

Homework 4

Due: December 22, 2021 in class

Note: No late homework will be accepted. You may discuss with your classmates but **you may not plagiarize.** You need to turn in **your analysis and also your code** (printout) written in Octave or Matlab.

Part A. (30%)

A.1 Show the discrete orthogonality of cosines

$$\sum_{j=0}^N \frac{1}{c_j} \cos k x_j \cos k' x_j = \begin{cases} 0 & \text{if } k \neq k' \\ \frac{1}{2} c_k N & \text{if } k = k' \end{cases}$$

where $x_j = \pi j/N$, $j = 0, 1, 2, \dots, N$ and

$$c_k = \begin{cases} 2 & \text{if } k = 0, N \\ 1 & \text{otherwise.} \end{cases}$$

(Hint: by substituting complex exponential representations for cosines.)

A.2 The discrete cosine series is defined by

$$f_j = \sum_{k=0}^N a_k \cos k x_j \quad j = 0, 1, 2, \dots, N,$$

where $x_j = \pi j/N$.

Prove that the coefficients of the series are

$$a_k = \frac{2}{N} \frac{1}{c_k} \sum_{j=0}^N \frac{1}{c_j} f_j \cos k x_j \quad k = 0, 1, 2, \dots, N.$$

Part B. (20%)

Differentiate the following functions using two methods: FFT and central difference formula

$$f'_j = \frac{f_{j+1} - f_{j-1}}{2h}.$$

When you use the central difference formula, compute the derivative only at the interior points but not at the boundary points. **For each method, use $N = 16$ and $N = 32$.** Plot your results based on FFT and central difference formula as symbols (for example, squares or triangles) and the exact derivative as a continuous line.

B.1

$$f(x) = \sin 3x + 3 \cos 6x \quad 0 \leq x < 2\pi$$

B.2

$$f(x) = 6x - x^2 \quad 0 \leq x < 2\pi$$

Which method works better in B.1? Which method works better in B.2? Can you explain the reason?

Part C. (30%)

Here are two functions $f(x)$ and $g(x)$ defined in the interval $(0, 2\pi)$, i.e.

$$f(x) = \sin(2x) + 0.1\sin(15x),$$

$$g(x) = \sin(2x) + 0.1\cos(15x).$$

C.1 Use $N = 32$ grid points, i.e. $x_j = 2\pi j/N$, $j = 0, 1, 2, \dots, N-1$. Compute $f_j = f(x_j)$, $g_j = g(x_j)$ and $H_j = f_j g_j$. Compute the FFT of H_j , i.e. \hat{H}_k , $k = -N/2, -N/2 + 1, \dots, -1, 0, 1, \dots, N/2 - 1$? What is the real function that \hat{H}_k represents?

C.2 Use $N = 32$ grid points and compute the FFT of f_j , i.e. \hat{f}_k , and the FFT of g_j , i.e. \hat{g}_k . Compute \hat{h}_m using the convolution sum

$$\hat{h}_k = \sum_{m=-N/2}^{N/2-1} \hat{f}_m \hat{g}_{k-m},$$

where $k = -N/2, -N/2 + 1, \dots, -1, 0, 1, \dots, N/2 - 1$. What is the real function that \hat{h}_k represents?

C.3 Use trigonometric identities to show the exact result of $E(x) = f(x)g(x)$. Use $N = 32$ grid points and compute $E_j = E(x_j)$ and the FFT of E_j , i.e. \hat{E}_k . Does \hat{E}_k represent $E(x)$ correctly? Do you see any difference among \hat{E}_k , \hat{H}_k and \hat{h}_k ? Which is correct? Why?

Part D. (20%)

We use the Chebyshev derivative matrix operator to differentiate $u(x) = 4(x^2 - x^4)e^{-x/2}$ in the range $-1 \leq x \leq 1$. Let vector \mathbf{x} represent the collocation points $x_j = \cos(\pi j/N)$, $j = 0, 1, 2, \dots, N$, and vector \mathbf{u} represent the values of $u(x)$ at the collocation points. Construct the $(N+1) \times (N+1)$ Chebyshev collocation derivative matrix \mathbf{D} using (6.46) or (6.47) in the textbook.

D.1 For $N = 7$, write down the vectors \mathbf{x} and \mathbf{u} , the derivative matrix \mathbf{D} , and the first derivative of $u(x)$ at the collocation points, i.e. \mathbf{u}' , via $\mathbf{u}' = \mathbf{D}\mathbf{u}$. Plot the first derivative \mathbf{u}' at the collocation points using symbols and the exact first derivative using a continuous line.

D.2 For $N = 7$, write down the vectors \mathbf{x} and \mathbf{u} , the second derivative matrix \mathbf{D}_2 , and the second derivative of $u(x)$ at the collocation points, i.e. \mathbf{u}'' , via $\mathbf{u}'' = \mathbf{D}_2\mathbf{u}$. Plot the second derivative \mathbf{u}'' at the collocation points using symbols and the exact second derivative using a continuous line.