General Relativity

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1 Lie derivatives and symmetries

Last lecture, we determined the action of the Lie derivative on vector fields. This is given by

$$\left(\mathcal{L}_{\vec{V}}\vec{W}\right)^{\mu} = [\vec{V},\vec{W}]^{\mu} = V^{\alpha}\partial_{\alpha}W^{\mu} - W^{\alpha}\partial_{\alpha}V^{\mu}. \tag{1.1}$$

The result of this operation is also a vector. We also have rules for Lie derivatives on other objects such as the metric, which is a (0,2)-tensor:

$$\left(\mathcal{L}_{\vec{V}}g\right)_{\mu\nu} = V^{\alpha}\partial_{\alpha}g_{\mu\nu} + \left(\partial_{\mu}V^{\alpha}\right)g_{\alpha\nu} + \left(\partial_{\nu}V^{\alpha}\right)g_{\mu\alpha}. \tag{1.2}$$

Now, imagine that we do a Lie derivative in a special direction. We can visualize a coordinate system on a manifold as a coordinate grid (Fig. 1). One of the coordinates, x_1, advances on one of the grid lines. Let's say we want to take the Lie derivative of $\vec{W}$.

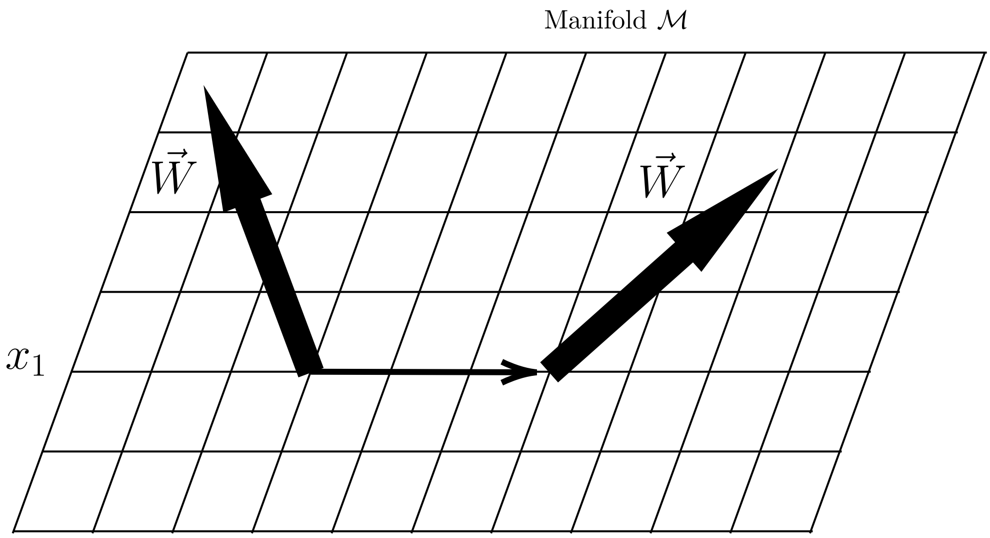

Figure 1. The Lie derivative gives us information on how the vector $\vec{W}$ changes as it moves along the coordinate x_1.

We are interested in only one coordinate, so we will work with the partial derivative along that coordinate as a vector,

$$\vec{\partial_1} = \left(\frac{\vec{\partial}}{\partial x_1}\right). \tag{1.3}$$

Since it is a vector, we can also bring it down into components:

$$\vec{V} = \vec{\partial_1} = V^{\mu}\vec{\partial_{\mu}}, \tag{1.4}$$

$$V^{\mu} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \delta^{\mu}_{1}. \tag{1.5}$$

The components of the Lie derivative acting on $\vec{W}$ in the x_1 direction are given by

$$\left(\mathcal{L}_{\vec{\partial}_1}\vec{W}\right)^\mu = \delta_1^\alpha \partial_\alpha W^\mu - W^\alpha \partial_\alpha \delta_1^\mu = \partial_1 W^\mu, \tag{1.6}$$

which is just the partial derivative in the x_1 direction. This can be used as a way to determine if a vector field does not change in a certain direction. For instance, let $W^\mu = W^\mu(x^0, x^2, x^3)$, then $\left(\mathcal{L}_{\vec{\partial}_1}\vec{W}\right)^\mu = 0$. As an example, let's imagine that we take the Lie derivative of the metric in the ϕ direction of the two-dimensional sphere S^2, as in Fig. 2. It's not hard to see that we can spin the 2-sphere in the ϕ direction and its shape would remain the same. Acting with a Lie derivative on the metric along ϕ would give zero, so we can say that it is a symmetry of our spacetime. In this case, we know that $g_{\mu\nu}$ will be a function of the polar angle θ only, according to our choice of coordinates.

