General Relativity

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PART

I

The Twin "Paradox"

1 Resolving the Twin "Paradox"

A commonly cited paradox in special relativity is known as the Twin Paradox. Consider two twins that start at the same space-time event A, travel along two different paths, and meet again at event C. In $\mathcal{O}$'s reference frame, they stay stationary and move along the worldline from A to C shown in Figure 1. $\mathcal{O}'$ moves with a velocity relative to $\mathcal{O}$ to B, changes direction, and moves with the opposite velocity relative to $\mathcal{O}$ to C.

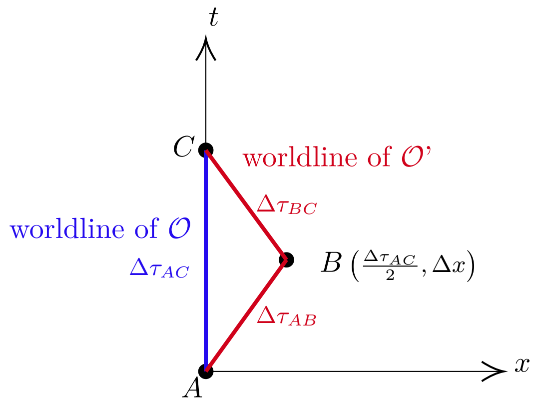

Figure 1. Twin paradox.

While twin $\mathcal{O}'$ travels along AB, each twin will rightly observe the other twin's clock ticking more slowly than their's. Similarly, they can make the same observation while twin $\mathcal{O}'$ travels from B to C. Since they both observe the other's clock running slower than theirs, would they not disagree on which of them is older when they reached point C?

We can calculate the proper time, or the amount of time elapsed that each of them measures as follows. Clearly, $\mathcal{O}$ measures the amount of time elapsed as $\Delta\tau_{AC} = \Delta t$. Observer $\mathcal{O}'$ measures

$$\Delta\tau_{ABC} = \Delta\tau_{AB} + \Delta\tau_{BC} = 2\Delta\tau_{AB} \tag{1.1}$$

Using the spacetime invariant, we can calculate $\Delta\tau_{AB}$:

$$\Delta\tau_{AB}^2 = \left(\frac{\Delta t}{2}\right)^2 - (\Delta x)^2 = \left(\frac{\Delta t}{2}\right)^2 \left(1 - \left(\frac{2\Delta x}{\Delta t}\right)^2\right) = \left(\frac{\Delta t}{2}\right)^2 (1 - v^2) \tag{1.2}$$

We can plug this back in to Equation 1.1 to get

$$\Delta\tau_{ABC} = 2\Delta\tau_{AB} = \Delta t(1 - v^2)^{1/2} < \Delta\tau_{AC} \tag{1.3}$$

So the twin that travels the path ABC is younger than the twin that travels the path AC when they meet back up at C!

Let's reconsider Figure 1 again, with observer $\mathcal{O}'$'s lines of simultaneity drawn in:

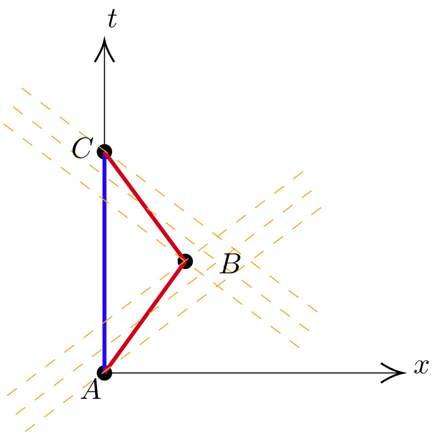

Figure 2. The twin paradox spacetime diagram with $\mathcal{O}'$'s lines of simultaneity drawn in orange.

Note the gap in the lines of simultaneity in twin $\mathcal{O}$'s reference frame. At B, as twin $\mathcal{O}'$ accelerates, from $\mathcal{O}'$'s perspective, their twin $\mathcal{O}$ appears to age very rapidly! This resolves the paradox, which occurs because $\mathcal{O}'$ is not an inertial observer.

PART

II

Proper Time Along a Curved Path

2 Calculating Proper Time

For two very nearby points,

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{2.1}$$

Since $\Delta\tau^2 = -\Delta s^2$, we can rewrite Equation 2.1 in terms of $\Delta\tau$ as:

$$d\tau^2 = -\eta_{\mu\nu} dx^\mu dx^\nu \tag{2.2}$$

Calculating $\Delta\tau$ for a curved path is very similar to the way we calculate the arc length of a curve in vector calculus. We will parameterize the curve and define a tangent vector $\frac{dx^\mu}{d\lambda}(\lambda)$. For a time-like path, the "length" of the path is equivalent to the proper time elapsed on that path.

Using this parameterization, we can calculate the "length" of the curve:

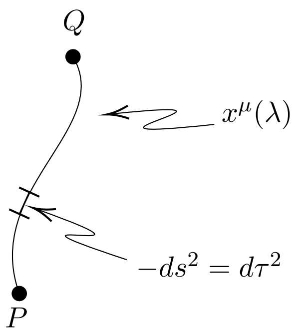

Figure 3. A infinitesimal segment of the curved path.

