General Relativity

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1 Covariant Derivative Again

Last lecture we built the covariant derivative by hand by demanding it met certain conditions. These conditions allowed us to uniquely define a covariant derivative that produces a new tensor when acting on a tensor field. In particular we defined it's action on a vector as

$$\nabla_\mu T^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\alpha} V^\alpha. \tag{1.1}$$

Our connection defined uniquely and known as the **Levi-Civita** connection.

Recall that he torsion free requirement of the connection gives us that

$$\Gamma^\rho_{[\mu\nu]} = 0 \implies \Gamma^\rho_{(\mu\nu)} = \Gamma^\rho_{\mu\nu}, \tag{1.2}$$

which left us with 4 degrees of freedom from ρ and 10 from $\mu\nu$ for a total 40 Γs to be determined. We further constrained this by demanding metric compatibility.

$$\nabla_\mu g_{\nu\rho} = 0 = \partial_\mu g_{\nu\rho} - \Gamma^\alpha_{\mu\nu} g_{\alpha\rho} - \Gamma^\alpha_{\mu\rho} g_{\nu\alpha} \tag{1.3}$$

By cyclically permuting the indices and adding the resulting expressions we can find an expression for the connection, we find

$$\begin{aligned} 0 = \nabla_\mu g_{\nu\rho} + \nabla_\nu g_{\rho\mu} + \nabla_\rho g_{\mu\nu} = \partial_\mu g_{\nu\rho} - \Gamma^\alpha_{\mu\nu} g_{\alpha\rho} - \Gamma^\alpha_{\mu\rho} g_{\nu\alpha} + \partial_\nu g_{\rho\mu} - \Gamma^\alpha_{\nu\rho} g_{\alpha\mu} \\ - \Gamma^\alpha_{\nu\mu} g_{\rho\alpha} + \partial_\mu g_{\mu\nu} - \Gamma^\alpha_{\rho\mu} g_{\alpha\nu} - \Gamma^\alpha_{\rho\nu} g_{\mu\alpha} \\ \implies \Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\alpha} \left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} \right). \end{aligned} \tag{1.4}$$

The $\Gamma^\rho_{\mu\nu}$s are known as the **Christoffel Symbols** and can be explicitly calculated using the first derivative of the metric.

Example. Consider the metric in plane-polar coordinates given by

$$ds^2 = dr^2 + r^2 d\phi^2.$$

We will calculate some of the Christoffel symbols for this metric. Letting capital Roman indices run from 1 to 2, $A = 1, 2$, it will also be useful for us to label the Christoffel symbols by coordinates.

$$\Gamma^r_{BC} = \frac{1}{2} g^{rA} \left(\partial_B g_{CA} + \partial_C g_{BA} - \partial_A g_{BC} \right) = \frac{1}{2} g^{rr} \left(\partial_B g_{Cr} + \partial_C g_{Br} - \partial_r g_{BC} \right)$$

Now we can read off from the metric

$$\begin{aligned} \Gamma^r_{\phi\phi} &= \frac{1}{2} g^{rr} \left(\partial_B g_{\phi r} + \partial_C g_{\phi r} - \partial_r g_{\phi\phi} \right) = \frac{1}{2} g^{rr} \left(-\partial_r r^2 \right) = -\frac{1}{2} \times 1 \times 2r = -r \\ \Gamma^r_{rr} &= \frac{1}{2} g^{rr} \left(\partial_r g_{rr} + \partial_r g_{rr} - \partial_r g_{rr} \right) = 0 \end{aligned}$$

2 Parallel Transport

Once you pick a particular connection and covariant derivative, we can ask how these objects "connect" us between spaces. The connection coefficients allow us to associate a vector $\vec{W} \in T_p$ to some other vector in T_q.

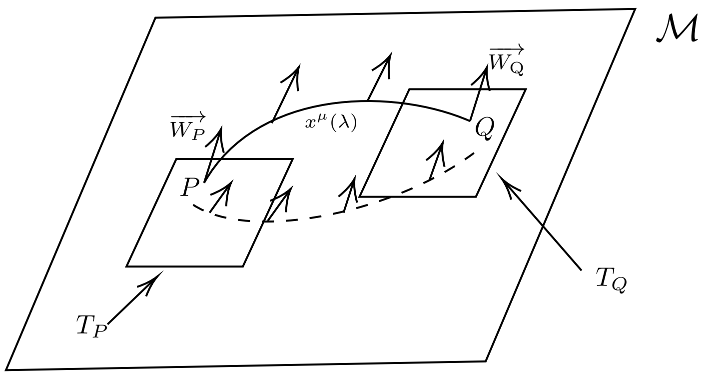

Figure 1. The parallel transport of $\vec{W}$ along x^μ to some transported $\vec{W}$ at point Q.

