EXERCISES 17

MULTIAGENT DECISION MAKING

17.1 Properties of Multiagent Environments

Exercise 17.GTDF

Define the following in your own words:

a. Multiagent system
b. Multibody planning
c. Coordination problem
d. Agent design
e. Mechanism design
f. Cooperative game
g. Non-cooperative game

a. Multiagent system: any environment with more than one actor.
b. Multibody planning: when a single central agent creates a joint plan for multiple physical units (such as a fleet of delivery robots).
c. Agent design: designing an agent that can act optimally in an environment (or at least approximate optimality).
d. Mechanism design: defining the rules of a game for collective good.
e. Cooperative game: a game in which it is possible for agents to enter into a binding agreement, without the possibility of reneging on the agreement.
f. Non-cooperative game: a game in which no binding agreement is possible, and agents must consider the other agent's strategies.

Exercise 17.CCGT

Give some examples, from movies or literature, of bad guys with a formidable army (robotic or otherwise) that inexplicably is under centralized control rather than more robust multiagent control, so that all the good guys have to do is defeat the one master controller in order to win.

Some examples include:

Movies: Independence Day, Edge of Tomorrow, Wizard of Oz, Tron, Logan's Run, I, Robot.

Literature: The Machine Stops, The Moon Is a Harsh Mistress.

Exercise 17.GLDB

The British TV game show called "GoldenBalls" had a game element in which two players independently choose whether to "split" or "steal" the jackpot. Read a description of the rules at `https://en.wikipedia.org/wiki/Golden_Balls#Split_or_Steal?`.

a. Formulate the game as a payoff matrix, assuming utility is equal to monetary reward, and analyze it using solution concepts and social welfare concepts.

b. What other factors might play a role in the formulation of utility? How does this change the game?

c. Now watch the video at `https://www.youtube.com/watch?v=S0qjK3TWZE8` or `http://tinyurl.com/ofruebv` and explain what is happening using game theory concepts.

a. The payoff matrix is:

	split	*steal*
split	$50, 50$	$0, 100$
steal	$100, 0$	$0, 0$

(steal) is a dominant strategy, so (steal, steal) is a dominant strategy equilibrium. The only outcome that is *not* a Nash equilibrium is (split, split). The only outcome that does not maximise utilitarian social welfare is (steal, steal) The outcomes (steal, steal) and (split, split) maximize egalitarian social welfare.

b. As always, the value of money might not be linear with the amount of money. In this game, since the TV show is seen by many viewers, players may derive utility from viewers collective perception of them: positive utility from being seen as being fair (splitting), or honest (keeping their word), or clever (influencing the opponent to arrive at a good outcome); or negative utility from being seen as greedy (stealing) or naive (letting their opponent steal from them). It is not clear how to add the value of viewer perception to the value of money.

c. bBoth players want to avoid the (steal, steal) dominant strategy equilibrium. In the video, Nick realizes that declaring "trust me, I'm going to play split" is not a credible threat. So Nick instead declares (1) "trust me, I'm going to play steal" and (2) "I will give you half the money," thereby *changing the game*. The first part is a credible threat because steal is a dominant strategy. The second part is only partially credible. But if Abraham believes the first part is credible, and believes the second part with anything more than probability 0, then Abraham is better off playing split than steal. Interestingly, Nick then plays split, despite his promise/threat, because Nick realizes that there is a chance Abraham will play steal. (Perhaps Nick could then convince Abraham to share some of the money, on the grounds that he demonstrated good faith.)

17.2 Non-Cooperative Game Theory

Exercise 17.SWGT

Either prove or disprove each of the following statements in the context of 2×2 games (you may find it helpful to do proofs by providing examples or counter examples).

a. If a player i has a dominant strategy in a game, then in every Nash equilibrium of that game player i will choose a dominant strategy.

b. Every dominant strategy equilibrium of a game is a Nash equilibrium.

c. Every Nash equilibrium of a game is a dominant strategy equilibrium.

d. If a game outcome maximises utilitarian social welfare, then is Pareto efficient.

e. If a game outcome is Pareto efficient, then it maximises utilitarian social welfare.

f. If all utilities in a game are positive, then any outcome that maximises the product of utilities of players is Pareto efficient.

g. If all utilities in a game are positive, then any Pareto efficient outcome of the game will maximise the product of utilities of players.

a. True.

b. True.

c. False.

d. True.

e. False.

f. True.

g. False.

Exercise 17.DOMQ

Show that a dominant strategy equilibrium is a Nash equilibrium, but not vice versa.

This question is simple a matter of examining the definitions. In a dominant strategy equilibrium $[s_1, \ldots, s_n]$, it is the case that for every player i, s_i is optimal for every combination t_{-i} by the other players:

$$\forall\, i \ \ \forall\, t_{-i} \ \ \forall\, s_i' \ \ [s_i, t_{-i}] \succsim [s_i', t_{-i}] \ .$$

In a Nash equilibrium, we simply require that s_i is optimal for the particular current combination s_{-i} by the other players:

$$\forall\, i \ \ \forall\, s_i' \ \ [s_i, s_{-i}] \succsim [s_i', s_{-i}] \ .$$

Therefore, dominant strategy equilibrium is a special case of Nash equilibrium. The converse does not hold, as we can show simply by pointing to the CD/DVD game, where neither of the Nash equilibria is a dominant strategy equilibrium.

