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Inequalities For rational functions with prescribed poles

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Abstract

Let R_n be the set of all rational function of the type $r(z) = p(z)/W(z)$, where $p(z)$ be a polynomial of degree not exceeding n and let $W(z) = \prod_{j-1}^{n}(z - a_j)$ where $|a_j| > 1$, $j = 1, 2, \ldots, n$. If $r, s \in R_n$ and $s(z)$ has n zeros in $|z| \leq 1$, then Xin Li [Proc. Amer. Math. Soc.,Vol. 139, No. 5 (2011), pp. 1659–1665] proved that

$$\left| r'(z) + \frac{\beta}{2} B'(z) r(z) \right| \leq \left| s'(z) + \frac{\beta}{2} B'(z) s(s) \right|.$$

Where $|r(z)| \leq |s(z)|$ for $|z| = 1$ and $|\beta| \leq 1$.

In this paper, we consider a class of rational function $r(z)$ with restricted zeros and present certain generalizations of above inequality.

Subject Classification: *(2010) 30A15, 30C10, 26D07.*

Keywords: *Rational Function, Derivative, Polar Derivative, Zeros.*

1. Introduction

Let P_n denote the class of all complex polynomials of degree at most n. Let $T_1 := \{z; |z| = 1\}$, D_{1-} denote the region inside T_1 and D_{1+} denote the region out side T_1. If $p \in P_n$, then according to the well known result of S. Bernstein [see (5)]

$$\max_{z \in T_1} |p'(z)| \leq n \max_{z \in T_1} |p(z)|. \tag{1.1}$$

If we restrict ourselves to the class of polynomial having all its zeros in $T_1 \cup D_{1+}$, then it was conjectured by P. Erdös and later on proved by P. D. Lax [9]

$$\max_{z \in T_1} |p'(z)| \leq \frac{n}{2} \max_{z \in T_1} |p(z)|, \tag{1.2}$$

and if all the zeros of $p \in P_n$ lies in $T_1 \cup D_{1-}$, then it was proved by P. Turán [13] that

$$\max_{z \in T_1} |p'(z)| \geq \frac{n}{2} \max_{z \in T_1} |p(z)|. \tag{1.3}$$

In 1930, Bernstein [5] revisited his inequality and established the following comparative result by assuming that $p(z)$ and $q(z)$ are polynomials such as $p(z)$ has at most of degree n and $q(z)$ has exactly n zeros in $T_1 \cup D_{1-}$ and

$$|p(z)| \leq |q(z)| \quad \text{for} \quad z \in T_1, \tag{1.4}$$

then

$$|p'(z)| \leq |q'(z)| \quad \text{for} \quad z \in T_1. \tag{1.5}$$

Aziz and Dawood [3] obtained the following inequality concerning the minimum modulus of $p(z)$ and its derivative $p'(z)$. They proved that, if $p \in P_n$ having all its zeros in $T_1 \cup D_{1-}$, then

$$\min_{z \in T_1} |p'(z)| \geq n \min_{z \in T_1} |p(z)|. \tag{1.6}$$

Malik and Vong [7] generalized the inequality (1.5) due to Bernstein [5] by showing that, for any real or complex number β with $|\beta| \leq 1$

$$|zp'(z) + n\frac{\beta}{2} p(z)| \leq |zq'(z) + n\frac{\beta}{2} q(z)|, \tag{1.7}$$

for $|z| = 1$. If we replace $q(z) = Mz^n$, where $M = \max_{z \in T_1} |p(z)|$, then we have the following generalized form of Bernstein's [5] inequality (1.1) due to jain [8]

$$|zp'(z) + n\frac{\beta}{2} p(z)| \leq n\left|1 + \frac{\beta}{2}\right| M. \tag{1.8}$$

In the same paper, Jain [8] proved the following generalized result.

Theorem A: *If* $p \in P_n$ *with* $\max_{z \in T_1} |p(z)| = 1$, *then for every* β *with* $|\beta| \leq 1$

$$\left|zp'(z) + \frac{n\beta}{2} p(z)\right| + \left|zq'(z) + \frac{n\beta}{2} q(z)\right| \leq n\left\{\left|1 + \frac{\beta}{2}\right| + \left|\frac{\beta}{2}\right|\right\}. \tag{1.9}$$

Where $q(z) = z^n \overline{p(1/\bar{z})}$.

