^t \pi\sigma \, dz_{t'}$$

$$\Rightarrow \quad \log W_t = \mathcal{N}\left(r + \pi(\mu - r) + \pi^2\sigma^2/2 \, , \, \pi^2\sigma^2 \right)$$

$$\mathbb{E}[U(x)] \approx U(\bar{x}) + \frac{1}{2} U''(\bar{x}) \cdot \sigma_x^2$$

$\bar{x} \equiv \mathbb{E}[x]$

$$U(x) = \log(x)$$

$$\therefore \quad U'(x) = \frac{1}{x}$$

$$\therefore \quad U''(x) = -\frac{1}{x^2}$$

$$\therefore \quad \mathbb{E}[\log(W_t)] = \log(\mu) - \frac{\sigma^2}{2\mu^2}$$

• Maximize $\mathbb{E}[U(W_t)]$ $\Leftrightarrow$ Maximize $\mathbb{E}[U(\log W_t)]$

$$\therefore \quad \pi^* = \underset{\pi}{\text{argmax}} \quad \log\left[r + \pi(\mu - r) + \frac{\pi^2\sigma^2}{2}\right]$$

$$- \frac{\pi^2\sigma^2}{2r + 2\pi(\mu-r) + \pi^2\sigma^2}$$

$\hookrightarrow$ This is messy. Consider instead using X_{CE}

$$\log(X_{CE}) = \log(\mu) - \frac{\sigma^2}{2\mu^2}$$

$$\therefore \quad X_{CE} = \frac{\exp[\log(\mu)]}{\exp[\sigma^2/(2\mu^2)]}$$

$$= \mu \exp[-\sigma^2/(2\mu^2)]$$

$$\therefore \quad \pi^* = \underset{\pi}{\text{argmax}} \left[r + \pi(\mu - r) + \frac{\pi^2\sigma^2}{2}\right] \exp\left\{\frac{-\pi^2\sigma^2}{2r + 2\pi(\mu-r) + \pi^2\sigma^2}\right\}$$

$U(x)$ is concave $\therefore$ X_{CE} is concave $\therefore$ π^* is π that satisfies:

$$0 = \frac{d}{d\pi}\left[r + \pi(\mu - r) + \frac{\pi^2\sigma^2}{2}\right] \exp\left\{\frac{-\pi^2\sigma^2}{2r + 2\pi(\mu-r) + \pi^2\sigma^2}\right\}$$

$$\therefore \quad 0 = [\mu - r + \pi\sigma^2] \exp\left\{\frac{-\pi^2\sigma^2}{2r + 2\pi(\mu-r) + \pi^2\sigma^2}\right\}$$

$$+ \left[r + \pi(\mu - r) + \frac{\pi^2\sigma^2}{2}\right] \exp\left\{\frac{-\pi^2\sigma^2}{2r + 2\pi(\mu-r) + \pi^2\sigma^2}\right\}$$

$$\cdot \left[\frac{-\pi\sigma^2}{r + \pi(\mu-r) + \pi^2\sigma^2/2} + \right.$$

$$\left. \frac{\pi^2\sigma^2}{(2r + 2\pi(\mu-r) + \pi^2\sigma^2)^2}\left(2(\mu - r + \pi\sigma^2)\right)\right]$$

$\hookrightarrow$ π^* is value of π that satisfies this above equation

Problem 3

- Objective: Select betting strategy to maximize expected wealth over long run.
  - ↳ Optimal solution known as Kelly Criterion - some constant fraction of one's wealth bet each turn (f^*)
    - ↳ Obtained by maximizing utility of wealth after a single turn, with $U(W) = \log(W)$

- Let W_0 be initial wealth and f be fraction of W_0 in bet.

