General Relativity

Class 8 — Feb. 7, 2020

Chris Layden

E-mail: cplayden777@utexas.edu

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PART

I

Classical Field Theory

General relativity is a classical field theory, where the metric $g_{\mu\nu}$ is a tensor field defined at points in spacetime. So here, we will begin to build up the theory of classical fields, deriving the Euler-Lagrange Equations for fields.

1 Review of Classical Mechanics

In classical mechanics, we define the **Lagrangian** as $L = T - V$, where T is the kinetic energy and V the potential energy of a particle or a system of particles. This Lagrangian is a function of only the position q and velocity $\dot{q}$ of the particle(s), for if L is a function of higher-order derivatives, nonphysical, acausal behavior often arises. Thus, $L = L(q, \dot{q})$. To determine the equations of motion for the particle(s), we search for critical points of the **action** S, where

$$S = \int dt\, L(q, \dot{q}) \ . \tag{1.1}$$

That is, we search for trajectories $q(t)$ for which any infinitesimal perturbation in $q(t)$ causes no change in S — trajectories for which $\delta S = 0$. Using the calculus of variations, it can be shown that the equations of motion that satisfy this criterion are those for which

$$\frac{dL}{dq} - \frac{d}{dt}\frac{dL}{d\dot{q}} = 0 \ . \tag{1.2}$$

These are called the **Euler-Lagrange Equations**.

2 An Introduction to Classical Field Theory

We seek to extend this discussion so that we may determine how a set of fields $\Phi^a(x^\mu)$ evolves. The fields are defined across spacetime — on an infinite number of points — so we cannot simply define a Lagrangian for the fields as we would for a system of particles. Instead, we now define a **Lagrangian density** $\mathcal{L}$ at each point in space, which is a function of the field and the first derivatives of the field at that point: $\mathcal{L} = \mathcal{L}(\Phi^a, \partial_\mu \Phi^a)$. The Lagrangian is now the integral of this density over space:

$$L = \int d^3x\, \mathcal{L} \ . \tag{2.1}$$

As before, the action S is defined as the Lagrangian integrated over time, so

$$S = \int dt\, L = \int d^4x\, \mathcal{L}(\Phi^a, \partial_\mu \Phi^a) \ . \tag{2.2}$$

In analogy to classical mechanics, we are now looking for evolutions of the field(s) $\Phi^a(x^\mu)$ for which any infinitesimal change to the evolution has no effect on the action, such that $\delta S = 0$.

So we would like to show how δS varies due to some infinitesimal perturbation in the field $\delta\Phi^a$. To do so, we will need to determine how $\mathcal{L}$ varies due $\delta\Phi^a$. Therefore, we must first observe how the variables of $\mathcal{L}$, Φ^a and $\partial_\mu\Phi^a$, change due to such a perturbation. The change in Φ^a is trivial:

$$\Phi^a \to \Phi^a + \delta\Phi^a \ . \tag{2.3}$$

While the change in $\partial_\mu\Phi^a$ is just the derivative of the change in Φ^a:

$$\partial_\mu\Phi^a \to \partial_\mu\Phi^a + \delta(\partial_\mu\Phi^a) = \partial_\mu\Phi^a + \partial_\mu\delta\Phi^a \ . \tag{2.4}$$

Note that we are only going to consider perturbations in Φ^a and $\delta\Phi^a$ that vanish at infinity; that is, we are only considering variations over some contained volume. This criterion will be important later in the derivation.

Now we may find how $\mathcal{L}$ changes due to the infinitesimal change $\delta\Phi^a$ by Taylor expanding $\mathcal{L}$:

$$\mathcal{L}(\Phi^a + \delta\Phi^a, \partial_\mu\Phi^a + \partial_\mu\delta\Phi^a) = \mathcal{L}(\Phi^a, \partial_\mu\Phi^a) + \frac{\partial\mathcal{L}}{\partial\Phi^a}\delta\Phi^a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^a)}\partial_\mu\delta\Phi^a \ . \tag{2.5}$$

We can therefore determine the change in the action, $S \to S + \delta S$ by plugging this Taylor expansion into Eq. 2.2 and subtracting the original S:

$$\delta S = \int d^4x \frac{\partial\mathcal{L}}{\partial\Phi^a}\delta\Phi^a + \int d^4x \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^a)}\partial_\mu\delta\Phi^a \ . \tag{2.6}$$