Jain [8] also proved the following result in the same manner, which is a generalization of inequality (1.2) due to Lax [9].

Theorem B: *If* $p \in P_n$, *having all zeros in* $T_1 \cup D_{1+}$ *and* $\max_{z \in T_1} |p(z)| = 1$, *then for any real or complex number* β *with* $|\beta| \leq 1$ *and* $z \in T_1$

$$\left|zp'(z) + \frac{n\beta}{2} p(z)\right| \leq \frac{n}{2}\left[\left|\frac{\beta}{2}\right| + \left|1 + \frac{\beta}{2}\right|\right]. \tag{1.10}$$

The result is best possible and equality hold in (1.10) *for* $p(z)=\alpha+\lambda z^n$ *with* $|\alpha|=|\lambda|=\dfrac{1}{2}$.

The following generalization of inequality (1.6) due to Aziz and Dawood [4] was given by Dewan and Hans [6].

Theorem C: *If* $p(z)$ *is a polynomial of degree* n, *having all its zeros in* D_{1-}, *then for any real or complex number* β *with* $|\beta|\le 1$

$$\min_{|z|=1}\left|zp'(z)+\frac{n\beta}{2}p(z)\right|\ge n\left|1+\frac{\beta}{2}\right|\min_{z\in T_1}|p(z)|. \tag{1.11}$$

The result is best possible and equality holds for $p(z)=me^{i\gamma}z^n, m>0$.

Dewan and Hans [6] also proved the following improvement of Theorem B by considering $m=\inf_{z\in T_1}|p(z)|$.

Theorem D: *If* $p\in P_n$, *having all zeros in* $T_1\cup D_{1+}$, *then for every real or complex number* β *with* $|\beta|\le 1$

$$\left|zp'(z)+\frac{n\beta}{2}p(z)\right|\le\frac{n}{2}\left[\left|\frac{\beta}{2}\right|+\left|1+\frac{\beta}{2}\right|\right]M-\frac{n}{2}\left[\left|1+\frac{\beta}{2}\right|-\left|\frac{\beta}{2}\right|\right]m \tag{1.12}$$

where $M=\max_{z\in T_1}|p(z)|$ *and* $m=\min_{z\in T_1}|p(z)|$. *Inequality* (1.12) *is sharp and equality holds for* $p(z)=\alpha+\lambda z^n$ *with* $|\alpha|=|\lambda|=\dfrac{1}{2}$.

2. Rational Function

Let $a_j\in\mathbb{C}$, $j=1,2,\ldots,n$ are n points in D_{1+}, where $\mathbb{C}$ is the set of all complex numbers. Now consider

$$W(z)=\prod_{j=1}^{n}(z-a_j),\qquad B(z)=\prod_{j=1}^{n}\frac{(1-\overline{a_j}z)}{(z-a_j)} \tag{2.1}$$

and

$$R_n=R_n(a_1,a_2,\ldots,a_n)=\frac{p(z)}{W(z)};\qquad p\in P_n. \tag{2.2}$$

Then R_n is set of all rational functions with poles $a_j, j=1,2,\ldots,n$, at most and with finite limit at infinity. It is clear that $B(z)\in R_n$ and for $z\in T_1, |B(z)|=1$. We set $\|f\|=\sup_{z\in T_1}|f(z)|$. Throughout this paper, we shall assuming that all the poles $a_j, j=1,2,\ldots,n$ are in D_{1+}.

