MATH 248
Methods of Proof in Mathematics

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THE FOLLOWING IS JUST A PREVIEW

Contents

Chapter 1

Set Theory

A set is a collection of objects together with a rule for deciding whether a given object is a member of one set or not.

Sets are usually denoted by capital letters $A, B, C, \ldots$.

If an object x is a member of the set A we write $x \in A$ "x is an element of A."

If x is not a member of A we write $x \notin A$ "x is not a member of A."

Sets are typically described using curly brackets $\{$(elements go here)$\}$.

Example 1.1.

$$A = \{1, 2, 3\}$$
$$B = \{2, 4, 6, \ \ldots \ , 20\}$$
$$C = \{5, 6, 7, \ \ldots\}$$
$$D = \{ \ \ldots \ , -2, -1, 0, 1, 2, \ \ldots\}$$

In most cases we will need to be more specific when defining our sets: $\{elements \mid rule\}$. This set notation is standard. The middle line"$|$" reads as "such that".

Standard Sets

Natural Numbers $\mathbb{N} = \{1, 2, 3, 4, \ \ldots\}$

Integers $\mathbb{Z} = \{ \ \ldots \ , -2, -1, 0, 1, 2, \ \ldots\}$

Rational Numbers $\mathbb{Q} = \{\frac{p}{q} \mid \ p, q \in \mathbb{Z},\ q \neq 0\}$

Real Numbers $\mathbb{R}$

Complex Numbers $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R},\ i = \sqrt{-1}\}$

Empty Set $\emptyset = \{\}$

Example 1.2. $A = \{x \in \mathbb{Z} \mid x \geq 5\} = \{5, 6, 7, \ldots\}$.

Example 1.3. $B = \{2x \mid x \in \mathbb{Z}\} = \{\ldots -4, -2, 0, 2, 4, \ldots\}$. (Even integers).

Example 1.4. $C = \{x \in \mathbb{R} \mid x > 3 \text{ and } x < 0\} = \emptyset$.

Example 1.5. From the previous examples (1.2-1.4)
$3 \notin A \quad 3 \notin B \quad 3 \notin C \quad 6 \in A \quad 6 \in B \quad 6 \notin C$.

If A has a finite number of elements, we use $|A|$ to denote the number of elements in A.

Example 1.6. If $A = \{2, 5, 7\}$, then $|A| = 3$.

Example 1.7. $|\emptyset| = 0$.

Example 1.8. If $B = \{3, \{2, 3\}, \emptyset, 4, \{1, 5\}\}$, then $|B| = 5$.

Example 1.9. $|\{\emptyset\}| = 1$.

Example 1.10. If $C = \{1, 3, 7, 3\} = \{1, 3, 7\}$, then $|C| = 3$.

Definition 1.11. Let A, B be sets. If every element of A is an element of B, then we say that A is a *subset* of B.
Notation: $A \subseteq B$ "A is a subset of B".

Example 1.12. $\{1, 2, 3\} \subseteq \mathbb{Z}$.

Example 1.13. $\{-1, 1\} \subseteq \mathbb{Z}$.

Example 1.14. $\{-1, 1\}\mathbb{N}$

Example 1.15. $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$.

Example 1.16.
$\{1\} \subseteq \{1, \{1\}\}$.
$\{1\} \in \{1, \{1\}\}$.

Example 1.17.
$\{1, \{2\}\} \nsubseteq \{\{1\}, 2\}$.
$\{1, \{2\}\} \nsupseteq \{\{1\}, 2\}$.

*Note: If there exists some $x \in A$ such that $x \notin B$, then $A \nsubseteq B$. As a consequence, if A is any set, $\emptyset \subseteq A$.
Reason: If $\emptyset \nsubseteq A$, there would be some $x \in \emptyset$ such that $x \notin A$. Also if A is any set,then $A \subseteq A$.

When discussing sets, we will often restrict our attention to subsets of a given set $\mathcal{U}$, called *the universe of discourse.* $\mathcal{U}$ may be spelled out explicitly, in other cases the universe will be clear from context.

Intervals $(\mathcal{U} = \mathbb{R})$
$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$
$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$
$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$
$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$
$(a, \infty) = \{x \in \mathbb{R} \mid x > a\}$
$(-\infty, a) = \{x \in \mathbb{R} \mid x < a\}$
$[a, \infty) = \{x \in \mathbb{R} \mid x \geq a\}$
$(-\infty, a] = \{x \in \mathbb{R} \mid x \leq a\}$
$(-\infty, \infty) = \mathbb{R}$

Definition 1.18. Sets A and B are *equal* if they have exactly the same members.