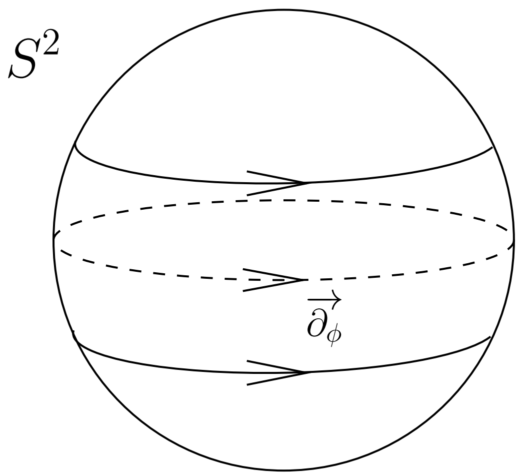

Figure 2. If the Lie derivative of the metric along the ϕ direction on the 2-sphere is zero, we know that $g_{\mu\nu} = g_{\mu\nu}(\theta)$.

With this analysis, we can draw two important conclusions:

1. If $\mathcal{L}_{\vec{K}} g = 0$, then there is a symmetry of the spacetime.
2. If g doesn't depend on a particular coordinate (say, x^1) then $\vec{K} = \vec{\partial}_1$ is a "Killing vector (field)" and there is a symmetry.

2 Covariant derivative

Once defined, the covariant derivative will give us a notion of how to take a directional derivative in spacetime as well as what we really mean by a geodesic. While we won't have the full story about how things behave in curved spacetime, we will know a lot about how objects move in curved spacetime.

Our approach will be to build a derivative that will satisfy certain conditions. We want to work with a derivative that, when acting on a given tensor field, it will produce a new tensor, i.e. $\nabla : (k, l) \to (k, l+1)$. The rank of the new tensor can be guessed by remembering that the partial derivative acting on a scalar function, which is a rank-(0,0) tensor, produces a rank-(0,1) tensor.

Now, we want this new mathematical object to behave both as a tensor as well as a derivative. We will ask six things to be satisfied by it in order to define it in an appropriate, unique way. These properties are:

1. Linearity: for two coefficients a, b and two tensor fields T, S we want

$$\nabla(aT + bS) = a\nabla T + b\nabla S \tag{2.1}$$

2. Leibniz rule:

$$\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S) \tag{2.2}$$

Let's see how this would work for a vector field $\vec{W}$:

$$\nabla\vec{W} = (\nabla_\mu W^\nu)\, \underset{\sim}{d}x^\mu \otimes \vec{\partial}_\nu. \tag{2.3}$$

We guess that the components will look like this

$$\nabla_\mu W^\nu = \partial_\mu W^\nu + (\Gamma_\mu)^\nu_{\ \alpha} W^\alpha, \tag{2.4}$$

where we introduced the four 4x4 matrices $(\Gamma_\mu)^\nu_{\ \alpha}$ in order to have a linear operator. We will insist that the connection coefficients $\Gamma^\rho_{\mu\nu}$ in just the right way such that

$$\nabla_{\mu'} W^{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \nabla_\mu W^\nu, \tag{2.5}$$

which also means that the connection coefficients do not transform as tensors since the partial derivative of a general tensor does not transform as such. The transformation of the coefficients is given by

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} - \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda}. \tag{2.6}$$

3. We want the covariant derivative to reduce to a partial derivative when it acts on a scalar function:

$$\nabla_\mu f = \partial_\mu f. \tag{2.7}$$

4. ∇ respects contractions, i.e.

$$\nabla_\mu \left(T^\alpha_{\ \nu\alpha}\right) = (\nabla T)_\mu^{\ \ \alpha}{}_{\nu\alpha} \tag{2.8}$$

At this point, we can extend this derivative to all objects. The natural question to ask is, how does the covariant derivative act on 1-forms?

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu + \tilde{\Gamma}^\alpha_{\mu\nu} \omega_\alpha. \tag{2.9}$$

The index order is not important as the connection coefficients are not really tensors. However, if necessary, the following convention can be used: $\Gamma^\rho_{\mu\nu} = \Gamma^\rho{}_{\mu\nu}$.