Since

$$B(z)=\prod_{j=1}^{n}\frac{(1-\overline{a_j}z)}{(z-a_j)}$$

therefore, for $z\in T_1$

$$|B'(z)|=\frac{zB'(z)}{B(z)}=\sum_{j=1}^{n}\frac{(|a_j|^2-1)}{|z-a_j|^2}>0. \tag{2.3}$$

Li, Mohapatra and Rodgriguez [11] extended the inequality (1.1) due to Bernstein [5] and (1.3) due to P.Turán [13] for rational functions and proved that for $r(z)\in R_n$ and $z\in T_1$

$$|r'(z)|\le|B'(z)|\|r\| \tag{2.4}$$

and

$$|r'(z)|\ge\frac{|B'(z)|}{2}\|r\|. \tag{2.5}$$

Recently, Xin Li [10] extended the inequality (1.5) due to Bernstein [5] for rational functions by showing that

Theorem E: *Let $r,s\in R_n$ and assume s has all its n zeros in $T_1\cup D_{1-}$ and*

$$|r(z)|\le|s(z)| \quad for \quad z\in T_1, \tag{2.6}$$

then

$$|r'(z)|\le|s'(z)| \quad for \quad z\in T_1. \tag{2.7}$$

3. Polar Derivative

For a complex number α and for $p\in P_n$, define

$$D_\alpha p(z)=np(z)+(\alpha-z)p'(z) \tag{3.1}$$

is known as polar derivative of $p(z)$. On dividing above equation by α and taking $\alpha\to\infty$, we obtained ordinary derivative of $p(z)$. That is

$$\lim_{\alpha\to\infty}\frac{D_\alpha p(z)}{\alpha}=p'(z). \tag{3.2}$$

Firstly, it was Aziz [2], who proved inequality (1.2) due to Lax [9] in terms of polar derivatives by showing that for $p\in P_n$, $p(z)\ne 0$ in D_{1-} and $|\alpha|\ge 1$

$$|D_\alpha p(z)|\le\frac{n}{2}(|\alpha z^{n-1}|+1)\max_{z\in T_1}|p(z)| \quad for \quad z\in T_1\cup D_{1+}. \tag{3.3}$$

The result is sharp and the equality holds at $p(z) = az^n + b$ *with* $|a| = |b|$.

Liman, Mohapatra and Shah [12] extended inequality (1.7) due to Malik and Vong [7] under the same hypothesis, in terms of polar derivative by proving that

$$\left|zD_\alpha p(z) + \frac{n\delta}{2}(|\alpha|-1)p(z)\right| \le \left|zD_\alpha q(z) + \frac{n\delta}{2}(|\alpha|-1)q(z)\right| \tag{3.4}$$

for every real or complex number δ with $|\delta| \le 1$.

Xin Li [10] pointed out that inequalities involving polynomials and their polar derivatives are a special case of the inequalities for rational functions by considering $a_i = \alpha$ for each $i = 1, 2, \ldots, n$, that is for $|a_i| = |\alpha| > 1$

$$r(z) = \frac{p(z)}{(z-\alpha)^n} \tag{3.5}$$

$$r'(z) = \left(\frac{p(z)}{(z-\alpha)^n}\right) = -\frac{D_\alpha p(z)}{(z-\alpha)^{n+1}} \tag{3.6}$$

and present the following reduction of inequality (2.7) of Theorem E in polar derivative of $p \in P_n$.

Theorem F: *Let* $p, q \in P_n$ *and* q *having all its zeros in* $z \in T_1 \cup D_{1^-}$. *If*

$$|p(z)| \le |q(z)| \quad for \quad z \in T_1,$$

then for any α *with* $|\alpha| \ge 1$

$$|D_\alpha p(z)| \le |D_\alpha q(z)| \quad for \quad z \in T_1, \tag{3.7}$$

where $q(z) = z^n \overline{p(1/\bar{z})}$.

4. Main Result

In this paper, we first prove following result concerning infima of rational function $r \in R_n$ and obtained some compact generalization for rational function.

Theorem 1: *If* $r \in R_n$ *has* n *zeros all in* D_{1^-}, *then for every* β *with* $|\beta| \le 1$

$$\inf_{z \in T_1}\left|zr'(z) + \frac{\beta}{2}|B'(z)|r(z)\right| \ge |B'(z)|\left|1 + \frac{\beta}{2}\right|m, \tag{4.1}$$

where $m = \inf_{z\in T_1} |r(z)|$. *The equality in* (4.1) *holds for* $r(z) = \lambda B(z), \lambda > 0$.

By taking $\beta = 0$ in inequality (4.1), we get the following result, which is basically extension of inequality (1.6) due to Aziz and Dawood [2] for rational function $r \in R_n$.