Exercise 17.RPSG

In the children's game of rock–paper–scissors each player reveals at the same time a choice of rock, paper, or scissors. Paper wraps rock, rock blunts scissors, and scissors cut paper. In the extended version rock–paper–scissors–fire–water, fire beats rock, paper, and scissors; rock, paper, and scissors beat water; and water beats fire. Write out the payoff matrix and find a mixed-strategy solution to this game.

In the following table, the rows are labelled by A's move and the columns by B's move, and the table entries list the payoffs to A and B respectively.

	R	P	S	F	W
R	0,0	–1,1	1,-1	–1,1	1,-1
P	1,-1	0,0	–1,1	–1,1	1,-1
S	–1,1	1,-1	0,0	–1,1	1,-1
F	1,-1	1,-1	1,-1	0,0	–1,1
W	–1,1	–1,1	–1,1	1,-1	0,0

Suppose A chooses a mixed strategy $[r : R;\ p : P;\ s : S;\ f : F;\ w : W]$, where $r + p + s + f + w = 1$. The payoff *to* A of B's possible pure responses are as follows:

$$
\begin{aligned}
R &: +p - s + f - w \\
P &: -r + s + f - w \\
S &: +r - p + f - w \\
F &: -r - p - s + w \\
W &: +r + p + s - f
\end{aligned}
$$

It is easy to see that no option is dominated over the whole region. Solving for the intersection of the hyperplanes, we find $r = p = s = 1/9$ and $f = w = 1/3$. By symmetry, we will find the same solution when B chooses a mixed strategy first.

Exercise 17.PIRT

Consider the following scenario:

Five pirates wish to divide the loot of a 100 gold pieces. They are democratic pirates, in their own way, and it is their custom to make such divisions in the following manner: The fiercest pirate makes a proposal about the division and everybody (including the proposer) votes on it. If 50 percent or more are in favor, the proposal is implemented. Otherwise the proposer is thrown overboard, and the procedure is repeated with the next fiercest pirate.

All the pirates enjoy throwing their fellows overboard, but given a choice they prefer more gold. Of course, they intensely dislike being thrown overboard themselves. (Specifically, we say that each pirate assigns a utility of 1 *to each gold piece, a utility of* $1/100$ *to throwing another pirate overboard, and a utility of* -1000 *to being thrown overboard). All pirates are rational and know that the* *other pirates are also rational. Moreover, no two pirates are equally fierce, so there is a precise order, known to all, of making proposals. The gold pieces are indivisible and arrangements to share pieces are not permitted, because no pirate trusts his mates.*

What proposal should the fiercest pirate make?

We proceed by backwards induction in order to derive the proposal made by the fiercest pirate; we call the first proposer 1, the second 2 and so on.

pirate 5 remains: Pirate 5 receives all the gold, plus $\frac{1}{100}$ for each pirate tossed overboard, for a total utility of $100+\frac{4}{100}$, while the rest get at most -999 (they get some utility from throwing other pirates overboard, before being thrown overboard themselves).

4 proposes: Even if pirate 5 rejects a division, pirate 4's vote is 50% of the total vote, so any division proposed by pirate 4 is accepted. Thus, the best choice for player 4 is to keep all the money while player 5 gets nothing.

3 proposes: In order not to get tossed overboard, pirate 3 must propose a division that will guarantee either 4 or 5 a better payoff than what they are receiving should only the two of them remain. He cannot sway pirate 4, as he can get $100+\frac{3}{100}$ by rejecting the offer, and the most pirate 3 can offer him is 100, for a total utility of $100+\frac{2}{100}$ (i.e., pirate 4 prefers to get 100 by throwing pirate 3 overboard rather than accepting his offer). However, pirate 3 can offer 1 to pirate 5 (and nothing to pirate 4)—pirate 5 will accept as this guarantees him at least some gold. This is obviously the best that pirate 3 can do, so he can guarantee 99 coins for himself.

2 proposes: In order to have his proposal accepted, pirate 2 needs the approval of just one other pirate. The pirate who is easiest to bribe (i.e. the one who guarantees the highest payoff to pirate 2) is pirate 4; by giving pirate 4 one gold coin (and nothing to pirates 3 and 5), pirate 2 makes pirate 4 accept the division.

1 proposes: In order to have his proposal accepted, pirate 1 needs two more votes. By giving 3 and 5 one coin each, he ensures that they will get strictly higher utility than what they can get if 2 proposes. Thus, in the unique subgame perfect Nash equilibrium of this game pirate 1 keeps 98 coins, offers 1 coin to pirates 3 and 5 and nothing to pirates 2 and 4, and this proposal is accepted since pirates 1, 3, and 5 vote for it. Note that no one gets thrown overboard!

Exercise 17.TFMG

Solve the game of *three*-finger Morra.

