General Relativity

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More on One Forms

PART

I

1 One Forms and the Gradient

In 3D, vectors can be decomposed via the Helmholtz decomposition into:

$$\bar{V} = \bar{\nabla}\Phi + \bar{\nabla} \times \bar{A} \tag{1.1}$$

[1] So, one forms are not uniquely represented by the gradient of some function since there is a component that is the curl of a field. However, we do not yet have a definition for the cross product in higher dimensions.
An example of a one form is the gradient:

[1] *(1.1) Where Φ is some scalar function and $\bar{A}$ is some vector field.*

$$\underset{\sim}{\omega} = \underset{\sim}{d}f \tag{1.2}$$

This can be used to build a more general definition for a one form:

$$\underset{\sim}{\omega} = \quad = \omega_\mu dx^\mu \tag{1.3}$$

in which dx^μ is the coordinate basis. Now, consider if $\omega_\mu = \partial_\mu f$. Then,

$$\partial_\mu \omega_\nu - \partial_\nu \omega_\mu = \partial_\mu \partial_\nu f - \partial_\nu \partial_\mu f \quad = 0 \tag{1.4}$$

This shows that not all one forms can be written as the gradient of a function.

2 Kinds of One Forms

- A "rotation-free" one form is defined as:

$$\underset{\sim}{\omega} = h\underset{\sim}{d}f \tag{2.1}$$

  for h and f as some other functions on the manifold. This type of one form can no longer be written as a gradient of a function. It is possible to show that $\omega_{[\mu}\partial_\nu\omega_{\rho]} = 0$, meaning that all indices are fully antisymmetrized [2] , where ω^μ is perpendicular to some hypersurface.

[2] *Recall the definition of antisymmetrization:*
$A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu})$
$A_{[\mu\nu\rho]} = \frac{1}{6}(A_{\mu\nu\rho} + A_{\rho\mu\nu} + A_{\nu\rho\mu} - A_{\rho\nu\mu} - A_{\mu\rho\nu} - A_{\nu\mu\rho})$
and so on.

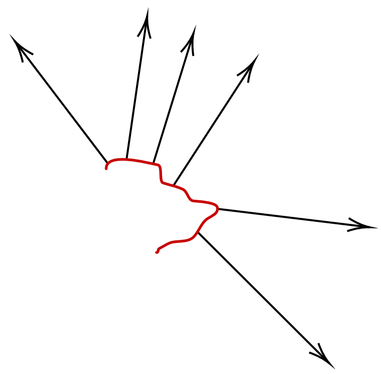

Figure 1. A vector field (in black) normal to some weirdly shaped surface (in red). It is possible to check if a vector field is normal to a hypersurface using $\omega_{[\mu}\partial_\nu\omega_{\rho]}$

- A "curl-free" one form is a special type of one form defined as:

$$\underset{\sim}{\omega} = \underset{\sim}{d}f \tag{2.2}$$

If a one form is "curl-free", then it obeys equation (1.4). So, does every one form which obeys equation (1.4) have the form $\omega_\mu = \partial_\mu f$? The answer is no, not always. This depends on the space that you are in. If you are looking locally, meaning in the neighborhood of the point, then yes, $\omega_\mu = \partial_\mu f$ holds. We call this a closed form, which implies that the form is exact. If you are looking globally, however, $\omega_\mu = \partial_\mu f$ does not hold, meaning that in some spaces, not all closed forms are exact.

PART

II

The Metric Tensor

The metric tensor in curved spacetime is:

$$g = g_{\mu\nu} \underset{\sim}{d}x^{\mu} \otimes \underset{\sim}{d}x^{\nu} \tag{2.3}$$

where $g_{\mu\nu}$ represents the components of g. This should not be confused with $\eta_{\mu\nu}$, which was used previously to represent the metric for flat space. While it is appropriate to use $g_{\mu\nu}$ as the metric for flat space as well as for curved space, $\eta_{\mu\nu}$ is reserved solely for flat space. [3]

[3] *Some properties of $g_{\mu\nu}$:*
$g_{(\mu\nu)} = g_{\mu\nu}$ *(symmetric)*
$g_{\mu\nu}$ *has 10 degrees of freedom.*