Corollary 1: *If* $r \in R_n$ *has* n *zeros all in* D_{1^-}, *then*

$$\inf_{z\in T_1} |r'(z)| \geq |B'(z)| \inf_{z\in T_1} |r(z)|. \tag{4.2}$$

The equality in (4.2) *holds for* $r(z) = \lambda B(z), \lambda > 0$.

The following result for the polar derivative is a spacial case of Theorem 1 applied to a single pole (when $a_i = \alpha$ with $|\alpha| \geq 1$, $i = 1, 2, \ldots, n$).

Corollary 2: *If* $p \in P_n$ *and having all its zeros in* D_{1^-}, *then for every real or complex number* β *with* $|\beta| \leq 1$

$$\begin{aligned}&\min_{z\in T_1}\left| -\frac{D_\alpha p(z)}{(z-\alpha)^{n+1}} + \frac{n\beta}{2}\frac{(|\alpha|^2-1)}{|z-\alpha|^2}\frac{p(z)}{(z-\alpha)^n}\right| \geq \\ &\frac{n(|\alpha|^2-1)}{|z-\alpha|^2}\left|1+\frac{\beta}{2}\right| \min_{z\in T_1}\left|\frac{p(z)}{(z-\alpha)^n}\right|,\end{aligned} \tag{4.3}$$

where $|\alpha| \geq 1$.

Next, we prove the following extension of inequality (1.7) due to Malik and Vong [7] for the rational function, which is similar to a result proved by Xin Li [10].

Theorem 2: *Let* $r, s \in R_n$ *and all the zeros of* $s(z)$ *lies in* $T_1 \cup D_{1^-}$. *If*

$$|r(z)| \leq |s(z)| \quad \text{for} \quad z \in T_1, \tag{4.4}$$

then for every real or complex number β *with* $|\beta| \leq 1$

$$\left| zr'(z) + \frac{\beta}{2}|B'(z)| r(z) \right| \leq \left| zs'(z) + \frac{\beta}{2}|B'(z)| s(z) \right| \quad \text{for} \quad z \in T_1. \tag{4.5}$$

If we take $s(z) = B(z)\|r\|$ in inequality (4.5) of Theorem 2, then the following result has been obtained, which extend inequality (1.8) for rational functions.

Corollary 3: *If* $r \in R_n$, *then for every real or complex* β *with* $|\beta| \leq 1$

$$\left| zr'(z) + \frac{\beta}{2}|B'(z)| r(z) \right| \leq \left| 1 + \frac{\beta}{2} \right| |B'(z)| \|r\| \tag{4.6}$$

for $z \in T_1$. *The equality in above holds for* $r(z) = \lambda B(z), \lambda > 0$.

Also we prove the following result, which is basically an extension of inequality (1.9) of Theorem A due to Jain [8] for the rational functions $r \in R_n$.

Theorem 3: *If* $r \in R_n$, *then for* $|\beta| \le 1$

$$\left|zr'(z) + \frac{\beta}{2}|B'(z)|r(z)\right| + \left|zr^{*'}(z) + \frac{\beta}{2}|B'(z)|r^*(z)\right| \le \left\{\left|1+\frac{\beta}{2}\right| + \left|\frac{\beta}{2}\right|\right\}|B'(z)|M \tag{4.7}$$

for $z \in T_1$, *where* $M = sup_{z\in T_1}|r(z)|$.

If we consider $r \in R_n$ having all its zero in $T \cup D_{1+}$ and $m = \inf_{z\in T_1}|r(z)|$, then $m \le |r(z)|$. For given complex number δ with $|\delta| \le 1$, we have from Rouche's Theorem that the polynomial $F(z) = r(z) - \delta m$ have all its zeros in $T \cup D_{1+}$ and $G(z) = B(z)\overline{F(1/\overline{z})} = B(z)\overline{r(1/\overline{z})} - \overline{\delta}B(z)m = r^*(z) - \overline{\delta}B(z)m$ has all it zeros in $T_1 \cup D_{1-}$. Applying Theorem 2 for the polynomial $G(z) = r^*(z) - \overline{\delta}B(z)m$, we have

$$\left|zr'(z) + \frac{\beta}{2}|B'(z)|r(z)\right| \le \left|zr^{*'}(z) + \frac{\beta}{2}|B'(z)|r^*(z)\right| - |B'(z)|\left\{\left|\frac{\beta}{2}\right| + \left|1+\frac{\beta}{2}\right|\right\}m. \tag{4.8}$$

Combining inequality (4.7) of Theorem 3 and inequality (4.8), we get the following result, which is similar to the result recently proved by Ahmad, Liman and Shah [1].