Our strategy now is to put the right side of Eq. 2.6 all in terms of $\delta\Phi^a$. The first term already fits this mold, but the second term has a derivative of $\delta\Phi^a$. We can integrate the second term by parts to put it in a friendlier form:

$$\delta S = \int d^4x \frac{\partial\mathcal{L}}{\partial\Phi^a}\delta\Phi^a - \int d^4x\, \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^a)}\right)\delta\Phi^a + \int d^4x\, \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi^a)}\delta\Phi^a\right) . \tag{2.7}$$

We are still left with one pesky term that does not have a simple factor $\delta\Phi^a$. But this term is now the integral of a total derivative, taken over an arbitrarily large volume V. By the Divergence Theorem, we can convert integrals of this form (i.e. an integral of a total derivative of some arbitrary vector $\vec{W}$) into integrals of flux over the boundary of the volume, δV:

$$\int\limits_V d^4x\, \partial_\mu W^\mu = \int\limits_{\delta V} d\Sigma\, n_\mu W^\mu \ . \tag{2.8}$$

Where $d\Sigma$ is an infinitesimal piece of the boundary and $n_\mu W^\mu$ denotes taking the flux of $\vec{W}$ through the boundary. In our case, $\vec{W}$ contains a factor $\delta\Phi^a$, which we have demanded must vanish at infinity. Thus, by simply making the volume of integration arbitrarily large, the third term in Eq. 2.7 can be set to zero. We can now pull out our factor $\delta\Phi^a$ from Eq. 2.7, and we are left with

$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \Phi^a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} \right) \right) \delta\Phi^a \ . \tag{2.9}$$

We now seek solutions for Φ^a for which δS is zero (i.e. where the action reaches a critical point). In this case, the right side of Eq. 2.9 must be zero for any choice of perturbation $\delta\Phi^a$. The only way to ensure this is true is for the expression in the parentheses to be zero:

$$\frac{\partial \mathcal{L}}{\partial \Phi^a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^a)} \right) = 0 \ . \tag{2.10}$$

These are our field equations, the equivalents of the Euler-Lagrange Equations for field theory.

3 Example

We now consider a specific Lagrangian density $\mathcal{L}$ and find the solutions of Eq. 2.10 for a single scalar field scalar field $\Phi(x^\mu)$. As we will see shortly, the following Lagrangian density is analogous to the classical Lagrangian $L = T - V$:

$$\mathcal{L} = -\frac{1}{2}\eta^{\alpha\beta}\, \partial_\alpha \Phi\, \partial_\beta \Phi - V(\Phi) \ . \tag{3.1}$$

The first term can be expanded in the following way:

$$-\frac{1}{2}\eta^{\alpha\beta}\, \partial_\alpha \Phi\, \partial_\beta \Phi = -\frac{1}{2}\Big(-(\partial_t \Phi)^2 + (\partial_x \Phi)^2 + (\partial_y \Phi)^2 + (\partial_z \Phi)^2 \Big) \ . \tag{3.2}$$

Thus the term $\frac{1}{2}(\partial_t \Phi)^2$ in Eq. 3.2 can be thought of as analogous to the classical kinetic energy $T = \frac{1}{2}mv^2$, and the spatial terms represent some gradient energy in the field.

We must now find the necessary derivatives of $\mathcal{L}$ to plug into the Eq. 2.10. Firstly,

$$\frac{\partial \mathcal{L}}{\partial \Phi} = -\frac{\partial V}{\partial \Phi} \ . \tag{3.3}$$

And applying Eq. 3.2, we see that

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi)} = (\partial_t \Phi, -\partial_x \Phi, -\partial_y \Phi, -\partial_z \Phi) = -\eta^{\mu\alpha}\, \partial_\alpha \Phi = -\partial^\mu \Phi \ . \tag{3.4}$$

Pluggin Eq. 3.3 and 3.4 into Eq. 2.10, we find

$$\frac{\partial V}{\partial \Phi} = -\partial_\mu \partial^\mu \Phi = \Box\, \Phi \ . \tag{3.5}$$

Where $\Box$, called the **d'Alembertian** operator, acts on Φ like so:

$$\Box\,\Phi = (\partial_t^2 - \nabla^2)\Phi \; . \tag{3.6}$$

Often the potential V may be written

$$V = \frac{1}{2}m^2\Phi^2 \; . \tag{3.7}$$

In which case Eq. 3.5 may be rewritten

$$\Box\Phi = m^2\Phi \; , \tag{3.8}$$

the famous Klein-Gordon equation.