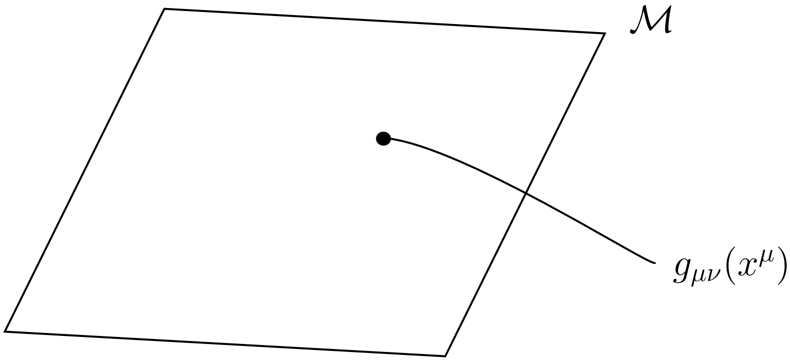

Figure 2. The manifold $\mathcal{M}$ on which we have a point x^{μ}. The metric $g_{\mu\nu}$ at that point is defined by $g_{\mu\nu}(x^{\mu})$.

We can define the determinant of the metric tensor as the following.

$$det(g_{\mu\nu}) = g \tag{2.4}$$

If $g \neq 0$, then the inverse matrix of $g_{\mu\nu}$, called $g^{\mu\nu}$, exists. $g^{\mu\nu}$ has the property that $g^{\mu\alpha} g_{\alpha\nu} = \delta^{\mu}{}_{\nu}$.

3 Raising and Lowering Indices

Since the metric tensor forms an invertible map between one forms and vectors, we can now define a way to raise and lower the indices of one forms and vectors,

$$\omega^{\mu} = g^{\mu\alpha}\omega_{\alpha} \tag{3.1}$$

$$V_{\mu} = g_{\mu\alpha}V^{\alpha} \tag{3.2}$$

Equation (3.1) defines a way to raise indices, going from one forms to vectors. Equation (3.2) defines a way to lower indices, going from vectors to one forms.

4 The Dot Product

We can also use the metric tensor to define the dot product.

$$\bar{U} \cdot \bar{V} = g(\bar{U}, \bar{V}) = g_{\mu\nu}U^{\mu}V^{\nu} \tag{4.1}$$

Using this definition of the dot product, we can define the magnitude of vectors.

$$V_{\mu}V^{\mu} = \begin{cases} < 0 & \text{"}timelike\text{"} \\ > 0 & \text{"}spacelike\text{"} \\ = 0 & \text{"}null\text{"/"}lightlike\text{"} \end{cases} \tag{4.2}$$

[4] for some $\bar{V}(f) = \frac{df}{d\lambda}$ where $\bar{V} = \frac{\bar{d}}{d\lambda}$. $\bar{V}$ is a directional derivative that can act on a function. It is fine to think about $\bar{V} = V^{\mu}\bar{e}_{(\mu)}$, but $\bar{e}_{(\mu)}$ is just a directional derivative now. For the one form $\underset{\sim}{w}$: $T_p \to \mathbb{R}$ acting on $\bar{V}$, $\underset{\sim}{w}(\bar{V}) = w_{\mu}V^{\mu}$. This one form is acting via contraction to map a derivative to the real numbers.

[4] *This is similar to our previous definitions of "timelike separated", "spacelike separated", and "lightlike separated" when we were discussing the spacetime interval Δs^2 in special relativity.*

5 Distance

We can also write down a definition for distance using the metric. For a spacelike curve, we define distance as proper distance s.

$$s = \int_{\lambda_a}^{\lambda_b} \sqrt{g_{\mu\nu}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}}\,d\lambda \tag{5.1}$$

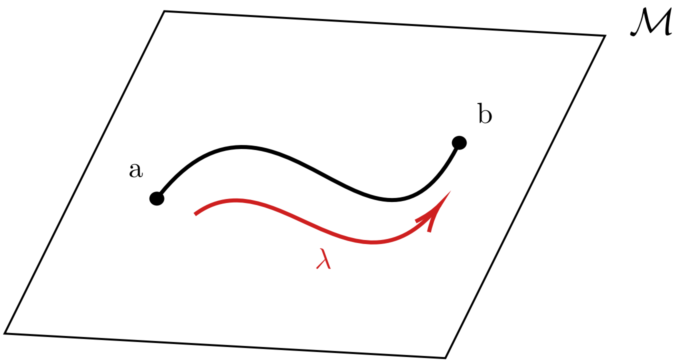

Figure 3. The manifold $\mathcal{M}$ on which we have a spacelike curve from points a to b. The curve is parameterized by λ.