Corollary 4: *If* $r \in R_n$ *has n zeros, all in* $T_1 \cup D_{1+}$, *then for every real and complex number* β *with* $|\beta| \le 1$

$$\left|zr'(z) + \frac{\beta}{2}|B'(z)|r(z)\right| \le \frac{|B'(z)|}{2}\left[\left\{\left|1+\frac{\beta}{2}\right| + \left|\frac{\beta}{2}\right|\right\}M - \left\{\left|1+\frac{\beta}{2}\right| - \left|\frac{\beta}{2}\right|\right\}m\right], \tag{4.9}$$

where $M=\sup_{z\in T_1}|r(z)|$ *and* $m=\inf_{z\in T_1}|r(z)|$. *The equality in above holds for* $r(z)=B(z)+ke^{i\delta}, k\geq 1$.

For $\beta=0$, Corollary 4 gives the following result, which was proved by Aziz and Shah [4].

Corollary 5: *If* $r\in R_n$ *has n zeros, all in* $T_1\cup D_{1+}$, *then*

$$|r'(z)|\leq \frac{|B'(z)|}{2}\left[M-m\right], \tag{4.10}$$

where $M=\sup_{z\in T_1}|r(z)|$ *and* $m=\inf_{z\in T_1}|r(z)|$. *The inequality* (4.11) *is sharp and equality holds for* $r(z)=B(z)+ke^{i\delta}, k\geq 1$.

By considering $r(z)=p(z)/(z-\alpha)^n$ in inequality (4.7) of Theorem 3, we get the following analogous result in terms of polar derivative of $p\in P_n$.

Corollary 6: *If* $p\in P_n$ *and* $M=\max_{z\in T_1}|p(z)|=1$, *then for every real or complex* β *with* $|\beta|\leq 1$

$$\left|-\frac{D_\alpha p(z)}{(z-\alpha)^{n+1}}+\frac{n\beta}{2}\frac{(|\alpha|^2-1)}{|z-\alpha|^2}\frac{p(z)}{(z-\alpha)^n}\right|+\left|-\frac{D_\alpha q(z)}{(z-\alpha)^{n+1}}+\frac{n\beta}{2}\frac{(|\alpha|^2-1)}{|z-\alpha|^2}\frac{q(z)}{(z-\alpha)^n}\right|$$
$$\leq \frac{n(|\alpha|^2-1)}{|z-\alpha|^2}\left\{\left|1+\frac{\beta}{2}\right|+\left|\frac{\beta}{2}\right|\right\}. \tag{4.11}$$

Where $q(z)=z^n\overline{p(1/\overline{z})}$ *and* $|\alpha|\geq 1$.

In the same manner, following result has been obtained by taking $r(z)=\dfrac{p(z)}{(z-\alpha)^n}$ in Corollary 4.

Corollary 7: *If* $p\in P_n$ *and having all its zeros in* $T_1\cup D_{1+}$, *then for every real or complex* β *with* $|\beta|\leq 1$

$$\left|-\frac{D_\alpha p(z)}{(z-\alpha)^{n+1}}+\frac{n\beta}{2}\frac{(|\alpha|^2-1)}{|z-\alpha|^2}\frac{p(z)}{(z-\alpha)^n}\right|$$
$$\leq \frac{n(|\alpha|^2-1)}{|z-\alpha|^2}\left[\left\{\left|1+\frac{\beta}{2}\right|+\left|\frac{\beta}{2}\right|\right\}\max_{z\in T_1}\left|\frac{p(z)}{(z-\alpha)^n}\right|\right.$$
$$\left.-\left\{\left|1+\frac{\beta}{2}\right|-\left|\frac{\beta}{2}\right|\right\}\min_{z\in T_1}\left|\frac{p(z)}{(z-\alpha)^n}\right|\right], \tag{4.12}$$

where $|\alpha|\geq 1$.