In order to parallel transport we choose a curve parametrized by λ, $x^\mu(\lambda)$ with a tangent given by $V^\mu = \frac{dx^\mu}{d\lambda}$. The equation

$$\nabla_{\vec{V}}\vec{W} = V^\beta\left(\partial_\beta W^\mu + \Gamma^\mu_{\beta\alpha}W^\alpha\right) = 0, \tag{2.1}$$

defines the parallel transport with respect to a given connection. This boils down to a set of 1^{st} order ODEs for the components of $\vec{W}$ with $\vec{W}^\mu|_p$ as the initial conditions.

Different connections define different rules of parallel transport – or how to keep a vector the 'same' during transport. Generally the solution given by parallel transport will be path dependent, unless however the curvature vanishes. This can be seen in the figure below.

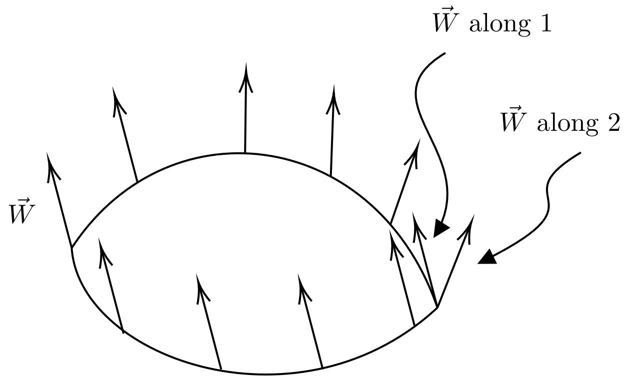

Figure 2. The parallel transport of $\vec{W}$ along two different curves will generally not result in the same transported vector.

For two vectors $\vec{A}$ and $\vec{B}$, we can show that under any metric compatible covariant derivative the parallel transport of the vectors will preserve the inner product (angle) between them. For the parallel transported vectors we have $\nabla_{\vec{V}}\vec{A} = 0$ and $\nabla_{\vec{V}}\vec{B} = 0$. The angle between $\vec{A}$ and $\vec{B}$ is defined as

$$A^\mu B^\nu g_{\mu\nu} = A^\mu B_\mu = \text{"angle"}. \tag{2.2}$$

Now computing the parallel transport along their inner product we find

$$\nabla_{\vec{V}}\left(A^\mu B_\mu\right) = A^\mu B^\nu \nabla_{\vec{V}}(g_{\mu\nu}) + A^\mu g_{\mu\nu} \nabla_{\vec{V}} B^\nu + A^\nu g_{\mu\nu} \nabla_{\vec{V}} B^\mu = 0 \tag{2.3}$$

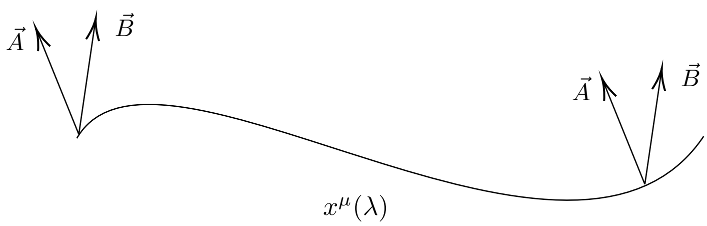

Figure 3. Metric compatible parallel transports preserve angles between vectors.

3 Geodesics

We are now in the position to look at trajectories on which particle motion occurs. Consider a vector $\vec{V}$ which is defined as a vector parallel transported along itself.

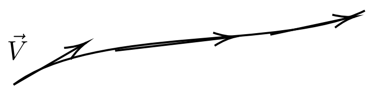

Figure 4. A vector $\vec{V}$ parallel transported along itself.