For a timelike curve, we define distance as proper time τ.

$$\tau = \int_{\lambda_a}^{\lambda_b} \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} d\lambda \tag{5.2}$$

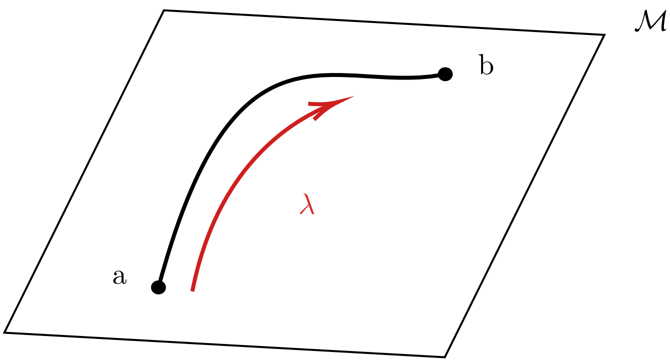

Figure 4. The manifold $\mathcal{M}$ on which we have a timelike curve from points a to b. The curve is parameterized by λ.

In both cases, we have assumed nothing about the parameterization λ. It is important to note that the proper time and proper distance along a specified curve between two points is invariant under coordinate changes.

$$V^\mu V_\mu = V^{\mu'} V_{\mu'} \tag{5.3}$$

A convenient shorthand for distance, ds, can be written as follows:

$$\begin{aligned} ds^2 &= g_{\mu\nu} \underset{\sim}{d}x^\mu \otimes \underset{\sim}{d}x^\nu \\ &= g_{\mu\nu} dx^\mu dx^\nu \end{aligned} \tag{5.4}$$

While the second line of equation (5.4) is an abuse of notation, this is safe to use if one acts as though dx^μ and dx^ν are just differential objects.

Example. Let us consider cartesian coordinates in 3D flat space. In this space,

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{5.5}$$

We can write the distance as:

$$\begin{aligned} ds^2 &= \underset{\sim}{d}x \otimes \underset{\sim}{d}x + \underset{\sim}{d}y \otimes \underset{\sim}{d}y + \underset{\sim}{d}z \otimes \underset{\sim}{d}z \\ &= dx^2 + dy^2 + dz^2 \end{aligned} \tag{5.6}$$

This is just the Pythagorean theorem!

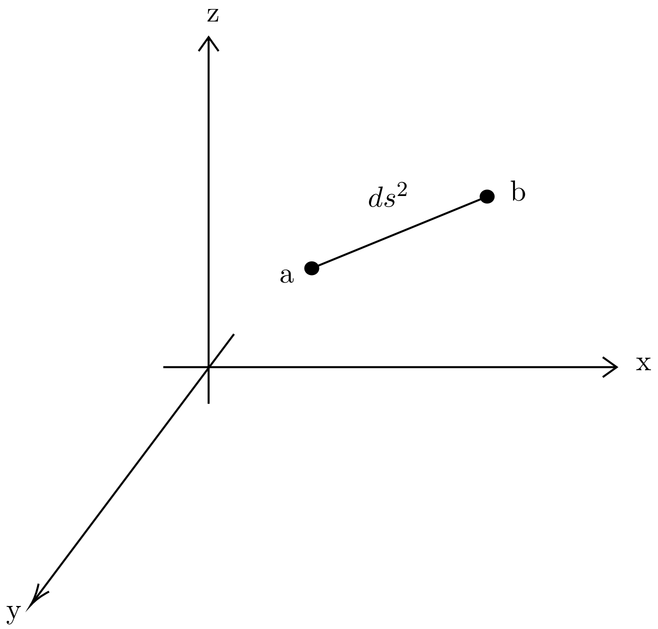

Figure 5. Two points a and b separated by a distance ds^2 in 3D flat space.

