- CME 241 Notes: OB Dynamics and Market Making
  - People who submit limit orders provide liquidity
  - People who submit market orders are liquidity takers
  - Market Makers = Dealers = Always willing prices they will buy and sell at.
    - ↳ Always show bid and ask at any point in time
    - ↳ Market makers need to constantly adjust prices and # of shares they are willing to buy and sell.
      - ↳ This is a stochastic control problem

- Today, consider single market maker trying to buy at bid and sell at ask and take home profit = spread

- Goal: Maximize Utility of gains
  - ↳ Assume a finite horizon T
  - ↳ If Buy/Sell limit orders too narrow = more frequent but small gains
  - ↳ If spread is wide, less frequent transactions, but larger gains
  - ↳ Market Maker also needs to manage potentially unfavorable inventory buildup and consequent unfavorable liquidation

    Lowering price to balance inventory
    - ↳ You can have negative or positive inventories, and your prices change accordingly

  - ↳ We will consider simplified setup with single bid/ask price

    Finite time steps: $t = 0, 1, \dots, T$

    $W_t \in \mathbb{R}$ = Cash

    $I_t \in \mathbb{Z}$ = Inventory

$S_t \in \mathbb{R}^+$ is the order book Mid Price at t

↳ Average of bid and ask price in market

$P_t^{(b)} \in \mathbb{R}^+$: Market Maker bid price (limit order)

$N_t^{(b)} \in \mathbb{Z}^+$: # of Shares willing to buy

Submit $P_t^{(b)}, N_t^{(b)}, \underbrace{P_t^{(a)}, N_t^{(a)}}_{\text{Asks}}$

Assume market maker can add or remove bids / asks costlessly at t

Denote $\delta_t^{(b)} : \overset{\text{Midprice}}{S_t} - P_t^{(b)}$; $\delta_t^{(a)} = P_t^{(a)} - S_t$

Our bid spread

Market gave shares

$X_t^{(b)} \in \mathbb{Z}_{\geq 0}$ denotes bid-share "hit" UP TO (accumulated) time t

$X_t^{(a)} \in \mathbb{Z}_{\geq 0}$ denotes ask-shares "lifted" UP TO time t

∴ <u>"Material" Balance:</u>

$$W_{t+1} = W_t + P_t^{(a)} \overbrace{\left(X_{t+1}^{(a)} - X_t^{(a)} \right)}^{\text{Lifted}} - P_t^{(b)} \left(X_{t+1}^{(b)} - X_t^{(b)} \right)$$

$$I_t : X_t^{(b)} - X_t^{(a)}$$

$$I_0 = 0$$

Accumulated Lifts

Accumulated Hits

• Goal is to maximize $\mathbb{E}\left[U\left(W_T + I_T S_T \right) \right]$ for concave $U(\cdot)$

Liquidation at the midprice (idealistic)

↳ Assumes no consumption in time horizon

↳ W_T, I_T, S_T are random variables!

$\therefore$ Use expectations

- MDP Activity at $0 \le t \le T-1$:

  Observed State $:: (S_t, W_t, I_t) \in \mathcal{S}_t$

  Actions $:: (P_t^{(b)}, N_t^{(b)}, P_t^{(a)}, N_t^{(a)})$

  ↳ See Feb. 12, slide 23 for remainder of formulation

  π_t^* is <u>deterministic</u>

  ↳ Assume $\gamma = 1$, but it does not matter because reward is at the end.