*Note: $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Definition 1.19. If $A \subseteq B$ and $A \neq B$, then A is a *proper subset* of B.
Notation: $A \subset B$ or $A \subsetneqq B$.

Example 1.20. $\{1, 2\}$ is a proper subset of $\{1, 2, 3\}$ but $\{1, 2, 3\}$ is not a proper subset of $\{1, 2, 3\}$.

Definition 1.21. Let A be a set. The *power set* of A is the set $\wp(A)$ consisting of all subsets of A.

Example 1.22.
$A = \{x, y\}$
$\wp(A) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$.

Example 1.23.
$B = \{2, \{2, 5\}, 7\}$
$\wp(B) = \{\emptyset, \{2\}, \{\{2, 5\}\}, \{7\}, \{2, \{2, 5\}\}, \{2, 7\}, \{\{2, 5\}, 7\}, B\}$.

Example 1.24. $\wp(\emptyset) = \{\emptyset\}$.

Example 1.25. $\wp(x) = \{\emptyset, \{x\}\}$.

We will see that if $|A| = n$ then $|\wp(A)| = 2^n$.

It can be useful to represent a set diagrammatically by a Venn diagram.

- Elements are represented by points.
- $D \subseteq B, D \nsubseteq A, x \in A, x \notin B$.
- Box represents $\mathcal{U}$.
- Subsets of $\mathcal{U}$ are represented by circles.

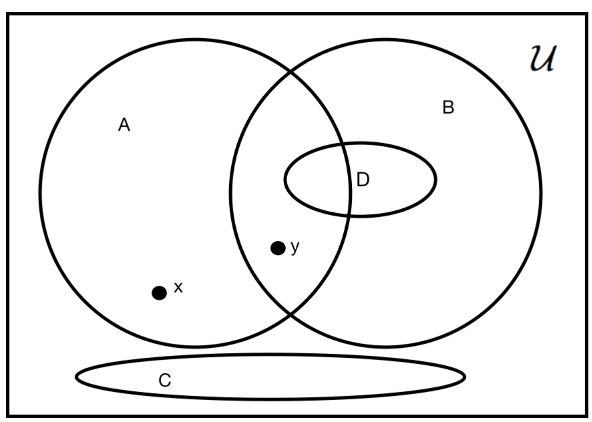

Figure 1.1: The diagram.

Set Operations

Definition 1.26. Let A and B be sets. The *union* of A and B is noted as $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.
The union lies in A or B or both.
*Note: In math, "or" is used **inclusively**.

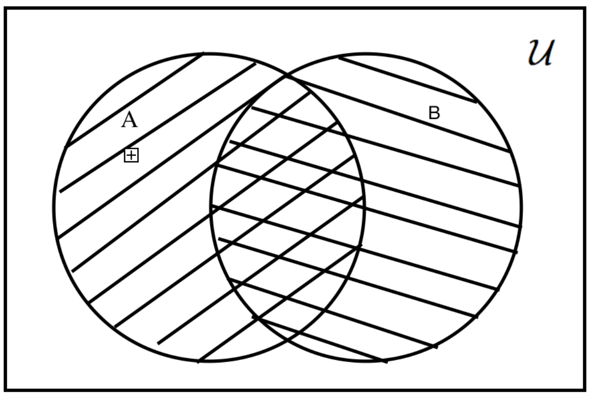

Figure 1.2: The union of two sets A and B.

Example 1.27.
$A = \{-1, 0, 1\}$ and $B = \{1, 2, 3\}$.
$A \cup B = \{-1, 0, 1, 2, 3\}$.

Example 1.28.
$A = \{1, 2\}$ and $B = \{0, 1, 2, 3\}$.
$A \cup B = \{0, 1, 2, 3\}$.

Example 1.29. For any set A, $A \cup \emptyset = A$ and $A \cup A = A$.

Example 1.30.
$A = [1, 2]$ and $B = [2, 3]$.
$A \cup B = [1, 3)$.