The payoff matrix for three-finger Morra is as follows:

	O: one	*O: two*	*O: three*
E: one	$E=2,O=-2$	$E=-3,O=3$	$E=4,O=-4$
E: two	$E=-3,O=3$	$E=4,O=-4$	$E=-5,O=5$
E: three	$E=4,O=-4$	$E=-5,O=5$	$E=6,O=-6$

Suppose E chooses a mixed strategy $[p_1 : one;\ p_2 : two;\ p_3 : three]$, where $p_1 + p_2 + p_3 = 1$. The payoff *to* E of O's possible pure responses are as follows:

$$\begin{aligned} one &: 2p_1 - 3p_2 + 4p_3 \\ two &: -3p_1 + 4p_2 - 5p_3 \\ three &: 4p_1 - 5p_2 + 6p_3 \end{aligned}$$

It is easy to see that no option is dominated over the whole region. Solving for the intersection of the hyperplanes, we find $p_1 = 1/4$. $p_2 = 1/2$, $p_3 = 1/4$. The expected value is 0.

Exercise 17.FBPK

In the game of football ("soccer" in the US), a player who is awarded a penalty kick scores about 3/4 of the time. Suppose we model a penalty kick as a game between two players, the shooter, S, and the goalkeeper, G. The shooter has 4 possible actions:

LC: Aim for left corner of the goal.
LM: Aim for left middle of the goal.
RM: Aim for right middle of the goal.
RC: Aim for right corner of the goal.

Aiming for a corner risks missing the net completely, but if the shot is on target it is difficult for the goalkeeper to make a save. Aiming more in the middle gives less risk of missing the net, but more chance for a save. The goalkeeper has 3 possible actions:

L: Lean to the shooter's left.
M: Stay in the middle.
R: Lean to the shooter's right.

Leaning to one side as the shooter strikes the ball makes it easier to reach a shot to that side, but harder to reach a shot to the other side. The expected percent of goals scored, is as follows:

	L	**M**	**R**
LC	65	80	85
LM	50	55	90
RM	90	55	50
RC	85	80	65

Football is a zero-sum game, so these are the values for S, and the negative of these are the values for G. What are equilibrium strategies for S and G and what is the outcome of the game?

Also, can you come up with a slightly different payoff matrix that results in equilibrium strategies where both players use at least three of their available actions at least some of the time?

The equilibrium strategy for S is a mixed strategy: [0.5: **LC**, 0.5: **RC**]. The equilibrium strategy for G is also a mixed strategy: [0.5: **L**, 0.5:**R**]. The outcome is **75** for S (and thus –75 for G).

	L	**M**	**R**
LC	67	80	85
LM	60	60	94
RM	94	60	60
RC	85	80	67

There are many possible answers to the second part of the question. Below is a payoff matrix with outcome 76.19 for S where there are two symmetric equilibrium strategies, one of which is [0.58: **LC**, 0.19: **RM**, 0.23: **RC**] for S and [0.48: **L**, 0.04: **M**, 0.48: **R**] for G.

Exercise 17.XFPS

Consider a 2 player game in which player 1 can choose A or B. The game ends if she chooses A, while it continues to player 2 if he chooses B. Player 2 can then choose C or D with the game ending if C is chosen, and continuing again to player 1 if D is chosen. Player 1 can then choose E or F, with the game ending either choice.

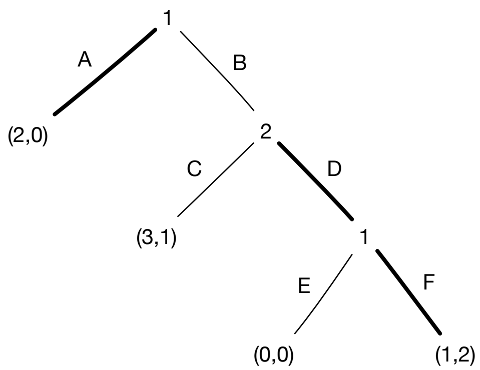
a. Model this as an extensive form game.

b. How many pure strategies does each player have?

c. Identity the subgames of this game.

d. Suppose that choice A gives utilities $(2, 0)$ (i.e., 2 to player A, 0 to player E), choice C gives $(3, 1)$, choice E gives $(0, 0)$, and F gives $(1, 2)$. Then what are the pure Nash equilibria of the game? What SPNE outcome(s) do you obtain through backwards induction?

a. The extensive form:



b. The players' pure strategies are given by:

$$\begin{aligned}\Sigma_1 &= \{AE, AF, BE, BF\} \\ \Sigma_2 &= \{C, D\}\end{aligned}$$

Hence, player 1 has four pure strategies and player 2 two.

c. Three: one for each decision node.

d. Modelling the game in normal form:

	C	D
AE	$2,0$	$\underline{2,0}$
AF	$2,0$	$\underline{2,0}$
BE	$\underline{3,1}$	$0,0$
BF	$3,1$	$1,2$

The pure Nash equilibria are:

$$(AE, D), \quad (AF, D), \quad (BE, C).$$

By backwards induction, (AF, D) can be seen to be the only the subgame perfect Nash equilibrium of this game.