Now, we will transform coordinates from (x, y, z) $\rightarrow$ (r, θ, ϕ). Let us denote (x, y, z) with x^i and (r, θ, ϕ) with $x^{i'}$. We will perform this transformation using the Jacobian transformation matrices $\frac{\partial x^{i'}}{\partial x^i}$. For i = 1:

$$\frac{\partial x^{1'}}{\partial x^1} = \frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2 + y^2 + z^2} = \frac{x}{r} \tag{5.7}$$

This same process can be done for i = 2 and i = 3. The distance in spherical polar coordinates is as follows:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \tag{5.8}$$

Now, we can write the metric and its inverse for this coordinate system.

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \tag{5.9}$$

$$g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2 \sin^2\theta} \end{pmatrix} \tag{5.10}$$

Example. Another example that we can consider is the 4D Friedmann–Lemaître–Robertson–Walker (FLRW) metric.

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2) \tag{5.11}$$

This looks like our spacetime interval Δs^2 for flat 4D spacetime except for the $a^2(t)$ term, which is called the "scale factor".

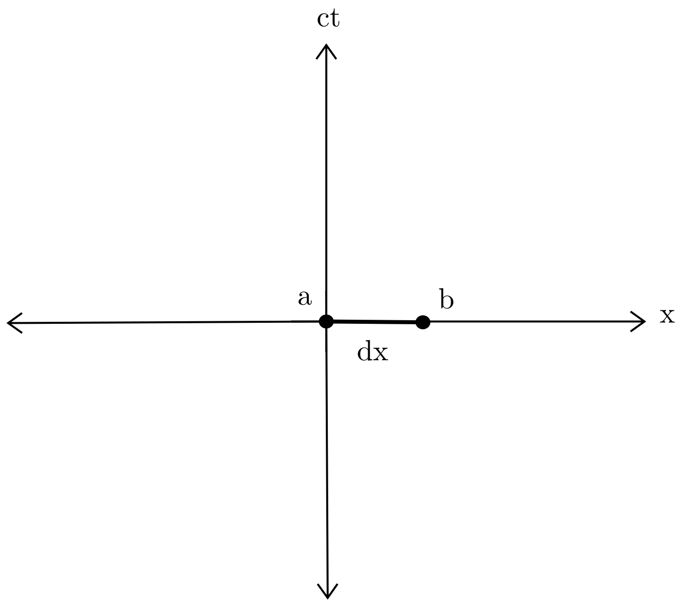

Figure 6. Two points in 4D spacetime separated by a distance in space, dx.

For Figure 6 above, $ds^2 = a^2(t)dx^2 = a^2dx^2 \to ds = adx$. This means that observers at fixed coordinates see the distance between themselves grow or shrink in time depending on the value of a, the scale factor.

6 Transformations

For a tensor T, the transformation of its components is defined as follows:

$$T_{\mu'\nu'} = \frac{\partial x^{\mu}}{\partial x^{\mu'}} \frac{\partial x^{\nu}}{\partial x^{\nu'}} T^{\mu\nu} \tag{6.1}$$

The components of the tensor change under transformation, however the tensor itself is invariant under coordinate transformation.

Example. Let us consider an example which is ***not*** a tensor: $\tilde{\epsilon}_{\mu\nu\rho\sigma}$.

$$\begin{aligned}\tilde{\epsilon}_{\mu'\nu'\rho'\sigma'} &= \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}\frac{\partial x^{\rho}}{\partial x^{\rho'}}\frac{\partial x^{\sigma}}{\partial x^{\sigma'}}\tilde{\epsilon}_{\mu\nu\rho\sigma} \\ &= det\left|\frac{\partial x^{\mu}}{\partial x^{\mu'}}\right|\tilde{\epsilon}_{\mu\nu\rho\sigma}\end{aligned} \tag{6.2}$$

There is a "scaling factor", $\left|\frac{\partial x^{\mu}}{\partial x^{\mu'}}\right|$, multiplying $\tilde{\epsilon}_{\mu\nu\rho\sigma}$ which shows that $\tilde{\epsilon}_{\mu\nu\rho\sigma}$ is not invariant under coordinate transformation. Thus, $\tilde{\epsilon}_{\mu\nu\rho\sigma}$ is not a tensor.