Let's figure out what the $\tilde{\Gamma}$ are by taking the derivative of the following contraction:

$$\begin{aligned}
\nabla_\mu(V^\alpha \omega_\alpha) &= \partial_\mu(V^\alpha \omega_\alpha) \\
&= (\nabla_\mu V^\alpha)\omega_\alpha + V^\alpha(\nabla_\mu \omega_\alpha) \\
&= \left(\partial_\mu V^\alpha + \Gamma^\alpha_{\mu\beta} V^\beta\right)\omega_\alpha + \left(\partial_\mu \omega_\alpha + \tilde{\Gamma}^\beta_{\mu\alpha}\omega_\beta\right) V^\alpha
\end{aligned}$$

$$\begin{aligned}&\Rightarrow 0 = V^{\alpha}\left(\Gamma^{\beta}_{\mu\alpha} + \tilde{\Gamma}^{\beta}_{\mu\alpha}\right)\omega_{\beta}\\ &\Rightarrow \tilde{\Gamma}^{\rho}_{\mu\nu} = -\Gamma^{\rho}_{\mu\nu}.\end{aligned} \tag{2.10}$$

So now we can determine how ∇ acts on any tensor. For instance, let $T^{\mu}_{\ \nu}$ be a (1,1)-tensor:

$$\nabla_{\mu} T^{\nu}_{\ \rho} = \partial_{\mu} T^{\nu}_{\ \rho} + \Gamma^{\nu}_{\mu\alpha} T^{\alpha}_{\ \rho} - \Gamma^{\alpha}_{\mu\rho} T^{\nu}_{\ \alpha}. \tag{2.11}$$

With what we know so far about the connection coefficients following the rules we have stated, we can actually build a tensor with them. Consider two different sets of coefficients Γ, $\hat{\Gamma}$ and take the difference

$$\begin{aligned}\nabla_{\mu} V^{\nu} - \hat{\nabla}_{\mu} V^{\nu} &= \partial_{\mu} V^{\nu} + \Gamma^{\nu}_{\mu\alpha} V^{\alpha} - \partial_{\mu} V^{\nu} - \hat{\Gamma}^{\nu}_{\mu\alpha} V^{\alpha}\\ &= \left(\Gamma^{\nu}_{\mu\alpha} - \hat{\Gamma}^{\nu}_{\mu\alpha}\right).\end{aligned}$$

Since $\nabla_{\mu} V^{\nu} - \hat{\nabla}_{\mu} V^{\nu}$ is a tensor, the object

$$S^{\nu}_{\mu\alpha} = \Gamma^{\nu}_{\mu\alpha} - \hat{\Gamma}^{\nu}_{\mu\alpha} \tag{2.12}$$

must be a tensor.

Before we state the next property, let's consider the following object associated to any given connection:

$$T^{\rho}_{\ \mu\nu} = \Gamma^{\rho}_{\ \mu\nu} - \Gamma^{\rho}_{\ \nu\mu} = 2\Gamma^{\rho}_{\ [\mu\nu]}. \tag{2.13}$$

This object is known as the **torsion tensor**, and clearly it is antisymmetric in its lower indices. A connection that is symmetric in its lower indices is known as "torsion-free". With this in mind, we can ask our connection coefficients to be

5. Torsionless or torsion-free:

$$T^{\rho}_{\ \mu\nu} = 0. \tag{2.14}$$

6. Finally, we want metric compatibility

$$\nabla_{\mu} g_{\nu\rho} = 0. \tag{2.15}$$

This makes the connection unique. There are several consequences to this, such as

$$\begin{aligned}\nabla_{\mu}\left(V^{\alpha} W_{\alpha}\right) &= \nabla_{\mu}\left(V^{\alpha} W^{\beta} g_{\alpha\beta}\right)\\ &= g_{\alpha\beta} \nabla_{\mu}\left(V^{\alpha} W^{\beta}\right)\\ &= \nabla_{\mu}\left(V_{\alpha} W_{\beta} g^{\alpha\beta}\right),\end{aligned}$$

$$\begin{aligned}\nabla_{\mu} g^{\rho\sigma} &= 0,\\ \nabla_{\mu} g &= 0,\\ \nabla_{\alpha} \epsilon_{\mu\nu\rho\sigma} &= 0,\end{aligned}$$

where g is the trace of the metric tensor and $\epsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita tensor.