Exercises from Chapter 2

Exercise 2.2

The truth table for the *exclusive* version of *or*:

P	Q	$P \oplus Q$
t	t	f
t	f	t
f	t	t
f	f	f

Exercise 2.4

The truth table for $\neg(P \Leftrightarrow Q)$:

P	Q	$P \Leftrightarrow Q$	$\neg(P \Leftrightarrow Q)$
t	t	t	f
t	f	f	t
f	t	f	t
f	f	t	f

As we can see, the the column for $\neg(P \Leftrightarrow Q)$ is identical to the column for $P \oplus Q$ in exercise 2.2. Then

$$x \oplus y = \neg\,(x \Leftrightarrow y) = \neg\,(x \equiv y) = x \not\equiv y$$

Thus the implementation of $(\oplus)$ is correct.

Exercise 2.9

$(P$	$\Leftrightarrow$	$Q)$	$\oplus$	Q	$\Leftrightarrow$	P
t	f	t	t	t	t	t
t	t	f	t	f	t	t
f	t	t	f	t	t	f
f	f	f	f	f	t	f

$(P \oplus Q) \oplus Q \Leftrightarrow P$ is a logical validity, thus $(P \oplus Q) \oplus Q$ is equivalent to P.

Ecercise 2.11

1. Law of double negation:

P	$\Leftrightarrow$	$\neg$	$\neg$	P
t	t	t	f	t
f	t	f	t	f

2. Laws of idempotence:

P	$\wedge$	P	$\Leftrightarrow$	P
t	t	t	t	t
f	f	f	t	f

P	$\vee$	P	$\Leftrightarrow$	P
t	t	t	t	t
f	f	f	t	f

3.

P	$\Rightarrow$	Q	$\Leftrightarrow$	$\neg$	P	$\vee$	Q
t	t	t	t	f	t	t	t
t	f	f	t	f	t	f	f
f	t	t	t	t	f	t	t
f	t	f	t	t	f	t	f

$\neg$	$(P$	$\Rightarrow$	$Q)$	$\Leftrightarrow$	P	$\wedge$	$\neg$	Q
f	t	t	t	t	t	f	f	t
t	t	f	f	t	t	t	t	f
f	f	t	t	t	f	f	f	t
f	f	t	f	t	f	f	t	f

4. Laws of contraposition:

$(\neg$	P	$\Rightarrow$	$\neg$	$Q)$	$\Leftrightarrow$	$(Q$	$\Rightarrow$	$P)$
f	t	t	f	t	t	t	t	t
f	t	t	t	f	t	f	t	t
t	f	f	f	t	t	t	f	f
t	f	t	t	f	t	f	t	f

$(P$	$\Rightarrow$	$\neg$	$Q)$	$\Leftrightarrow$	$(Q$	$\Rightarrow$	$\neg$	$P)$
t	f	f	t	t	t	f	f	t
t	t	t	f	t	f	t	f	t
f	t	f	t	t	t	t	t	f
f	t	t	f	t	f	t	t	f

$(\neg$	P	$\Rightarrow$	$Q)$	$\Leftrightarrow$	$(\neg$	Q	$\Rightarrow$	$P)$
f	t	t	t	t	f	t	t	t
f	t	t	f	t	t	f	t	t
t	f	t	t	t	f	t	t	f
t	f	f	f	t	t	f	f	f

5.

$(P$	$\Leftrightarrow$	$Q)$	$\Leftrightarrow$	$((P$	$\Rightarrow$	$Q)$	$\wedge$	$(Q$	$\Rightarrow$	$P))$	$\Leftrightarrow$	$((P$	$\wedge$	$Q)$	$\vee$	$(\neg$	P	$\wedge$	$\neg$	$Q))$
t	t	t	t	t	t	t	t	t	t	t	t	t	t	t	t	f	t	f	f	t
t	f	f	t	t	f	f	f	f	t	t	t	t	f	f	f	f	t	f	t	f
f	f	t	t	f	t	t	f	t	f	f	t	f	f	t	f	t	f	f	f	t
f	t	f	t	f	t	f	t	f	t	f	t	f	f	f	t	t	f	t	t	f