- <u>Avellaneda - Stoikov Continuous Time Formulation</u> of previously described MDP

  - Observe continuous hits / lifts by Poisson Process

    ↳ Let $\lambda_t^{(b)} dt$, $\lambda_t^{(a)} dt$ be mean $dX_t^{(b)}$, $dX_t^{(a)}$

    ↳ $\lambda_t^{(b)}$: likelihood of hit on submission
    ↳ $\lambda_t^{(a)}$: likelihood of lift on submission

    $\Big\}$ Depends on Spread

    $\therefore$ Define $f^{(b)}$, $f^{(a)}$ be decreasing functions of spread δ

  - In continuous time, poisson probabilities are essentially Bernoulli

    ↳ In infinitesimal time, one share or zero shares are transacted

    $P(\text{buy or sell} > 1 \text{ share}) \approx 0$

    $\therefore$ Treat $dX^{(b)}$ and $dX^{(a)}$ as Bernoulli variables over dt

    $\therefore$ Only focus on $N_t^{(b)}$ and $N_t^{(a)} = 1$, since this is all that can be traded in dt

$\therefore$ Action space reduces to $\left(\delta_t^{(b)}, \delta_t^{(a)}\right)$

Prices, relative to midpoint

↳ Assume CARA utility function for $a > 0$

$\therefore$ $U(x) = -e^{-ax}$ where $a > 0$ is coeff. of risk aversion

- Hamilton - Jacobi - Bellman (HJB) Equation:

Optimal Value Function $V^*(t, S_t, W_t, I_t)$

$V^*(t, S_t, W_t, I_t) = \max\limits_{\delta_t^{(b)}, \delta_t^{(a)}}$ ... (slide 25)

Max at $\underline{\underline{\text{all}}}$ times t

$$\Rightarrow \max_{\delta_t^{(b)}, \delta_t^{(a)}} \mathbb{E}\left[V^* \right] = 0$$

- Change to V^* comprised of 3 components:

  1. Due to movements in time
  2. Due to changes in midprice
  3. Due to randomness of hits/lifts

↳ Slide 26 is expansion of dV^* due to these factors

↳ include $(dz_t)^2$ because for stochastic processes

↳ Change in V^* due to hitting and lifting affected by 3 possible outcomes

↳ Hit $\Rightarrow$ lower cash and increase inventory by 1

↳ Lift $\Rightarrow$ ↑ cash, inventory down by 1

↳ Neither hit nor lift should not affect V^*

Slide 27: Final equation on slide shows changes in value function with hits and misses

↳ Separable max over actions

Slide 28: Replace λ with $f(\delta)$

Make guess form of V^* based on structure of terminal state

↳ Not tested on ability to find guess form

↳ Slide 28: Job is to solve for θ

↳ Substituting θ into slide 28 PDE

↳ Slide 29

↳ Includes discrete differential of θ due to hit/lift

- Differences in θ due to hit/lift is known as the Indifference Bid and Ask Prices

↳ States optimal value function should not change when buying or selling 1 share

↳ Big concept in economics.

$Q^{(b)}(t, S_t, I_t)$: Indifference bid price: Price to buy 1 share such that optimal VF does not change with guarantee of immediate sale

↳ Substitute $Q^{(b)}$ and $Q^{(a)}$ into equation on slide 29

Slide 32: Set partial derivatives of each max argument equal to zero

↳ Equations 6, 7 are implicit equations for $\delta^{(b)*}$ and $\delta^{(a)*}$ cannot be evaluated analytically

↳ Solving Numerically

Follow instructions on slide 33-34

↳ Assignment is to solve steps up to slide 34

- Intuition : Indifference Midprice is an adjustment to S_t (midprice) based on inventory risk
  $\uparrow Q_t^{(m)}$

  ↳ Center interval on slide 35 about $Q_t^{(m)}$

  $P_t^{(b)*}$ and $P_t^{(a)*}$ move in tandem with $Q_t^{(m)}$

- Slide 36 : Make simplifying assumption for $f^{(b)}(\delta)$ and $f^{(a)}(\delta)$

  ↳ Slide 37 apply linear approximation of $e^{\cdots}$

- Slide 38 : Assume θ is an asymptotic expansion and keep only up to quadratic term

  ↳ Collect terms for I , I^2 and solve separately

- Slide 44 : $Q^{(m)}$ carries all influence of inventory I