5. Lemma

For the proof of above results we required following two lemmas due to Li, Mohapatra, and Rodgriguez [11].

Lemma 1: *Let A and B be any two complex numbers. Then*

(i). *If $|A|\geq|B|$ and $B\neq 0$, then $A\neq \nu B$ for all complex number ν with $|\nu|<1$.*

(ii). *Conversely, if $A\neq \nu B$ for all complex number ν with $|\nu|<1$, then $|A|\geq|B|$.*

Lemma 2: *If $r\in R_n$ has n zeros all in $T_1\cup D_{1-}$, then for $z\in T_1$*

$$|zr'(z)|\geq \frac{|B'(z)|}{2}|r(z)|. \tag{5.1}$$

6. Proof of Theorem

Proof of Theorem 1: If $r(z)$ has a zero in T_1, then Theorem 1 is trivial. So, we consider that $r(z)$ has all its zeros in D_{1-}. If $m=\inf_{z\in T_1}|r(z)|$, then $m>0$ and $|r(z)|\geq m$ for $z\in T_1$. Therefore, for any δ with $|\delta|<1$, then the function $F(z)=r(z)-\delta mB(z)$ of degree n, has all its zeros in D_{1-}. By Lemma 2

$$|zF'(z)|\geq \frac{|B'(z)|}{2}|F(z)| \tag{6.1}$$

i.e.

$$|zr'(z)-\delta mzB'(z)|\geq \frac{|B'(z)|}{2}|r(z)-\delta mB(z)|. \tag{6.2}$$

Since $F(z)=r(z)-\delta mB(z)\neq 0$ in $T_1\cup D_{1+}$, then for any complex number β with $|\beta|\leq 1$, we have from (i) of Lemma 1

$$T(z)=z\{r'(z)-\delta mB'(z)\}+\beta\frac{|B'(z)|}{2}\{r(z)-\delta mB(z)\}\neq 0 \tag{6.3}$$

i.e.

$$T(z)=\left\{zr'(z)+\beta\frac{|B'(z)|}{2}r(z)\right\}-\delta\left\{zB'(z)+\frac{\beta}{2}|B'(z)|B(z)\right\}m\neq 0$$

in $T_1\cup D_{1+}$. Now from (ii) of Lemma 1, we obtained for $|\delta|<1$

$$\left|zr'(z)+\beta\frac{|B'(z)|}{2}r(z)\right|\geq\left|zB'(z)+\frac{\beta}{2}|B'(z)|B(z)\right|m. \tag{6.4}$$

Now from (2.3), $|B'(z)|B(z) = zB'(z)$ for $z \in T_1$, therefore inequality (6.4) becomes

$$\left|zr'(z) + \beta\frac{|B'(z)|}{2}r(z)\right| \geq \left|1+\frac{\beta}{2}\right||B'(z)|m. \tag{6.5}$$

When a zero of $r(z)$ lie on T_1 and $|\beta| \leq 1$, the case can follow by using argument of continuity. Which complete Theorem 1.

Proof of Theorem 2: Since $s(z)$ has all its zeros in $T_1 \cup D_{1-}$ and $|r(z)| \leq |s(z)|$ for $z \in T_1$, then from Rouche's Theorem for any $|\lambda| < 1$, $F(z) = \lambda r(z) + s(z)$ has as many zeros in $T_1 \cup D_{1-}$ as $s(z)$. Hence from Lemma 2, we have for $z \in T_1$

$$|zF'(z)| \geq \frac{|B'(z)|}{2}|F(z)|, \tag{6.6}$$

Since $F(z) \neq 0$ in $T_1 \cup D_{1+}$ and $|B'(z)| \neq 0$ (see formula 14 in [10]), then from Lemma 1 (i) for every real or complex β with $|\beta| < 1$

