Linear Programming I

January 19, 2025

Introduction

A model, which is used for optimum allocation of scarce or limited resources to competing products or activities under such assumptions as certainty, linearity, fixed technology, and constant profit per unit, is **linear programming.**

Linear Programming is one of the most versatile, powerful and useful techniques for making managerial decisions.

As a decision making tool, it has demonstrated its value in various fields such as production, finance, marketing, research and development and personnel management. Determination of optimal product mix (a combination of products, which gives maximum profit), transportation schedules, Assignment problem and many more.

PROPERTIES OF LINEAR PROGRAMMING MODEL

Any linear programming model (problem) must have the following properties:

- The relationship between variables and constraints must be linear.
- The model must have an objective function.
- The model must have structural constraints.
- The model must have non-negativity constraint.

Example

A company manufactures two products X and Y, which require, the following resources. The resources are the capacities machine M_1, M_2, and M_3. The available capacities are $50, 25$,and 15 hours respectively in the planning period. Product X requires 1 hour of machine M_2 and 1 hour of machine M_3. Product Y requires 2 hours of machine M_1, 2 hours of machine M_2 and 1 hour of machine M_3. The profit contribution of products X and Y are Rs.5/- and Rs.4/- respectively.

Machines	Products		Availability in hours
	X	Y	
M_1	0	2	50
M_2	1	2	25
M_3	1	1	15
Profit in Rs. Per unit	5	4	

Table: Machine Availability and Profit for Products

Let the company manufactures x units of X and y units of Y. As the profit contributions of X and Y are Rs.5/- and Rs. 4/- respectively. The objective of the problem is to maximize the profit Z, hence objective function is:

$$Z = 5x + 4y \quad \leftarrow \text{Objective Function}$$

This should be done so that the utilization of machine hours by products x and y should not exceed the available capacity. This can be shown as follows:

$$\begin{aligned} M_1 : \quad & 0x + 2y && \leq 50 \\ M_2 : \quad & 1x + 2y && \leq 25 \qquad \leftarrow \text{Linear Structural Constraints} \\ M_3 : \quad & 1x + 1y && \leq 15 \end{aligned}$$

But the company can stop production of x and y or can manufacture any amount of x and y. It cannot manufacture negative quantities of x and y. Hence we have write

$$x, y \geq 0 \quad \leftarrow \text{Non-negativity Contraints}$$

Basic Assumptions

The following are some important assumptions made in formulating a linear programming model:

1. It is assumed that the decision maker here is completely certain (i.e., deterministic conditions) regarding all aspects of the situation, i.e., availability of resources, profit contribution of the products, technology, courses of action and their consequences etc.
2. It is assumed that the relationship between variables in the problem and the resources available i.e., constraints of the problem exhibits linearity.
3. We assume here fixed technology. Fixed technology refers to the fact that the production requirements are fixed during the planning period and will not change in the period.
4. It is assumed that the profit contribution of a product remains constant, irrespective of level of production and sales.

Basic Assumptions

5. It is assumed that the decision variables are continuous. It means that the companies manufacture products in fractional units.
6. It is assumed that only one decision is required for the planning period. This condition shows that the linear programming model is a static model, which implies that the linear programming problem is a single stage decision problem.
7. All variables are restricted to nonnegative values

General Linear Programming Problem

$Z = c_1x_1 + c_2 \ x_2 + \dots c_n \ x_n$ subjects to the conditions, → OBJECTIVE FUNCTION

$a_{11}x_1 + a_{12} \ x_2 + \ a_{13} \ x_3 \ + \dots + a_{1j} \ x_j \ + \dots + .. a_{1n} \ x_n \ (\geq, =, \leq) \ b_1$

$a_{21} \ x_1 \ + a_{22} \ x_2 + a_{23} \ x_3 \ + \ \dots\dots\dots\dots + a_{2j} \ x_j \ + \dots\dots + a_{2n}x_n \ (\geq, =, \leq) \ b_2$

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$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 \ + \dots + a_{mj} \ x_j \dots + a_{mn} \ x_n \ (\geq, =, \leq) \ b_m$

Structural Constraints

and all x_j are = 0 → NON NEGETIVITY CONSTRINT.

Where $j = 1, 2, 3, \dots n$

Where all $c_j \ s$, b_i s and $a_{ij} \ s$ are constants and $x_j \ s$ are decision variables.

Steps in Formulating the Linear Programming Problem

1. Identify the unknown decision variables to be determined and assign symbols to them.
2. Identify all the restrictions or constraints in the problem and express them as linear equations or inequalities of decision variables.
3. Identify the objective or aim and represent it also as a linear function of decision variables.

Examples of Formulating a Linear Programming Problem
(to be answered during class)

Problem 1

A retail store stocks two types of shirts A and B. These are packed in attractive cardboard boxes. During a week the store can sell a maximum of 400 shirts of type A and a maximum of 300 shirts of type B. The storage capacity, however, is limited to a maximum of 600 of both types combined. Type A shirt fetches a profit of Rs. 2/- per unit and type B a profit of Rs. 5/- per unit. How many of each type the store should stock per week to maximize the total profit? Formulate a mathematical model of the problem.

Problem 2

A ship has three cargo holds, forward, aft and center. The capacity limits are:
Forward 2000 tons, 100,000 cubic meters
Center 3000 tons, 135,000 cubic meters
Aft 1500 tons, 30,000 cubic meters.

