FA5

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```r
knitr::opts_chunk$set(echo = TRUE)
```

Problem 8.17 – Mean and Variance of X

Midwestern University has 1/3 of its students taking **9 credit hours**, 1/3 taking **12 credit hours**, and 1/3 taking **15 credit hours**.
We define X as the number of credit hours.
Thus, the probability distribution is:

X	p(X)
9	1/3
12	1/3
15	1/3

The mean and variance are calculated as follows:

$$\mu_X = \sum Xp(X) = 9(1/3) + 12(1/3) + 15(1/3) = 12$$

$$\sigma_X^2 = \sum X^2 p(X) - \mu_X^2 = (81 + 144 + 225)(1/3) - 144 = 150 - 144 = 6$$

Hence, the distribution is **uniform**.

```r
X <- c(9, 12, 15)
pX <- c(1/3, 1/3, 1/3)

mu_X <- sum(X * pX)
var_X <- sum(X^2 * pX) - mu_X^2

mu_X
```

```
## [1] 12
```

```r
var_X
```

```
## [1] 6
```

Problem 8.18 – Sampling Distribution of the Mean

We now take **all samples of size n = 2 (with replacement)** from this population.

Each pair (e.g., (9,9), (9,12), (9,15), etc.) is a **sample**.
For each sample, we compute its **sample mean ($\bar{x}$)**.

Sample	Mean ($\bar{x}$)	Probability p($\bar{x}$)	$\bar{x} \times$ p($\bar{x}$)	$\bar{x}^2 \times$ p($\bar{x}$)
(9,9)	9	1/9	1.0000	9
(9,12)	10.5	2/9	2.3333	24.5
(9,15)	12	1/9	1.3333	16
(12,9)	10.5	2/9	2.3333	24.5
(12,12)	12	1/9	1.3333	16
(12,15)	13.5	2/9	3.0000	40.5
(15,9)	12	1/9	1.3333	16
(15,12)	13.5	2/9	3.0000	40.5
(15,15)	15	1/9	1.6667	25

The total probability = 1.
Now we can find:

$$\mu_{\bar{X}} = \sum(\bar{X} \cdot p(\bar{X})) = 12$$

$$\sigma^2_{\bar{X}} = \sum(\bar{X}^2 \cdot p(\bar{X})) - \mu^2_{\bar{X}} = 12 - 12^2 = 3$$

Hence, $\sigma^2_{\bar{X}} = \frac{\sigma^2_X}{2} = \frac{6}{2} = 3$.

```r
samples <- expand.grid(X, X)
xbar <- rowMeans(samples)

# Probability for each unique mean value
xbar_values <- unique(xbar)
p_xbar <- table(xbar) / length(xbar)

# Expected mean and variance of sampling distribution
mu_xbar <- sum(as.numeric(names(p_xbar)) * p_xbar)
var_xbar <- sum((as.numeric(names(p_xbar))^2) * p_xbar) - mu_xbar^2