$$T(z) = zF'(z) + \beta\frac{|B'(z)|}{2}F(z) \neq 0,$$

i.e.,

$$T(z) = z\{\lambda r'(z) + s'(z)\} + \beta\frac{|B'(z)|}{2}\{\lambda r(z) + s(z)\} \neq 0,$$

in $T_1 \cup D_{1+}$. This implies that

$$\lambda\left\{zr'(z) + \beta\frac{|B'(z)|}{2}r(z)\right\} \neq -\left\{zs'(z) + |B'(z)|\frac{\beta}{2}s(z)\right\}, \tag{6.7}$$

using (ii) of Lemma 1 in (6.7), we gets for λ with $|\lambda| < 1$

$$\left|zs'(z) + \frac{|B'(z)|}{2}s(z)\right| \geq \left|zr'(z) + \beta\frac{|B'(z)|}{2}r(z)\right|. \tag{6.8}$$

Using the continuity in zeros, we can obtained the inequality when some zeros of $s(z)$ lie on T_1 and $|\beta| \leq 1$. Theorem 2 is completed.

Proof of Theorem 3: For $|\alpha| > 1$ and $M = \sup_{z \in T_1}|r(z)|$, the polynomial

$$F(z) = r(z) - \alpha M \tag{6.9}$$

has no zero in D_{1-}, Define

$$G(z) = B(z)\overline{F(1/\overline{z})} = r^*(z) - \overline{\alpha}B(z)M \tag{6.10}$$

has all its zeros in $z \in T_1 \cup D_{1^-}$. Therefore, using Theorem 2 for $G(z)$, we have

$$\left|zF'(z)+\frac{\beta}{2}|B'(z)|F(z)\right| \le \left|zG'(z)+\frac{\beta}{2}|B'(z)|G(z)\right| \tag{6.11}$$

i.e.

$$\left|zr'(z)+\frac{\beta}{2}|B'(z)|\{r(z)-\alpha M\}\right| \quad \le$$
$$\left|z\{r^{*'}(z)-\bar{\alpha}B'(z)M\}+\frac{\beta}{2}|B'(z)|\{r^{*}(z)-\bar{\alpha}B(z)M\}\right| \tag{6.12}$$

From (2.3) $|B'(z)|B(z) = zB'(z)$ for $z \in T_1$, then inequality (6.12) becomes

$$\left|\left\{zr'(z)+\frac{\beta}{2}|B'(z)|r(z)\right\}-\alpha\frac{\beta}{2}B'(z)M\right| \quad \le$$
$$\left|\left\{zr^{*'}(z)+\frac{\beta}{2}|B'(z)|r^{*}(z)\right\}-\bar{\alpha}\left\{1+\frac{\beta}{2}\right\}B'(z)M\right| \tag{6.13}$$

Since $r^{*'}(z)$ is a polynomial of degree n, then from Corollary 3 for $z \in T_1$

$$\left|zr^{*'}(z)+\frac{\beta}{2}|B'(z)|r^{*}(z)\right| \quad \le \quad \left|1+\frac{\beta}{2}\right||B'(z)|\sup_{z\in T_1}|r^{*'}(z)|$$
$$= \quad \left|1+\frac{\beta}{2}\right||B'(z)|M. \tag{6.14}$$

By suitable choice of argument of α and for $z \in T_1$, we have

$$\left|zr'(z)+\frac{\beta}{2}|B'(z)|r(z)\right|-|\alpha||B'(z)||\frac{\beta}{2}|M \quad \le$$
$$|\alpha|\left|1+\frac{\beta}{2}\right||B'(z)|M-\left|r^{*'}(z)+\frac{\beta}{2}|B'(z)|r^{*}(z)\right|. \tag{6.15}$$

Taking $|\alpha| \to 1$, we obtained

$$|r'(z)+\frac{\beta}{2}B(z)r(z)| \quad + \quad |r^{*'}(z)+\frac{\beta}{2}B(z)r^{*}(z)|$$
$$\le \quad \left\{|1+\frac{\beta}{2}|+|\frac{\beta}{2}|\right\}|B'(z)|M. \tag{6.16}$$

Which follows Theorem 3. Theorem 3 holds for the case of some zeros of $r(z)$ lies on T_1 and $|\beta|\leq 1$ by using the argument of continuity of zeros.

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