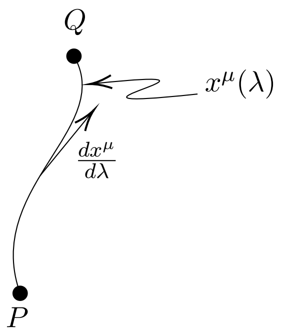

Figure 4. A curved path in spacetime and its tangent vector.

$$\left(\frac{d\tau}{d\lambda}\right)^2 = \eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} \tag{2.3}$$

Rearranging and integrating Equation 2.3, we get

$$\Delta\tau_{PQ} = \int_{\mathcal{C}} d\lambda\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} \tag{2.4}$$

The result of Equation 2.4 is a measurable quantity: it is the time elapsed as measured by an observer that travels on that path from P to Q. [1]

3 Proof That a Straight Line Maximizes Proper Time

Now that we can calculate the proper time along a path, we can ask the question: which path extremizes $\Delta\tau$? To answer this question, consider two events P and Q and the set of paths between them.

Define S as follows:

$$S \equiv \Delta\tau_{PQ} = \int_{\mathcal{C}} d\lambda\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} \tag{3.1}$$

For convenience, we will define L as follows:

[1] *For a space-like path,*

$$\Delta s_{PQ} = \int_P^Q d\lambda\sqrt{\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}$$

. For a path that is neither space-like nor time-like, we will break it into segments that are space-like, time-like, or null. We will always talk about these segments separately. Since no observer can travel along such a path, it is unlikely that this situation will come up in this class.

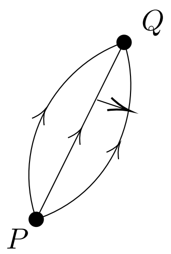

Figure 5. Possible paths between two events P and Q and a small displacement from one possible path to another.

$$L \equiv \sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} \implies S = \int\limits_{\mathcal{C}} L d\lambda \tag{3.2}$$

Since S depends on the path taken between P and Q, let's consider a small variation in S, δS. To find the extrema of S, we need to solve

$$\delta S = \delta \int\limits_{\mathcal{C}} L d\lambda = 0 \tag{3.3}$$

From Equation 3.3, we will get the Euler-Lagrange equations for L: [2]

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu} = 0 \tag{3.4}$$

By the definition of L, we know that

$$L^2 = \left(\frac{dt}{d\lambda}\right)^2 - \left(\frac{dx}{d\lambda}\right)^2 - \left(\frac{dy}{d\lambda}\right)^2 - \left(\frac{dz}{d\lambda}\right)^2 \tag{3.5}$$

From Equation 3.5, we can see that the second term of Equation 3.4 is equal to 0. So all we are left with is the first term. Let's first consider the second part of the first term. [3]

$$\begin{aligned}
\frac{\partial L}{\partial \dot{x}^\mu} &= -\frac{1}{2L}\frac{\partial}{\partial \dot{x}^\mu}\left(\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta\right) \\
&= -\frac{1}{2L}\left(\eta_{\alpha\beta}\delta_\mu{}^\alpha\dot{x}^\beta + \eta_{\alpha\beta}\delta_\mu{}^\beta\dot{x}^\alpha\right) \\
&= -\frac{1}{2L}\left(\eta_{\mu\beta}\dot{x}^\beta + \eta_{\alpha\mu}\dot{x}^\alpha\right) \\
&= -\frac{1}{2L}\left(\eta_{\mu\beta}\dot{x}^\beta + \eta_{\mu\alpha}\dot{x}^\alpha\right) \\
&= -\frac{1}{2L}\left(2\eta_{\alpha\mu}\dot{x}^\alpha\right) \\
&= -\frac{1}{L}\eta_{\alpha\mu}\dot{x}^\alpha
\end{aligned} \tag{3.6}$$

Returning to our definition of L (Equation 3.2),

[2] *In Equation 3.4, $\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$. Also note that Equation 3.4 is really four equations: one for time and one for each spatial coordinate.*

[3] *Note that $\frac{\partial \dot{x}^\alpha}{\partial \dot{x}^\mu} = \delta_\mu{}^\alpha$.*

$$S = \Delta\tau = \int_{\mathcal{C}} d\lambda L \implies d\tau = L d\lambda = \frac{d\tau}{d\lambda} = L \implies \frac{1}{L}\frac{d}{d\lambda} = \frac{d}{d\tau} \tag{3.7}$$

Combining the results of Equations 3.6 and 3.7 give the following result:

$$\frac{dL}{d\dot{x}^\mu} = -\eta_{\alpha\mu}\frac{dx^\alpha}{d\tau} \tag{3.8}$$

Substituting this into Equation 3.4 gives us the following result:

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = \frac{d}{d\lambda}\left(-\eta_{\alpha\mu}\frac{dx^\alpha}{d\tau}\right) = 0 \tag{3.9}$$

We can multiply both sides by $\frac{1}{L}$ to get

$$\frac{1}{L}\frac{d}{d\lambda}\left(-\eta_{\alpha\mu}\frac{dx^\alpha}{d\tau}\right) = \eta_{\alpha\mu}\frac{d^2x^\alpha}{d\tau^2} = 0 \tag{3.10}$$

Finally, we can multiply both sides by the the inverse of the metric to get

$$\frac{d^2x^\alpha}{d\tau^2} = 0 \tag{3.11}$$

The only $x^\mu\tau$ that satisfy this condition are straight lines in spacetime. So far we have only proven that straight lines give extrema for $\Delta\tau$. But in Part I, we showed that the twin who moved in a straight line measured a larger proper time than the other twin. Therefore, straight lines must maximize $\Delta\tau$.

In conclusion, straight lines (observers with constant velocity) measure the maximal proper time, $\Delta\tau$.