mu_xbar
```

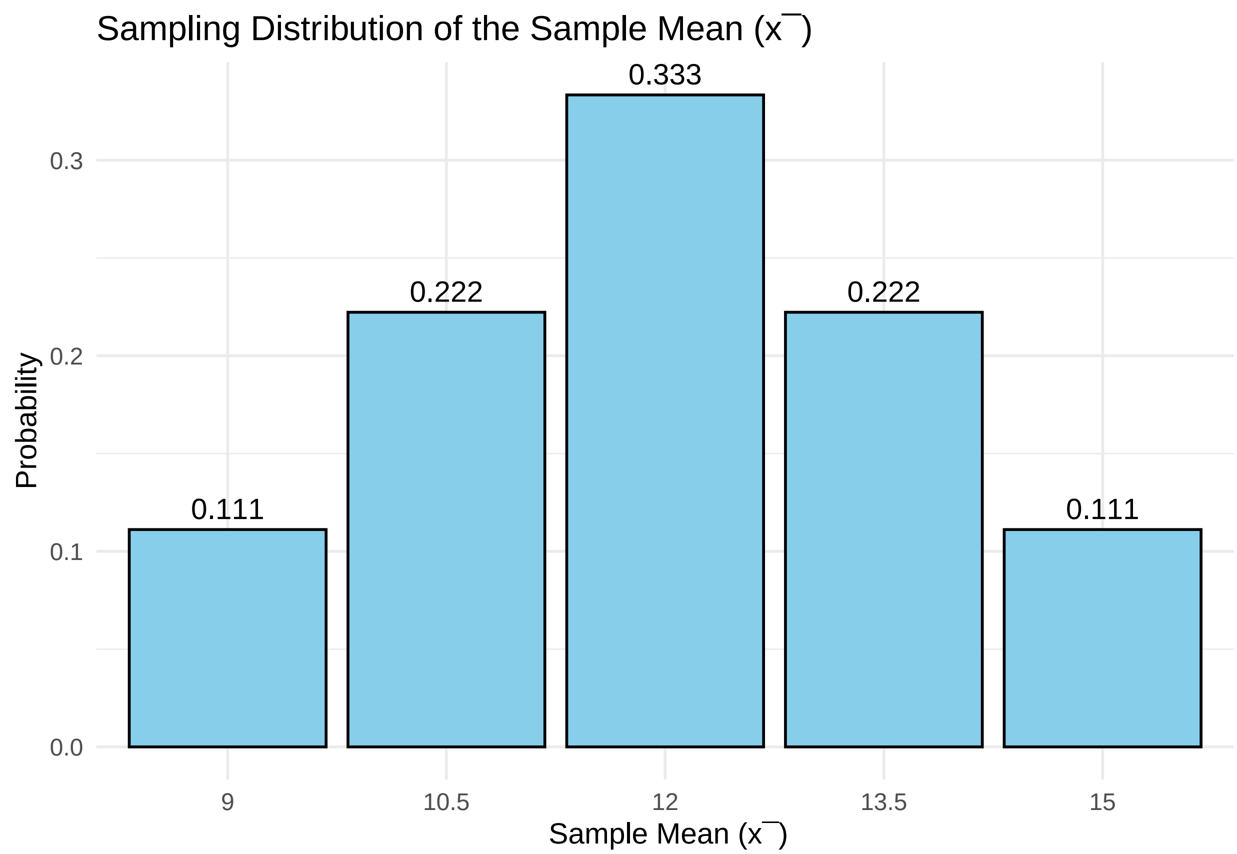
```
## [1] 12
```

```r
var_xbar
```

```
## [1] 3
```

```r
library(ggplot2)

# Create data frame for plotting
df <- data.frame(
  xbar = as.numeric(names(p_xbar)),
  probability = as.numeric(p_xbar)
)

# Plot distribution
ggplot(df, aes(x = factor(xbar), y = probability)) +
  geom_bar(stat = "identity", fill = "skyblue", color = "black") +
```

```r
geom_text(aes(label = round(probability, 3)), vjust = -0.5) +
labs(
  title = "Sampling Distribution of the Sample Mean (x̄)",
  x = "Sample Mean (x̄)",
  y = "Probability"
) +
theme_minimal()
```



Step-by-Step Explanation of Each Column

1. **Sample (A, B)** – All possible combinations of X values when sampling with replacement (9, 12, 15). There are 9 total combinations (3×3).

2. **$\bar{x}$ (Mean)** – The mean of each sample. Example: (9,15) (9+15)/2 = 12.

3. **$p(\bar{x})$** – The probability of each unique mean occurring. Since all combinations are equally likely (1/9 each), $p(\bar{x})$ = (number of occurrences of $\bar{x}$)/9.

4. **$\bar{x} \times p(\bar{x})$** – Used to compute the expected mean of the sampling distribution.

5. **$\bar{x}^2 \times p(\bar{x})$** – Used to compute the variance of the sampling distribution.

Verification

```r
c(mu_X = mu_X, mu_xbar = mu_xbar, var_X = var_X, var_xbar = var_xbar)
```

```
##     mu_X  mu_xbar    var_X var_xbar
##       12       12        6        3
```

We find that:
$\mu_X = \mu_{\bar{X}} = 12$
$\sigma^2_{\bar{X}} = \frac{\sigma^2_X}{2} = 3$
Thus, the theory holds true.

Problem 8.21:

```r
population <- c(3, 7, 11, 15)

pop_mean <- mean(population)
pop_sd <- sd(population)  # sample SD by default; use population SD instead
pop_sd <- sqrt(mean((population - pop_mean)^2))  # population SD formula
pop_mean; pop_sd
```

```
## [1] 9
```

```
## [1] 4.472136
```

```r
samples <- expand.grid(population, population)
sample_means <- rowMeans(samples)
table_means <- as.data.frame(table(sample_means))
table_means$Probability <- table_means$Freq / nrow(samples)
table_means
```