Exercise 17.XFTP

Consider the following scenario:

Two players ($N = \{1, 2\}$) must choose between three outcomes $\Omega = \{a, b, c\}$. The rule they use is the following: Player 1 goes first, and vetoes one of the outcomes. Player two then chooses one of the remaining two outcomes.

Suppose that player preferences are given by:

$$a \succ_1 b \succ_1 c$$
$$c \succ_2 b \succ_2 a$$

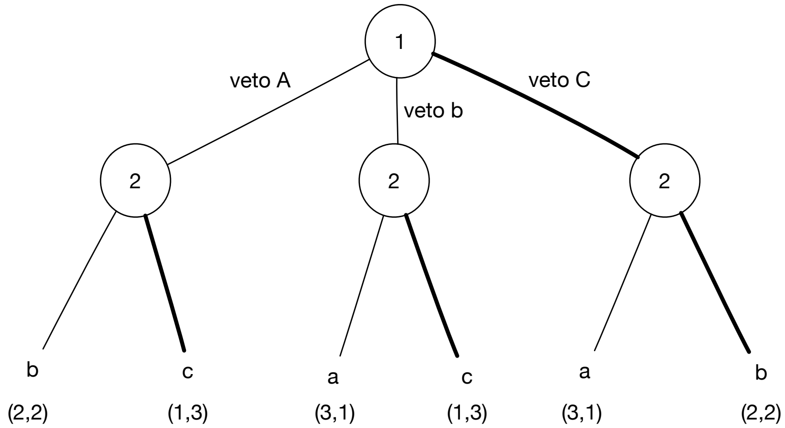
i) Express this scenario as an extensive form game.

ii) Identify any Subgame Perfect Nash Equilibria.

First let's fix utility functions to make things clear:

$$u_1(a) = 3 \quad u_1(b) = 2 \quad u_1(c) = 1$$
$$u_2(a) = 1 \quad u_2(b) = 2 \quad u_2(c) = 3$$

The following figure then illustrates the extensive form game (numbers inside decision nodes indicate the decision maker for that node).



Darker lines coming out from each node indicate subgame perfect Nash equilibria choices for that decision node.

Via backwards induction, the unique subgame perfect equilibrium is thus where player 1 vetoes c, and player 2 then chooses b. With our utility functions, this results in both players getting a payoff of 2.

Exercise 17.PRDG

In the *Prisoner's Dilemma*, consider the case where after each round, Ali and Bo have probability X meeting again. Suppose both players choose the perpetual punishment strategy (where each will choose $refuse$ unless the other player has ever played $testify$). Assume neither player has played $testify$ thus far. What is the expected future total payoff for choosing to $testify$ versus $refuse$ when $X = .2$? How about when $X = .05$? For what value of X is the expected future total payoff the same whether one chooses to $testify$ or $refuse$ in the current round?

For $X = 0.2$, the payoffs for $testify$ and $refuse$ are

$$0 + \sum_{t=1}^{\infty} 0.2^t \cdot (-5) = -1.25; \qquad \sum_{t=0}^{\infty} 0.2^t \cdot (-1) = -1.25\,.$$

and for $X = 0.05$ they are

$$0 + \sum_{t=1}^{\infty} 0.05^t \cdot (-5) \approx -0.263; \qquad \sum_{t=0}^{\infty} 0.05^t \cdot (-1) = -1.053\,.$$

The payoffs are identical for $X = 0.2$.

Exercise 17.FEDG

The following payoff matrix, from Bernstein (1996), shows a game between politicians and the Federal Reserve.

	Fed: contract	Fed: do nothing	Fed: expand
Pol: contract	$F = 7, P = 1$	$F = 9, P = 4$	$F = 6, P = 6$
Pol: do nothing	$F = 8, P = 2$	$F = 5, P = 5$	$F = 4, P = 9$
Pol: expand	$F = 3, P = 3$	$F = 2, P = 7$	$F = 1, P = 8$

Politicians can expand or contract fiscal policy, while the Fed can expand or contract monetary policy. (And of course either side can choose to do nothing.) Each side also has preferences for who should do what—neither side wants to look like the bad guys. The payoffs shown are simply the rank orderings: 9 for first choice through 1 for last choice. Find the Nash equilibrium of the game in pure strategies. Is this a Pareto-optimal solution? You might wish to analyze the policies of recent administrations in this light.

We apply iterated strict dominance to find the pure strategy. First, *Pol: do nothing* dominates *Pol: contract*, so we drop the *Pol: contract* row. Next, *Fed: contract* dominates *Fed: do nothing* and *Fed: expand* on the remaining rows, so we drop those columns. Finally, *Pol: expand* dominates *Pol: do nothing* on the one remaining column. Hence the only Nash equilibrium is a dominant strategy equilibrium with *Pol: expand* and *Fed: contract*. This is not Pareto optimal: it is worse for both players than the four strategy profiles in the top right quadrant.