6. Laws of commutativity:

P	$\wedge$	Q	$\Leftrightarrow$	Q	$\wedge$	P
t	t	t	t	t	t	t
t	f	f	t	f	f	t
f	f	t	t	t	f	f
f	f	f	t	f	f	f

P	$\vee$	Q	$\Leftrightarrow$	Q	$\vee$	P
t	t	t	t	t	t	t
t	t	f	t	f	t	t
f	t	t	t	t	t	f
f	f	f	t	f	f	f

7. DeMorgan laws:

$\neg$	$(P$	$\wedge$	$Q)$	$\Leftrightarrow$	$\neg$	P	$\vee$	$\neg$	Q
f	t	t	t	t	f	t	f	f	t
t	t	f	f	t	f	t	t	t	f
t	f	f	t	t	t	f	t	f	t
t	f	f	f	t	t	f	t	t	f

$\neg$	$(P$	$\vee$	$Q)$	$\Leftrightarrow$	$\neg$	P	$\wedge$	$\neg$	Q
f	t	t	t	t	f	t	f	f	t
f	t	t	f	t	f	t	f	t	f
f	f	t	t	t	t	f	f	f	t
t	f	f	f	t	t	f	t	t	f

8. Laws of associativity:

P	$\wedge$	$(Q$	$\wedge$	$R)$	$\Leftrightarrow$	$(P$	$\wedge$	$Q)$	$\wedge$	R
t	t	t	t	t	t	t	t	t	t	t
t	f	t	f	f	t	t	t	t	f	f
t	f	f	f	t	t	t	f	f	f	t
t	f	f	f	f	t	t	f	f	f	f
f	f	t	t	t	t	f	f	t	f	t
f	f	t	f	f	t	f	f	t	f	f
f	f	f	f	t	t	f	f	f	f	t
f	f	f	f	f	t	f	f	f	f	f

P	$\vee$	$(Q$	$\vee$	$R)$	$\Leftrightarrow$	$(P$	$\vee$	$Q)$	$\vee$	R
t	t	t	t	t	t	t	t	t	t	t
t	t	t	t	f	t	t	t	t	t	f
t	t	f	t	t	t	t	t	f	t	t
t	t	f	f	f	t	t	t	f	t	f
f	t	t	t	t	t	f	t	t	t	t
f	t	t	t	f	t	f	t	t	t	f
f	t	f	t	t	t	f	f	f	t	t
f	f	f	f	f	t	f	f	f	f	f

9. Distribution laws:

P	$\wedge$	$(Q$	$\vee$	$R)$	$\Leftrightarrow$	$(P$	$\wedge$	$Q)$	$\vee$	$(P$	$\wedge$	$R)$
t	t	t	t	t	t	t	t	t	t	t	t	t
t	t	t	t	f	t	t	t	t	t	t	f	f
t	t	f	t	t	t	t	f	f	t	t	t	t
t	f	f	f	f	t	t	f	f	f	t	f	f
f	f	t	t	t	t	f	f	t	f	f	f	t
f	f	t	t	f	t	f	f	t	f	f	f	f
f	f	f	t	t	t	f	f	f	f	f	f	t
f	f	f	f	f	t	f	f	f	f	f	f	f

P	$\vee$	$(Q$	$\wedge$	$R)$	$\Leftrightarrow$	$(P$	$\vee$	$Q)$	$\wedge$	$(P$	$\vee$	$R)$
t	t	t	t	t	t	t	t	t	t	t	t	t
t	t	t	f	f	t	t	t	t	t	t	t	f
t	t	f	f	t	t	t	t	f	t	t	t	t
t	t	f	f	f	t	t	t	f	t	t	t	f
f	t	t	t	t	t	f	t	t	t	f	t	t
f	f	t	f	f	t	f	t	t	f	f	f	f
f	f	f	f	t	t	f	f	f	f	f	t	t
f	f	f	f	f	t	f	f	f	f	f	f	f