```
##   sample_means Freq Probability
## 1            3    1      0.0625
## 2            5    2      0.1250
## 3            7    3      0.1875
## 4            9    4      0.2500
## 5           11    3      0.1875
## 6           13    2      0.1250
## 7           15    1      0.0625
```

```r
mean_sampling <- sum(as.numeric(as.character(table_means$sample_means)) * table_means$Probability)
sd_sampling <- sqrt(sum((as.numeric(as.character(table_means$sample_means)) - mean_sampling)^2 * table_r
mean_sampling; sd_sampling
```

```
## [1] 9
```

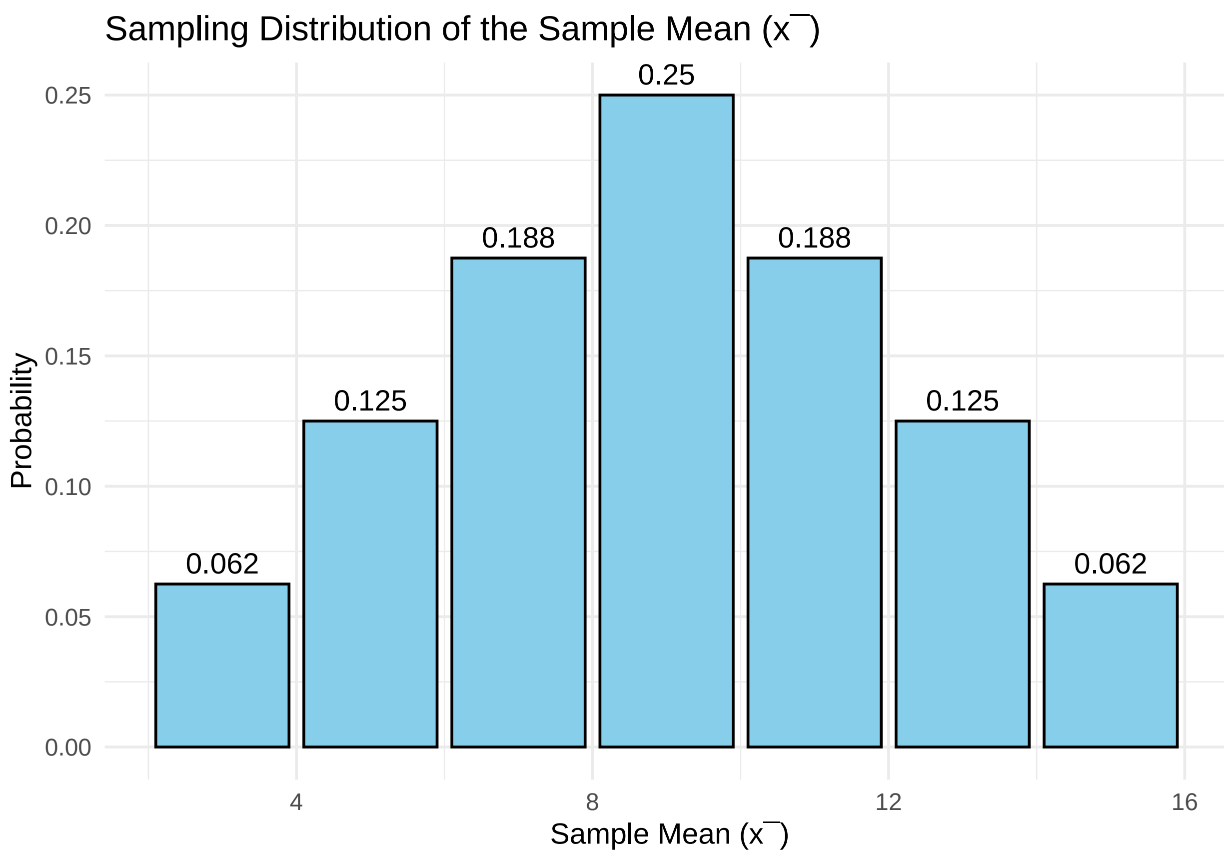
```
## [1] 3.162278
```

```r
theoretical_mean <- pop_mean
theoretical_sd <- pop_sd / sqrt(2)

theoretical_mean; theoretical_sd
```

```
## [1] 9
```

```
## [1] 3.162278
```

```r
library(ggplot2)

ggplot(table_means, aes(x = as.numeric(as.character(sample_means)),
                        y = Probability)) +
  geom_col(fill = "skyblue", color = "black") +
  geom_text(aes(label = round(Probability, 3)), vjust = -0.5) +
  labs(title = "Sampling Distribution of the Sample Mean (x̄)",
       x = "Sample Mean (x̄)",
       y = "Probability") +
  theme_minimal()
```