Definition 1.31. Let A and B be sets. The *intersection* of A and B is the set $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

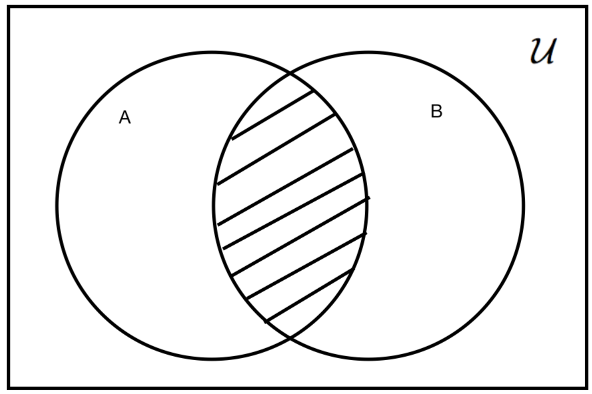

Figure 1.3: The intersection of two sets A and B.

Example 1.32.
$A = \{-1, 0, 1\}$ and $B = \{1, 2, 3\}$.
$A \cap B = \{1\}$.

Example 1.33.
$A = \{1, 2\}$ and $B = \{0, 1, 2, 3\}$.
$A \cap B = \{1, 2\}$.

Example 1.34. For any set A, $A \cap \emptyset = \emptyset$ and $A \cap A = A$.

Example 1.35.
$A = [1, 2]$ and $B = [2, 3]$.
$A \cap B = \{2\}$.

Definition 1.36. Let A and B be sets. The difference of A with B is $A - B = \{x \mid x \in A \text{ and } x \notin B\}$.

Example 1.37.
$A = \{-1, 0, 1\}$ and $B = \{1, 2, 3\}$.
$A - B = \{-1, 0\}$ and $B - A = \{2, 3\}$.

Example 1.38.
$A = \{1, 2\}$ and $B = \{0, 1, 2, 3\}$.
$A - B = \emptyset$ and $B - A = \{0, 3\}$.

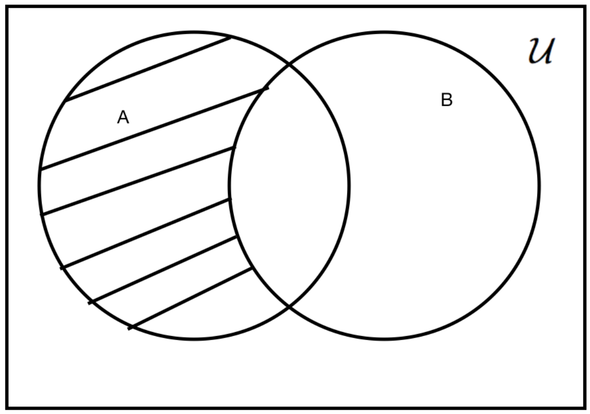

Figure 1.4: The difference of A with B.

Example 1.39. If A is any set, then $A - \emptyset = A$, $\emptyset - A = \emptyset$ and $A - A = \emptyset$.

Example 1.40.
$[1, 2] - [2, 3) = [1, 2)$.
$[2, 3) - [1, 2] = (2, 3)$.

Definition 1.41. Let A be a set in the universe $\mathcal{U}$. The *compliment* of A is $\overline{A} = \mathcal{U} - A$.

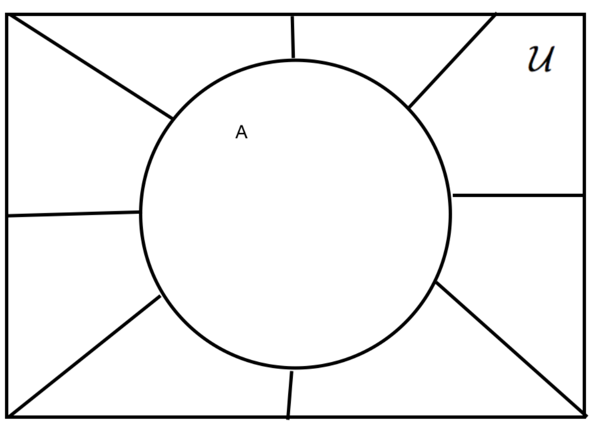

Figure 1.5: The compliment of A.

Example 1.42.
$\mathcal{U} = \mathbb{Z}$ and $A = \mathbb{N}$.
$\overline{A} = \{\dots, -2, -1, 0\}$.

Example 1.43.
$\mathcal{U} = \mathbb{R}$ and $A = [-1, 1]$.
$\overline{A} = (-\infty, -1) \cup (1, \infty)$.

Example 1.44.
$\mathcal{U} = \mathbb{R}$ and $A = \mathbb{Q}$.
$\overline{A} = \mathbb{R} - \mathbb{Q}$ (irrational numbers).

Example 1.45. $\overline{\mathcal{U}} = \emptyset$.

Example 1.46. $\overline{\emptyset} = \mathcal{U}$.

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