Exercise 2.13

Checks for the principles from Theorem 2.12:

```
import TAMO
test'1a = ¬ True ≡ False
test'1b = ¬ False ≡ True
test'2  = logEquiv1 (λp → p ⟹ False) (λp → ¬ p)
test'3a = logEquiv1 (λp → p ∨ True) (λp → True)
test'3b = logEquiv1 (λp → p ∧ False) (λp → False)
test'4a = logEquiv1 (λp → p ∨ False) id
test'4b = logEquiv1 (λp → p ∧ True) id
test'5  = logEquiv1 (λp → p ∨ ¬ p) (λp → True)
test'6  = logEquiv1 (λp → p ∧ ¬ p) (λp → False)
```

Exercise 2.15

Contradiction tests for propositional functions with one, two and three variables:

```
bools = [True, False]

contra1    :: (Bool → Bool) → Bool
contra1 bf = and [¬ (bf p) | p ← bools]

contra2    :: (Bool → Bool → Bool) → Bool
contra2 bf = and [¬ (bf p q) | p ← bools, q ← bools]

contra3    :: (Bool → Bool → Bool → Bool) → Bool
contra3 bf = and [¬ (bf p q r) | p ← bools, q ← bools, r ← bools]
```

Exercise 2.16

Useful denials for every sentence of Exercise 2.31:

1. The equation $x^2 + 1 = 0$ does not have a solution.

2. A largest natural number exists.
3. The number 13 is not prime.
4. The number n is not prime.
5. There are a finite number of primes.

Exercise 2.17

A denial for the statement that $x < y < z$ (where $x, y, z \in \mathbb{R}$):
$x < y < z \equiv x < y \wedge y < z$. Thus we can use the First Law of DeMorgan.

$$\neg(x < y < z) \equiv \neg(x < y \wedge y < z) \equiv x \geqslant y \vee y \geqslant z$$

Exercise 2.18

1.
$$\begin{aligned}(\Phi \Leftrightarrow \Psi) &\equiv ((\Phi \Rightarrow \Psi) \wedge (\Psi \Rightarrow \Phi)) && \text{by Theorem 2.10, 5}\\ &\equiv ((\neg\Psi \Rightarrow \neg\Phi) \wedge (\neg\Phi \Rightarrow \neg\Psi)) && \text{by contraposition}\\ &\equiv ((\neg\Phi \Rightarrow \neg\Psi) \wedge (\neg\Psi \Rightarrow \neg\Phi)) && \text{by commutativity of } \wedge\\ &\equiv (\neg\Phi \Leftrightarrow \neg\Psi) && \text{by Theorem 2.10, 5}\end{aligned}$$

2.
$$\begin{aligned}(\neg\Phi \Leftrightarrow \Psi) &\equiv ((\neg\Phi \Rightarrow \Psi) \wedge (\Psi \Rightarrow \neg\Phi)) && \text{by Theorem 2.10, 5}\\ &\equiv ((\neg\Psi \Rightarrow \Phi) \wedge (\Phi \Rightarrow \neg\Psi)) && \text{by contraposition}\\ &\equiv ((\Phi \Rightarrow \neg\Psi) \wedge (\neg\Psi \Rightarrow \Phi)) && \text{by commutativity of } \wedge\\ &\equiv (\Phi \Leftrightarrow \neg\Psi) && \text{by Theorem 2.10, 5}\end{aligned}$$

Exercise 2.19

$\Phi \equiv \Psi$ means that, no matter the truth values of $P, Q, \ldots$ occuring in the formulas, the formulas Φ and Ψ produce the same truth values: either both are true, or both are false. In both situations, $\Phi \Leftrightarrow \Psi$ is true. Thus $(\Phi \equiv \Psi) \Rightarrow (\Phi \Leftrightarrow \Psi)$.