For a curve $x^\mu(\lambda)$ with tangent $V^\mu = \frac{dx^\mu}{d\lambda}$. Then if $\nabla_{\vec{V}}\vec{V} = 0$, then the curve $x^\mu(\lambda)$ is called a **geodesic**. To use words words a geodesic is a curve whose tangent is parallel transported along itself. Written out in component form, $V^\alpha \nabla_\alpha V^\mu = 0$, which leads to the geodesic equation

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0 \tag{3.1}$$

which is a 2^{nd} order ODE for the curve $x^\mu(\lambda)$ itself. Note, that sometimes the definition of a covariant derivative along a curve is written as

$$\frac{D}{d\lambda} = V^\mu \nabla_\mu = \frac{dx^\mu}{d\lambda} \nabla_\mu . \tag{3.2}$$

In some sense this is Newton's equation written for general relativity. For gravitational forces, rather than using $F = ma$ to solve for the trajectories of particles we use $\frac{D}{d\lambda}\frac{x^\mu}{d\lambda} = 0$. Note that this will not be equal to zero but rather $\frac{F^\mu}{m}$ for non-gravitational forces where F^μ denotes the four-force and m the mass of the particle.

We are now in the place to define the concept of a **test particle**. A test particle is one so small that it does not influence the spacetime around it.

Particles in force free motion travel on geodesics.

Since ∇_ν preserves inner products on geodesics, particles will remain timelike or null on their trajectories.

4 Symmetries

Before we dive into symmetries let's first take stock of all the kinds of derivatives we have. Both the exterior derivative $\underset{\sim}{d}$ and the Lie derivative $\mathcal{L}_{\vec{V}}$ return tensors. For a torsion free covariant derivative we can modify both of these definitions be replacing partial derivatives with covariant derivatives. The derivatives then take the form

$$\begin{aligned} \underset{\sim}{d}\underset{\sim}{\omega} &= \frac{1}{p!}\left(\nabla_\nu \omega_{\mu_1} ... dx^\nu \wedge dx^{\mu_1} \wedge ...\right) \\ \mathcal{L}_{\vec{V}} \vec{W} &= \left[\vec{V}, \vec{W}\right] = V^\alpha \nabla_\alpha W^\mu - W^\alpha \nabla_\alpha V^\mu \end{aligned}$$

Given a vector $\vec{K}$ we take the Lie derivative of the metric

$$\left(\mathcal{L}_{\vec{K}} g\right)_{\mu\nu} = K^\alpha \nabla_\alpha g_{\mu\nu} + \left(\nabla_\mu K^\alpha\right) g_{\alpha\nu} + \nabla_\nu K^\alpha g_{\alpha\mu} \tag{4.1}$$

The first term vanishes by metric compatibility leaving us with

$$\left(\mathcal{L}_{\vec{K}} g\right)_{\mu\nu} = \nabla_\mu K_\nu + \nabla_\nu K_\mu. \tag{4.2}$$

If $g_{\mu\nu}$ unchanged by the Lie derivative along $\vec{K}$, we have a symmetry of the spacetime and can write

$$\mathcal{L}_{\vec{K}} g_{\mu\nu} = \nabla_\mu K_\nu + \nabla_\nu K_\mu = 0, \tag{4.3}$$

where $\vec{K}$ is a Killing vector and 4.3 is known as "Killing's equation".

Figure 5. A Killing vector field $\vec{K}$ that leaves a metric unchanged is a symmetry of the spacetime.

Consider for example $\frac{dx^\mu}{d\tau} = U^\mu$ and $P^\mu = mU^\mu$, where U^μ is tangent to a geodesic. We can compute

$$\begin{aligned} U^\alpha \nabla_\alpha \left(P_\mu K^\mu\right) &= P_\mu U^\alpha \nabla_\alpha K^\mu + \underbrace{\left(mU^\alpha \nabla_\alpha U_\mu\right)}_{=0 \text{ by geo. eqn.}} K^\mu && (4.4) \\ &= mU^\alpha U^\mu \nabla_\alpha K_\nu && (4.5) \\ &= 0 && (4.6) \end{aligned}$$

Where the last line is zero by Killing's equation. This shows that the scalar $K^\beta P_\beta$ is conserved along a geodesic for any particle.

Another example is in the Schwarzschild metric ∂_t is a Killing vector $\vec{\partial}_t$.