Exercise 17.PIRI

Consider the following scenario:

There are two pirates operating among three islands A, B, and C. On each island, two treasures are buried: a large one worth 2 and another smaller one worth 1. The prevailing winds in the area are such that from island A you can only reach island B, from island B only island C, and from island C only island A. Once on an island, the pirates only have time to excavate one treasure, before heading to the next island. The pirates are not on good terms, and if they are at the same island at the same time they will fight, and as a result, they will find no treasures (but will not suffer any additional damage). If they do not meet, both pirates will visit all three islands and each pirate will find three treasures. Each of the pirates has to decide where to start their treasure hunt.

a. Model this setting as a normal-form game.

b. Compute a mixed Nash equilibrium of this game and argue that it is unique.

c. Compute players' expected utilities in this equilibrium.

Note that, as soon as we know where each pirate starts, we can compute their payoffs. Thus, for each player the pure strategy space is $\{A, B, C\}$ and the payoff matrix is given by

```
      A     B     C
A    0,0   4,5   5,4
B    5,4   0,0   4,5
```

```
C   4,5   5,4   0,0
```

Next, we will compute the strategy of the first pirate. Suppose the second pirate chooses A with probability p and B with probability q. The first pirate's payoff from choosing A is $0 \cdot p + 4 \cdot q + 5(1 - p - q)$; his payoff from choosing B is $5 \cdot p + 0 \cdot q + 4(1 - p - q)$; his payoff from choosing C is $4 \cdot p + 5 \cdot q + 0(1 - p - q)$. All these payoffs must be equal. We obtain $5 - 5p - q = 4 + p - 4q$ and hence $6p - 3q = 1$, and $5 - 5p - q = 4p + 5q$ and hence $9p + 6q = 5$. From this we obtain $21p = 7$ and hence $p = 1/3$; substituting this into first equation we get $q = 1/3$. Thus, in the unique fully mixed Nash equilibrium each player chooses each strategy with probability $1/3$.

To compute the expected payoff of player 1 in the fully mixed Nash equilibrium, we can assume that this player chooses A (as he gets the same payoff from each pure strategy). Then his expected payoff is $(0 + 5 + 4)/3 = 3$.

Exercise 17.TURO

Avi and Bailey are friends, and enjoy a night out together in the pub. They each will independently decide to go either to the Turf or the Rose.

Avi mildly prefers the Rose over the Turf and would get a utility of 2 from going to the Rose with Bailey, but only 1 from the Turf.

Bailey has a very strong preference for the Turf and would get a utility of 5 for going to the Turf with Avi, and only 1 for the Rose.

If they go to different pubs, they both receive a utility of 0.

a. Formally express this scenario as a strategic form game.

b. Compute all Nash equilibria (pure and mixed) for the game.

c. Compute the expected utility of each player in each equilibrium.

		Bailey	
		T	B
Avi	T	5 1	0 0
	B	0 0	1 2

There are two pure Nash Equilibria: (T, T), which pays off Avi with 1 and Bailey with 5, and (R, R), which pays off Avi with 2 and Bailey with 1.

There is one mixed strategy equilibrium in which Avi plays T with probability $1/6$ and Bailey plays T with probability $2/3$.

Expected utilities:

$$
\begin{aligned}
EU_A(p,q) &= \frac{2}{3} * 1 + \frac{1}{3} * 0 = \frac{2}{3} \\
EU_B(p,q) &= \frac{1}{6} * 5 + \frac{5}{6} * 0 = \frac{5}{6}
\end{aligned}
$$

17.3 Cooperative Game Theory

Exercise 17.CGNE

Define the core of a cooperative game $G = (N, v)$, and show by way of example that the core of a cooperative game may be empty.

There are many games with an empty core. A standard one is as a game with three players and characteristic function $v(C) = 1$ iff $|C| \geq 2$ and $v(C) = 0$ otherwise.

The core is the set of all payoff vectors $(p_1, \ldots, p_n)$ such that $p_i \geq 0$ for all $i \in [n]$, and for all $C \subseteq [n]$ we have $p_C \geq v(C)$, where $p_C = \sum_{i \in C} p_i$.

Exercise 17.WVCG

Recall that a weighted voting game is a cooperative game defined by a structure $G = [q; w_1, \ldots, w_n]$ where q is the quota, the players are $N = \{1, \ldots, n\}$, the value w_i is the weight of player i, and the characteristic function of the game is defined as follows:

$$v(C) = \begin{cases} 1 & \text{if } \sum_{i \in C} w_i \geq q \\ 0 & \text{otherwise.} \end{cases}$$

Compute the players' Shapley values and describe the outcomes in the core in the following weighted voting games:

- $G = [13; 7, 7, 3, 3]$
- $G = [10; 3, 3, 3, 3, 1, 1]$

- First, consider a player i with weight 7; he is pivotal whenever he is preceded by any two players, or when he is preceded by the other player of weight 7. There are $3! = 6$ permutations where i is third, and $2! = 2$ more where he is preceded by the other player of weight 7. Thus, the Shapley value of the players of weight 7 is $\frac{6+2}{4!} = \frac{1}{3}$. By symmetry, both players of weight 7 have a Shapley value of $\frac{1}{3}$; by efficiency, the Shapley value of the two players of weight 3 totals $1 - 2 \times \frac{1}{3} = \frac{1}{3}$. Since they are symmetric as well, the Shapley value of each of them is $\frac{1}{6}$.
- Consider a player of weight 1. he is pivotal if and only if he appears in position 4, after three players of weight 3. There are 4 ways to choose the player of weight 3 who appears after him, 3!=6 ways to order the three players of weight 3 who appear before him, and 2 ways to order the two players (of weight 3 and 1) who appear after him, i.e., $4 \times 6 \times 2 = 48$ permutations. Thus, the Shapley value of each player of weight 1 is $\frac{48}{6!} = \frac{1}{15}$. By efficiency, the sum of Shapley values of the players of weight 3 is $1 - \frac{2}{15} = \frac{13}{15}$, so by symmetry the Shapley value of each of these players is $\frac{13}{60}$.