The following
cargoes are offered, the ship owners may accept all or any part of each commodity:

Commodity	Amount in tons.	Volume/ton in cubic meters	Profit per ton in Rs.
A	6000	60	60
B	4000	50	80
C	2000	25	50

In order to preserve the trim of the ship the weight in each hold must be proportional to the capacity in tons. How should the cargo be distributed so as to maximize profit? Formulate this as linear programming problem.

Problem 3

A patient consult a doctor to check up his ill health. Doctor examines him and advises him that he is having deficiency of two vitamins, vitamin A and vitamin D. Doctor advises him to consume vitamin A and D regularly for a period of time so that he can regain his health. Doctor prescribes tonic X and tonic Y, which are having vitamin A, and D in certain proportion. Also advises the patient to consume at least 40 units of vitamin A and 50 units of vitamin Daily. The cost of tonics X and Y and the proportion of vitamin A and D that present in X and Y are given in the table below. Formulate l.p.p. to minimize the cost of tonics.

Vitamins	Tonics		Daily requirements in units
	X	Y	
A	2	4	40
D	3	2	50
Cost in Rs. per unit	5	3	

Problem 4

A machine tool company conducts a job-training programme at a ratio of one for every ten trainees. The training programme lasts for one month. From past experience it has been found that out of 10 trainees hired, only seven complete the programme successfully. (The unsuccessful trainees are released). Trained machinists are also needed for machining. The company's requirement for the next three months is as follows:
January: 100 machinists, February: 150 machinists and March: 200 machinists.

In addition, the company requires 250 trained machinists by April. There are 130 trained machinists available at the beginning of the year. Pay roll cost per month is:
Each trainee Rs. 400/- per month.
Each trained machinist (machining or teaching): Rs. 700/- p.m.
Each trained machinist who is idle: Rs.500/- p.m.
(Labour union forbids ousting trained machinists).

Build a l.p.p. for produce the minimum cost hiring and training schedule and meet the company's requirement.

METHODS OF SOLVING A LINEAR PROGRAMMING PROBLEM

- The Graphical Method when we have two decision variables in the problem. (To deal with more decision variables by graphical method will become complicated, because we have to deal with planes instead of straight lines. Hence in graphical method let us limit ourselves to two variable problems.
- The Systematic Trial and Error method, where we go on giving various values to variables until we get optimal solution. This method takes too much of time and laborious, hence this method is not discussed here.
- The Vector method. In this method each decision variable is considered as a vector and principles of vector algebra is used to get the optimal solution. This method is also time consuming, hence it is not discussed here.
- The Simplex method. When the problem is having more than two decision variables, simplex method is the most powerful method to solve the problem. It has a systematic programme, which can be used to solve the problem.

Graphical Method
(will be demonstrated in class)

Characteristics of a graphical solution to lpp

- Generally the method is used to solve the problem, when it involves two decision variables.
- For three or more decision variables, the graph deals with planes and requires high imagination to identify the solution area.
- Always, the solution to the problem lies in first quadrant.
- This method provides a basis for understanding the other methods of solution.

Example A

A company manufactures two products, X and Y by using three machines A, B, and C. Machine A has 4 hours of capacity available during the coming week. Similarly, the available capacity of machines B and C during the coming week is 24 hours and 35 hours respectively. One unit of product X requires one hour of Machine A, 3 hours of machine B and 10 hours of machine C. Similarly one unit of product Y requires 1 hour, 8 hour and 7 hours of machine A, B and C respectively. When one unit of X is sold in the market, it yields a profit of Rs. 5/- per product and that of Y is Rs. 7/- per unit. Solve the problem by using graphical method to find the optimal product mix.

Example B

The cost of materials A and B is Re.1/- per unit respectively. We have to manufacture an alloy by mixing these to materials. The process of preparing the alloy is carried out on three facilities X, Y and Z. Facilities X and Z are machines, whose capacities are limited. Y is a furnace, where heat treatment takes place and the material must use a minimum given time (even if it uses more than the required, there is no harm). Material A requires 5 hours of machine X and it does not require processing on machine Z. Material B requires 10 hours of machine X and 1 hour of machine Z. Both A and B are to be heat treated at last one hour in furnace Y. The available capacities of X, Y and Z are 50 hours, 1 hour and 4 hours respectively. Find how much of A and B are mixed so as to minimize the cost.

Example C

Maximize $Z = 0/5a + 1b$

$1a + 1b \geq 0$
$-0.5a + 1b \leq 1$
$a, b \geq 0$

Example D

A company manufactures two products X and Y. The profit contribution of X and Y are Rs.3/- and Rs. 4/- respectively. The products X and Y require the services of four facilities. The capacities of the four facilities A, B, C, and D are limited and the available capacities in hours are 200 Hrs, 150 Hrs, and 100 Hrs. and 80 hours respectively. Product X requires 5, 3, 5 and 8 hours of facilities A, B, C and D respectively. Similarly the requirement of product Y is 4, 5, 5, and 4 hours respectively on A, B, C and D. Find the optimal product mix to maximise the profit.

Example E

Maximize $Z = 2a + 4b$

$$1a - 1b \leq -1$$
$$-1a + 1b \leq 0$$
$$a, b \geq 0$$