Since $\Phi \Leftrightarrow \Psi$ is only true when Φ and Ψ have the same truth values, we also have $(\Phi \Leftrightarrow \Psi) \Rightarrow (\Phi \equiv \Psi)$. By part 5 of Theorem 2.10 we conclude that $(\Phi \equiv \Psi) \Leftrightarrow (\Phi \Leftrightarrow \Psi)$.

Exercise 2.20

1. $\neg P \Rightarrow Q$ and $P \Rightarrow \neg Q$ are not equivalent. The truth table below shows that $(\neg P \Rightarrow Q) \Leftrightarrow (P \Rightarrow \neg Q)$ is not a logical validity:

$(\neg$	P	$\Rightarrow$	$Q)$	$\Leftrightarrow$	$(P$	$\Rightarrow$	$\neg$	$Q)$
f	t	t	t	f	t	f	f	t
f	t	t	f	t	t	t	t	f
t	f	t	t	t	f	t	f	t
t	f	f	f	f	f	t	t	f

Verifying with Haskell:

$$\mathit{logEquiv2}\ (\lambda p\ q \to \neg\ p \Longrightarrow q)\ (\lambda p\ q \to p \Longrightarrow \neg\ q)$$
False

2. $\neg P \Rightarrow Q$ and $Q \Rightarrow \neg P$ are not equivalent. By part 4 of Theorem 2.10 we see that $Q \Rightarrow \neg P$ is the contrapositive of $P \Rightarrow \neg Q$, which we have already shown to be non-equivalent to $\neg P \Rightarrow Q$.

Verifying with Haskell:

$$\mathit{logEquiv2}\ (\lambda p\ q \to \neg\ p \Longrightarrow q)\ (\lambda p\ q \to q \Longrightarrow \neg\ p)$$
False

3. $\neg P \Rightarrow Q$ and $\neg Q \Rightarrow P$ are equivalent, by part 4 of Theorem 2.10.

Verifying with Haskell:

$$\mathit{logEquiv2}\ (\lambda p\ q \to \neg\ p \Longrightarrow q)\ (\lambda p\ q \to \neg\ q \Longrightarrow p)$$
True

4. $P \Rightarrow (Q \Rightarrow R)$ and $Q \Rightarrow (P \Rightarrow R)$ are equivalent:

$$\begin{aligned}
(P \Rightarrow (Q \Rightarrow R)) &\equiv \neg P \vee (\neg Q \vee R) && \text{by Theorem 2.10, part 3}\\
&\equiv \neg P \vee \neg Q \vee R && \text{by associativity of } \vee\\
&\equiv \neg Q \vee \neg P \vee R && \text{by commutativity of } \vee\\
&\equiv \neg Q \vee (\neg P \vee R) && \text{by associativity of } \vee\\
&\equiv (Q \Rightarrow (P \Rightarrow R)) && \text{by Theorem 2.10, part 3}
\end{aligned}$$

Verifying with Haskell:

$$\mathit{logEquiv3}\ (\lambda p\ q\ r \to p \Longrightarrow (q \Longrightarrow r))\ (\lambda p\ q\ r \to q \Longrightarrow (p \Longrightarrow r))$$
True

5. $P \Rightarrow (Q \Rightarrow R)$ and $(P \Rightarrow Q) \Rightarrow R$ are not equivalent:

$$\begin{aligned}
(P \Rightarrow (Q \Rightarrow R)) &\equiv \neg P \vee (\neg Q \vee R) && \text{by Theorem 2.10, part 3} \\
&\equiv \neg P \vee \neg Q \vee R && \text{by associativity of } \vee \\
&\equiv \neg(P \wedge Q) \vee R && \text{by DeMorgan} \\
&\equiv (P \wedge Q) \Rightarrow R && \text{by Theorem 2.10, part 3}
\end{aligned}$$

Now, if $(P \wedge Q) \Rightarrow R$ and $(P \Rightarrow Q) \Rightarrow R$ are equivalent, then $P \wedge Q$ and $P \Rightarrow Q$ must also be equivalent. We check using a truth table:

P	$\wedge$	Q	$\Leftrightarrow$	$(P$	$\Rightarrow$	$Q)$
t	t	t	t	t	t	t
t	f	f	t	t	f	f
f	f	t	f	f	t	t
f	f	f	f	f	t	f

As we can see from the $\Leftrightarrow$ column, the two are not equivalent. Thus $P \Rightarrow (Q \Rightarrow R)$ and $(P \Rightarrow Q) \Rightarrow R$ are not equivalent.