Exercise 17.CGOT

Suppose we are given a cooperative game $G = (\{1, 2\}, v)$ with characteristic function v defined by:

$$v(C) = \begin{cases} 1 & \text{if } C \text{ contains exactly one player} \\ 0 & \text{otherwise.} \end{cases}$$

Show that weighted voting games cannot capture this "singleton" game: we will not be able to find a quota q and weights w_i such that for all coalitions C, $\sum_{i \in C} w_i \geq q$ iff C contains exactly one player.

Hint: You can give a counterexample involving a game with just two players.

Suppose for sake of contradiction that there does exist a weighted voting game to represent the singleton game, and let q, w_1, w_2 be the quota and respective weights. We must have:

$$\begin{aligned} 0 &< q \\ w_1 &\geq q \\ w_2 &\geq q \\ w_1 + w_2 &< q \end{aligned}$$

It immediately follows that w_1 and w_2 must be > 0.

Since $w_1 \geq q$, then $w_1 + w_2 \geq q$. Contradiction: no values w_1, w_2, q can satisfy these properties. (This is, incidentally, the same as the proof that perceptrons cannot capture XOR.)

Exercise 17.LOWG

In the Landowner and Workers game there is a landowner ℓ and n workers $w_1, \ldots, w_n$. A group of workers may lease the land from the landowner and grow vegetables on it. Their productivity depends on the group size: a group of k workers can grow $f(k)$ tons of vegetables, where f is an increasing function of k, $f(0) = 0$. The characteristic function of this game is then given by $v(C) = f(|C| - 1)$ if $\ell \in C$ and $v(C)$ otherwise. Compute the Shapley values of all players in this game.

When the landowner is in position k, $k = 1, \ldots, n+1$, his predecessors have a value of 0 without him, and with him they have a value of $f(k-1)$. This means that the landowner contributes $f(k-1)$ to all permutations in which he is in position k. There are exactly $n!$ such permutations, so the total contribution to the Shapley value of the landowner from the permutations in which he is in position k is $\frac{n!}{(n+1)!} f(k-1) = \frac{f(k-1)}{n+1}$. Thus, in total, the Shapley value of the owner is simply $\frac{1}{n+1} \sum_{k=1}^{n+1} f(k-1)$, or the average value of f over the points $0, \ldots, n$. Let us denote by φ_w the Shapley value of a worker and by φ_L the value of the landowner; all workers are symmetric so they all must have the same value. By efficiency, $n\varphi_w + \varphi_L = f(n)$, so $\varphi_w = \frac{f(n) - \varphi_L}{n}$, which equals $\frac{f(n) - \frac{1}{n+1} \sum_{k=0}^{n} f(k)}{n}$.

17.4 Making Collective Decisions

Exercise 17.DAUC

A Dutch auction is similar in an English auction, but rather than starting the bidding at a low price and increasing, in a Dutch auction the seller starts at a high price and gradually lowers the price until some buyer is willing to accept that price. (If multiple bidders accept the price, one is arbitrarily chosen as the winner.) More formally, the seller begins with a price p and gradually lowers p by increments of d until at least one buyer accepts the price. Assuming all bidders act rationally, is it true that for arbitrarily small d, a Dutch auction will always result in the bidder with the highest value for the item obtaining the item? If so, show mathematically why. If not, explain how it may be possible for the bidder with highest value for the item not to obtain it.

This question really has two answers, depending on what assumption is made about the probability distribution over bidder's private valuations v_i for the item.

In a Dutch auction, just as in a first-price sealed-bid auction, bidders must estimate the likely private values of the other bidders. When the price is higher than v_i, agent i will not bid, but as soon as the price reaches v_i, he faces a dilemma: bid now and win the item at a higher price than necessary, or wait and risk losing the item to another bidder. In the standard models of auction theory, each bidder, in addition to a private value v_i, has a probability density $p_i(v_1, \ldots, v_n)$ over the private values of all n bidders for the item. In particular, we consider **independent private values**, so that the distribution over the other bidders' values is independent of v_i. Each bidder will choose a bid—i.e., the first price at which they will bid if that price is reached—through a bidding function $b_i(v_i)$.