```r
data.frame(
  Description = c("Population Mean ( )",
                  "Population SD ( )",
                  "Mean of Sampling Dist. ( x̄)",
                  "SD of Sampling Dist. ( x̄)"),
  Value = c(pop_mean, pop_sd, mean_sampling, sd_sampling)
)
```

```
##                    Description    Value
## 1          Population Mean ( ) 9.000000
## 2            Population SD ( ) 4.472136
## 3 Mean of Sampling Dist. ( x̄) 9.000000
## 4   SD of Sampling Dist. ( x̄) 3.162278
```

Problem 8.34:

```r
#(a) Probability that less than 40% will be boys
# exact binomial
p_less_80_exact <- pbinom(79, size = 200, prob = 0.5)
p_less_80_exact
```

```
## [1] 0.001817474
```

```r
#(b) Probability that between 43% and 57% will be girls
# exact binomial: P(86 <= X <= 114)
p_between_exact <- pbinom(114, size = 200, prob = 0.5) - pbinom(85, size = 200, prob = 0.5)
p_between_exact
```

```
## [1] 0.9599628
```

```r
#(c) Probability that more than 54% will be boys
# exact binomial: P(X >= 109)
p_more_108_exact <- 1 - pbinom(108, size = 200, prob = 0.5)
p_more_108_exact
```

```
## [1] 0.1146233
```

Problem 8.49:

```r
x <- c(6, 9, 12, 15, 18)
p <- c(0.1, 0.2, 0.4, 0.2, 0.1)
mu <- sum(x * p)
sigma2 <- sum((x - mu)^2 * p)

mu
```

```
## [1] 12
```

```r
sigma2
```

```
## [1] 10.8
```

```r
samples <- expand.grid(x1 = x, x2 = x) # all 25 combinations
samples$mean <- (samples$x1 + samples$x2) / 2
samples$prob <- p[match(samples$x1, x)] * p[match(samples$x2, x)]

head(samples, 10)
```

```
##    x1 x2 mean prob
## 1   6  6  6.0 0.01
## 2   9  6  7.5 0.02
## 3  12  6  9.0 0.04
## 4  15  6 10.5 0.02
## 5  18  6 12.0 0.01
## 6   6  9  7.5 0.02
## 7   9  9  9.0 0.04
## 8  12  9 10.5 0.08
## 9  15  9 12.0 0.04
## 10 18  9 13.5 0.02
```

```r
sum(samples$prob)
```

```
## [1] 1
```

```r
knitr::kable(samples, caption = "All 25 samples (with replacement), their means, and probabilities")
```

Table 3: All 25 samples (with replacement), their means, and probabilities

x1	x2	mean	prob
6	6	6.0	0.01
9	6	7.5	0.02
12	6	9.0	0.04
15	6	10.5	0.02
18	6	12.0	0.01
6	9	7.5	0.02
9	9	9.0	0.04
12	9	10.5	0.08
15	9	12.0	0.04
18	9	13.5	0.02
6	12	9.0	0.04
9	12	10.5	0.08
12	12	12.0	0.16
15	12	13.5	0.08
18	12	15.0	0.04
6	15	10.5	0.02
9	15	12.0	0.04
12	15	13.5	0.08
15	15	15.0	0.04
18	15	16.5	0.02
6	18	12.0	0.01
9	18	13.5	0.02
12	18	15.0	0.04
15	18	16.5	0.02
18	18	18.0	0.01

```r
sampling_dist <- aggregate(prob ~ mean, data = samples, sum)
knitr::kable(sampling_dist, caption = "Sampling Distribution of the Sample Mean")
```

Table 4: Sampling Distribution of the Sample Mean

mean	prob
6.0	0.01
7.5	0.04
9.0	0.12
10.5	0.20
12.0	0.26
13.5	0.20
15.0	0.12
16.5	0.04
18.0	0.01

```r
mu_xbar <- sum(sampling_dist$mean * sampling_dist$prob)
var_xbar <- sum((sampling_dist$mean - mu_xbar)^2 * sampling_dist$prob)

mu_xbar
```

```
## [1] 12
```

```r
var_xbar
```

```
## [1] 5.4
```

```r
sigma2 / 2
```

```
## [1] 5.4
```