↳ Two outcomes for W after a single bet:

① Winning Turn $\Rightarrow$ $W = f \cdot W_0 (1 + \alpha) + (1-f) W_0 = f W_0 \alpha + W_0$

② Losing Turn $\Rightarrow$ $W = f \cdot W_0 (1 - \beta) + (1-f) W_0 = -f W_0 \beta + W_0$

$\therefore$ Two outcomes for $U(W)$ after a single bet:

① Winning Turn $\Rightarrow$ $U(W) = \log(f W_0 \alpha + W_0)$

② Losing Turn $\Rightarrow$ $U(W) = \log(-f W_0 \beta + W_0)$

$\therefore$ Expected Utility:

$$\mathbb{E}[\log W] = p \log(f W_0 \alpha + W_0) + (1-p) \log(-f W_0 \beta + W_0)$$

↳ Derivative of $\mathbb{E}[\log W]$ w.r.t. f:

$$\frac{d}{df} \mathbb{E}[\log W] = \frac{d}{df} \left\{ p \log(f W_0 \alpha + W_0) + (1-p) \log(-f W_0 \beta + W_0) \right\}$$

$$= \frac{p \cancel{W_0} \alpha}{\cancel{W_0}(f\alpha + 1)} - \frac{(1-p) \cancel{W_0} \beta}{\cancel{W_0}(1 - f\beta)}$$

→ Second derivative of $\mathbb{E}[\log W]$ w.r.t. f :

$$\frac{d}{df}\left\{ \frac{p W_0 \alpha}{W_0(f\alpha+1)} - \frac{(1-p) W_0 \beta}{W_0(1-f\beta)} \right\}$$

$$= -\frac{p W_0^2 \alpha^2}{(W_0(f\alpha+1))^2} - \frac{(1-p) W_0^2 \beta^2}{(W_0(1-f\beta))^2}$$

$p \in [0,1] \geq 0$
$W_0^2 > 0$
$\alpha^2 > 0$
$(W(f\alpha+1))^2 > 0$

∴ Term is negative

$p \in [0,1]$ ∴ $(1-p) \geq 0$
$W_0^2 > 0$
$\beta^2 > 0$
$(W_0(1-f\beta))^2 > 0$

∴ Term is negative

⇒ Since both terms of the second derivative are negative, the second derivative is negative, and $\mathbb{E}[U(W)]$ is concave

∴ $E[U(x)]$ is maximized where $\frac{d}{df}\mathbb{E}[U(W)] = 0$

∴ f^* satisfies :

$$0 = \frac{p \cancel{W_0} \alpha}{\cancel{W_0}(f^*\alpha+1)} - \frac{(1-p)\cancel{W_0}\beta}{\cancel{W_0}(1-f^*\beta)}$$

$$\therefore \frac{(1-p)\beta}{1-f^*\beta} = \frac{p\alpha}{f^*\alpha+1}$$

$$\Rightarrow p\alpha - p\alpha\beta f^* = (1-p)\beta\alpha f^* + (1-p)\beta$$

$$\Rightarrow p\alpha - (1-p)\beta = f^*\left(p\cancel{\alpha}\beta + (1-\cancel{p})\beta\alpha\right)$$

$$\therefore f^* = \frac{p}{\beta} - \frac{(1-p)}{\alpha} = \frac{p\alpha - \beta + p\beta}{\alpha\beta}$$

$$\therefore \boxed{f^* = \frac{p(\alpha+\beta)-\beta}{\alpha\beta}}$$

- Interpretation of Results:

  - $\alpha, \beta \in \mathbb{R}^+ \therefore \uparrow p \Rightarrow \uparrow f^*$

    As the probability of success increases, you want to bet a larger fraction of your wealth in the game with equivalent returns

  - $\alpha, \beta \in \mathbb{R}^+, \ p \in [0,1] \therefore \uparrow \alpha \Rightarrow \uparrow f^*$

    ↳ Easiest to see in form:

    $$f^* = \frac{p}{\beta} - \frac{(1-p)}{\alpha}$$

    As the returns from a winning game increases, you want to bet more at constant β, p.

  - $\alpha, \beta \in \mathbb{R}^+, \ p \in [0,1] \therefore \uparrow \beta \Rightarrow \downarrow f^*$

    ↳ Easiest to see in form:

    $$f^* = \frac{p}{\beta} - \frac{(1-p)}{\alpha}$$

    As the losses in a losing game increases, you want to bet less at constant α, p.