Independent private values

We are interested in finding a Nash equilibrium (technically a **Bayes–Nash equilibrium** in which each bidder's bidding function is optimal given the bidding functions of the other agents. Under risk-neutrality, optimality of a bid b is determined by the expected payoff, i.e., the probability of winning the auction with bid b times the profit when paying that amount for the item. Now, agent i wins the auction with bid b if all the other bids are less than b; let the probability of this happening be $W_i(b)$ for whatever fixed bidding functions the other bidders use. ($W_i(b)$ is thus a cumulative probability distribution and nondecreasing in b; under independent private values, it does not depend on v_i.) Then we can write the expected payoff for agent i as

$$Q_i(v_i, b) = W_i(b)(v_i - b)$$

and the optimality condition in equilibrium is therefore

$$\forall\, i, b \;\; W_i(b_i(v_i))(v_i - b_i(v_i)) \geq W_i(b)(v_i - b) \; . \tag{17.1}$$

We now prove that the bidding functions $b_i(v_i)$ must be **monotonic**, i.e., *nondecreasing* in the private valuation v_i. Let v and v' be two different valuations, with $b = b_i(v)$ and $b' = b_i(v')$. Applying Equation (17.1) twice, first to say that (v, b) is better than (v, b') and then to say that (v', b') is better than (v', b), we obtain

$$\begin{aligned} W_i(b)(v - b) \; &\geq \; W_i(b')(v - b') \\ W_i(b')(v' - b') \; &\geq \; W_i(b)(v' - b) \end{aligned}$$

Rearranging, these become

$$\begin{aligned} v(W_i(b) - W_i(b')) \; &\geq \; W_i(b)b - W_i(b')b' \\ v'(W_i(b') - W_i(b)) \; &\geq \; W_i(b')b' - W_i(b)b \end{aligned}$$

Adding these equations, we have

$$(v' - v)(W_i(b') - W_i(b)) \geq 0$$

from which it follows that if $v' > v$, then $W_i(b') \geq W_i(b)$. Monotonicity does not follow immediately, however; we have to handle two cases:

- If $W_i(b') > W_i(b)$, or if W_i is strictly increasing, then $b' \geq b$ and $b_i(\cdot)$ is monotonic.
- Otherwise, $W_i(b') = W_i(b)$ and W_i is flat between b and b'. Now if W_i is flat in any interval $[x, y]$, then an optimal bidding function will prefer x over any other bid in the interval since that maximizes the profit on winning without affecting the probability of winning; hence, we must have $b' = b$ and again $b_i(\cdot)$ is monotonic.

Intuitively, the proof amounts to the following: if a higher valuation could result in a lower bid, then by swapping the two bids the agent could increase the *sum* of the payoffs for the two bids, which means that *at least one* of the two original bids is suboptimal.

Returning to the question of efficiency—the property that the item goes to the bidder with the highest valuation—we see that it follows immediately from monotonicity in the case where the bidders' prior distributions over valuations are **symmetric** or identically distributed.*

Vickrey (1961) proves that the auction is *not* efficient in the asymmetric case where one player's distribution is uniform over $[0, 1]$ and the other's is uniform over $[a, b]$ for $a > 0$. Milgrom (1989) provides another, more transparent example of inefficiency: Suppose Alice has a known, fixed value of \$101 for an item, while Bob's value is \$50 with probability 0.8 and \$75 with probability 0.2. Given that Bob will never bid higher than his valuation, Alice can see that a bid of \$51 will win *at least* 80% of the time, giving an expected profit of *at least* $0.8 \times (\$101 - \$51) = \$40$. On the other hand, any bid of \$62 or more cannot yield an expected profit at most \$39, regardless of Bob's bid, and so is dominated by the bid of \$51. Hence, in any equilibrium, Alice's bid at most \$61. Knowing this, Bob can bid \$62 whenever his valuation is \$75 and be sure of winning. Thus, with 20% probability, the item goes to Bob, whose valuation for it is lower than Alice's. This violates efficiency.

Besides efficiency in the symmetric case, monotonicity has another important consequence for the analysis of the Dutch (and first-price) auction : it makes it possible to derive

* Vickrey (1961) proved that under this assumption, the Dutch auction is efficient. Vickrey's argument in Appendix III for the monotonicity of the bidding function is similar to the argument above but, as written, seems to apply only to the uniform-distribution case he was considering. Indeed, much of his analysis beginning with Appendix II is based on an inverse bidding function, which implicitly *assumes* monotonicity of the bidding function. Many other authors also begin by assuming monotonicity, then derive the form of the optimal bidding function, and then show it is monotonic. This proves the existence of an equilibrium with monotonic bidding functions, but not that all equilibria have this property.

the exact form of the bidding function. As it stands, Equation (17.1) is difficult or impossible to solve because the cumulative distribution of the other bidders' bids, $W_i(b)$, depends on their bidding functions, so all the bidding functions are coupled together. (Note the similarity to the Bellman equations for an MDP.) With monotonicity, however, we can define W_i in terms of the known valuation distributions. Assuming independence and symmetry, and writing $b_i^{-1}(b)$ for the inverse of the (monotonic) bidding function, we have

$$Q_i(v_i, b) = (P(b_i^{-1}(b)))^{n-1}(v_i - b)$$

where $P(v)$ is the probability that an individual valuation is less than v. At equilibrium, where b maximizes Q_i, the first derivative must be zero:

$$\frac{\partial Q}{\partial b} = 0 = \frac{(n-1)(P(b_i^{-1}(b)))^{n-2}p(b_i^{-1}(b))(v_i - b)}{b_i'(b_i^{-1}(b))} - (P(b_i^{-1}(b)))^{n-1}$$

where we have used the fact that $df^{-1}(x)/dx = 1/f'(f^{-1}(x))$.

