Problem 1: Derive Solution to Merton's Portfolio problem for the case of $\log(\cdot)$ utility function.

- Consider 1 risky asset and 1 riskless asset.
- Goal: Maximize finite-horizon aggregated expected utility of consumption
  - ↪ Assume: Current Wealth $W_0 > 0$, finite horizon is T more years, risky asset has normally distributed returns $N(\mu, \sigma^2)$, asset quantities are continuous, trading is frictionless, consumption of any fractional amount of wealth at any time, bequest $B(T) = \epsilon$ where $0 < \epsilon << 1$, discount rate $\gamma = 1$
  - ↪ Decision: Allocation and consumption at each time.

- Model of Wealth at time t:

consumption at t

$$dW_t = ((\pi_t \cdot (\mu - r) + r) W_t - c_t)\,dt + \pi_t \sigma W_t \, dz_t$$

Wealth allocation to risky asset

returns of riskless asset

- Utility Function:

$$U(c_t) = \log(c_t)$$

$$B(T)\,U(W_T) = \epsilon \log(W_T)$$

∴ Return at time t is given by accumulated discounted reward:

$$\int_t^T e^{-\rho(s-t)} \log(c_s)\,ds + e^{-\rho(T-t)} \epsilon \log(W_t)$$

ρ is utility discount factor

↪ Find policy to maximize expected return

$$\text{policy}: (t, W_t) \longrightarrow (\pi_t, c_t)$$

↪ $c_t \geq 0$

↪ π_t unconstrained

- Apply HJB Equation to obtain:

$$\max_{\pi_t, c_t} \left\{ \mathbb{E}_t \left[dV^*(t, W_t) + \log(c_t)\, dt \right] \right\} = \rho V^*(t, W_t)\, dt$$

- Apply Ito's Lemma on dV^*:

Eq 1 $\longrightarrow$ $$\max_{\pi_t, c_t} \left\{ \frac{\partial V^*}{\partial t} + \frac{\partial V^*}{\partial W_t}\left((\pi_t(\mu - r) + r)W_t - c_t\right) + \frac{\partial^2 V^*}{\partial W_t^2} \frac{\pi_t^2 \sigma^2 W_t^2}{2} + \log(c_t) \right\} = \rho V^*(t, W_t)$$

$\hookrightarrow$ Terminal Condition: $V^*(T, W_T) = \epsilon \log(W_T)$

$\hookrightarrow$ Let $\Phi(t, W_t; \pi_t, c_t) =$

$$\frac{\partial V^*}{\partial t} + \frac{\partial V^*}{\partial W_t} \partial W_t + \frac{\partial^2 V^*}{\partial W_t^2} \frac{\pi_t^2 \sigma^2 W_t^2}{2} + \log(c_t)$$

$$\therefore \frac{\partial \Phi}{\partial \pi_t} = W_t(\mu - r)\frac{\partial V^*}{\partial W_t} + \frac{\partial^2 V^*}{\partial W_t^2} \pi_t \sigma^2 W_t^2$$

$\therefore$ π_t^* satisfies

$$(\mu - r)\frac{\partial V^*}{\partial W_t} + \frac{\partial^2 V^*}{\partial W_t^2} \pi_t^* \sigma^2 W_t = 0$$

$$\therefore \pi_t^* = \frac{-\frac{\partial V^*}{\partial W_t}(\mu - r)}{\frac{\partial^2 V^*}{\partial W_t^2} \sigma^2 W_t}$$

$$\therefore \frac{\partial \Phi}{\partial c_t} = -\frac{\partial V^*}{\partial W_t} + \frac{1}{c_t}$$

$\therefore$ c_t^* satisfies: $$-\frac{\partial V^*}{\partial W_t} + \frac{1}{c_t^*} = 0$$

$$\therefore \quad c_t^* = \left(\frac{\partial V^*}{\partial W_t}\right)^{-1} \qquad \therefore \quad \log c_t^* = -\log\left(\frac{\partial V^*}{\partial t}\right)$$

- Substituting π_t^*, c_t^* into Eq. 1 yields :

$$\text{Eq 2} \longrightarrow \frac{\partial V^*}{\partial t} - \frac{(\mu-r)^2}{2\sigma^2}\frac{\left(\frac{\partial V^*}{\partial W_t}\right)^2}{\left(\frac{\partial^2 V^*}{\partial W_t^2}\right)} + \frac{\partial V^*}{\partial W_t} r W_t - \cancel{\frac{\partial V^*}{\partial W_t}\left(\frac{\partial V^*}{\partial W_t}\right)^{-1}}^{\,1} - \log\left[\left(\frac{\partial V^*}{\partial W_t}\right)\right] = \rho V^*(t, W_t)$$

$\hookrightarrow$ Terminal Condition : $V^*(T, W_T) = \epsilon \log(W_T)$

- <u>Guess Solution Form:</u> $V^*(t, W_t) = A\log(W_t) + f(t) = A\log(W_t\, g(t))$

$$\therefore \quad \frac{\partial V^*}{\partial t} = \frac{A}{\cancel{W_t}\, g(t)} \cancel{W_t}\, g'(t)$$

By inspection w/ Terminal Case

$A = \epsilon, \quad g(T) = 1$

$$\therefore \quad \frac{\partial V^*}{\partial W_t} = \frac{A}{W_t \cancel{g(t)}} \cancel{g(t)} \cancel{\frac{\partial W_t}{\partial W_t}}$$

$$\therefore \quad \frac{\partial^2 V^*}{\partial W_t^2} = -\frac{A}{W_t^2}$$

$\therefore$ Substituting guess form into Eq. 2 :

$$\frac{A\, g'(t)}{g(t)} + \frac{(\mu-r)^2}{2\sigma^2}\frac{\left(\frac{A}{\cancel{W_t}}\right)^{\cancel{2}}}{\left(+\frac{\cancel{A}}{\cancel{W_t^2}}\right)} + \frac{A}{\cancel{W_t}} r \cancel{W_t} - 1 - \log\left[\frac{A}{W_t}\right] = \rho\,\epsilon\log(W_t\, g(t))$$

$$\Rightarrow \quad \frac{A\, g'(t)}{g(t)} + A\frac{(\mu-r)^2}{2\sigma^2} + Ar - 1 + \log\left[\frac{W_t}{A}\right] - \rho\,\epsilon\log(W_t\, g(t)) = 0$$

Problem 3:

- Spend a fraction α of your day working and the remaining fraction $(1-\alpha)$ learning.
- Each minute spent working earns $f(s)$ dollars where s is your skill level
- Each minute spent learning improves skill level by $g(s)$