For an equilibrium bidding function, of course, $b_i^{-1}(b) = v_i$; substituting this and simplifying, we find the following differential equation for b_i:

$$b_i'(v_i) = (v_i - b_i(v_i)) \cdot (n-1)p(v_i)/P(v_i)\ .$$

To find concrete solutions we also need to establish a boundary condition. Suppose v_0 is the lowest possible valuation for the item; then we must have $b_i(v_0) = v_0$ (Milgrom and Weber, 1982). Then the solution, as shown by McAfee and McMillan (1987), is

$$b_i(v_i) = v_i - \frac{\int_{v_0}^{v_i}(P(v))^{n-1}dv}{(P(v_i))^{n-1}}\ .$$

For example, suppose p is uniform in $[0,1]$; then $P(v) = v$ and $b_i(v_i) = v_i \cdot (n-1)/n$, which is the classical result obtained by Vickrey (1961).

Exercise 17.AAUC

Imagine an auction mechanism that is just like an ascending-bid auction, except that at the end, the winning bidder, the one who bid b_{max}, pays only $b_{max}/2$ rather than b_{max}. Assuming all agents are rational, what is the expected revenue to the auctioneer for this mechanism, compared with a standard ascending-bid auction?

In such an auction it is rational to continue bidding as long as winning the item would yield a profit, i.e., one is willing to bid up to $2v_i$. The auction will end at $2v_o + d$, so the winner will pay $v_o + d/2$, slightly less than in the regular version.

Exercise 17.NHLT

Teams in the National Hockey League historically received 2 points for winning a game and 0 for losing. If the game is tied, an overtime period is played; if nobody wins in overtime, the game is a tie and each team gets 1 point. But league officials felt that teams were playing too conservatively in overtime (to avoid a loss), and it would be more exciting if overtime produced a winner. So in 1999 the officials experimented in mechanism design: the rules were changed, giving a team that loses in overtime 1 point, not 0. It is still 2 points for a win and 1 for a tie.

a. Was hockey a zero-sum game before the rule change? After?

b. Suppose that at a certain time t in a game, the home team has probability p of winning in regulation time, probability $0.78 - p$ of losing, and probability 0.22 of going into overtime, where they have probability q of winning, $.9 - q$ of losing, and .1 of tying. Give equations for the expected value for the home and visiting teams.

c. Imagine that it were legal and ethical for the two teams to enter into a pact where they agree that they will skate to a tie in regulation time, and then both try in earnest to win in overtime. Under what conditions, in terms of p and q, would it be rational for both teams to agree to this pact?

d. Longley and Sankaran (2005) report that since the rule change, the percentage of games with a winner in overtime went up 18.2%, as desired, but the percentage of overtime games also went up 3.6%. What does that suggest about possible collusion or conservative play after the rule change?

Every game is either a win for one side (and a loss for the other) or a tie. With 2 for a win, 1 for a tie, and 0 for a loss, 2 points are awarded for every game, so this is a constant-sum game.

If 1 point is awarded for a loss in overtime, then for some games 3 points are awarded in all. Therefore, the game is no longer constant-sum.

Suppose we assume that team A has probability r of winning in regular time and team B has probability s of winning in regular time (assuming normal play). Furthermore, assume team B has a probability q of winning in overtime (which occurs if there is a tie after regular time). Once overtime is reached (by any means), the expected utilities are as follows:

$$\begin{aligned} U_A^O &= 1 + p \\ U_B^O &= 1 + q \end{aligned}$$

In normal play, the expected utilities are derived from the probability of winning plus the probability of tying times the expected utility of overtime play:

$$\begin{aligned} U_A &= 2r + (1 - r - s)(1 + p) \\ U_B &= 2s + (1 - r - s)(1 + q) \end{aligned}$$

Hence A has an incentive to agree if $U_A^O > U_A$, or

$$1 + p > 2r + (1 - r - s)(1 + p) \quad \text{or} \quad rp - r + sp + s > 0 \quad \text{or} \quad p > \frac{r - s}{r + s}$$

and B has an incentive to agree if $U_B^O > U_B$, or

$$1 + q > 2s + (1 - r - s)(1 + q) \quad \text{or} \quad sq - s + rq + r > 0 \quad \text{or} \quad q > \frac{s - r}{r + s}$$

When both of these inequalities hold, there is an incentive to tie in regulation play. For any values of r and s, there will be values of p and q such that both inequalities hold.

For an in-depth statistical analysis of the actual effects of the rule change and a more sophisticated treatment of the utility functions, see “Overtime! Rules and Incentives in the National Hockey League” by Stephen T. Easton and Duane W. Rockerbie, available at `http://people.uleth.ca/˜rockerbie/OVERTIME.PDF`.