Verifying with Haskell:

$$\textit{logEquiv3}\ (\lambda p\ q\ r \rightarrow p \Longrightarrow (q \Longrightarrow r))\ (\lambda p\ q\ r \rightarrow (p \Longrightarrow q) \Longrightarrow r)$$
False

6. $(P \Rightarrow Q) \Rightarrow P$ and P are equivalent:

$(P$	$\Rightarrow$	$Q)$	$\Rightarrow$	P	$\Leftrightarrow$	P
t	t	t	t	t	t	t
t	f	f	t	t	t	t
f	t	t	f	f	t	f
f	t	f	f	f	t	f

Verifying with Haskell:

$$\textit{logEquiv2}\ (\lambda p\ q \rightarrow (p \Longrightarrow q) \Longrightarrow p)\ (\lambda p\ q \rightarrow p)$$
True

7. $P \vee Q \Rightarrow R$ and $(P \Rightarrow R) \wedge (Q \Rightarrow R)$ are equivalent:

$$\begin{aligned}
(P \vee Q \Rightarrow R) &\equiv \neg(P \vee Q) \vee R && \text{by Theorem 2.10, part 3} \\
&\equiv (\neg P \wedge \neg Q) \vee R && \text{by DeMorgan} \\
&\equiv (\neg P \vee R) \wedge (\neg Q \vee R) && \text{by distribution} \\
&\equiv (P \Rightarrow R) \wedge (Q \Rightarrow R) && \text{by Theorem 2.10, part 3}
\end{aligned}$$

Verifying with Haskell:

$$\textit{logEquiv3}\ (\lambda p\ q\ r \rightarrow p \vee q \Longrightarrow r)\ (\lambda p\ q\ r \rightarrow (p \Longrightarrow r) \wedge (q \Longrightarrow r))$$
True

Exercise 2.21

1. Let Φ be defined as $P \vee \neg Q$. Now Φ has the desired truth table:

P	$\vee$	$\neg$	Q	Φ
t	t	f	t	t
t	t	t	f	t
f	f	f	t	f
f	t	t	f	t

2. There are a total of $2^4 = 16$ truth tables for 2-letter formulas.

3. All 16 truth tables, with formulas, are listed in table **??** on page **??**.

4. I do not know if there is a general method for finding these formulas. There probably is, seeing as constructing the above formulas was very easy, but I am unable to precisely define the process.

5. There would be a total of $2^8 = 256$ truth tables for 3-letter formulas. As for the question of a general method of finding the formulas, see the previous answer.

Exercise 2.26

$$\exists x, y \in \mathbb{Q}\ (x < y) \tag{1}$$
$$\forall x \in \mathbb{R}\ \exists y \in R\ (x < y) \tag{2}$$
$$\forall x \in \mathbb{Z}\ \exists m, n \in \mathbb{N}\ (x = m - n) \tag{3}$$

Exercise 2.27

$$\forall x\ (x \in \mathbb{Q} \Rightarrow \exists m, n\ (m \in \mathbb{Z} \wedge n \in \mathbb{Z} \Rightarrow \neq 0 \wedge x = m/n)) \tag{1}$$
$$\forall x, y\ (x \in F \wedge y \in D \Rightarrow (Oxy \Rightarrow Bxy)) \tag{2}$$

Exercise 2.31

1. The equation $x^2 + 1 = 0$ has a solution:

$$\exists x\ (x^2 + 1 = 0)$$

2. A largest natural number does not exist:

$$\forall x \in \mathbb{N}\ \exists y \in \mathbb{N}\ (x < y)$$

P	Q	Φ
t	t	f
t	f	f
f	t	f
f	f	f

$\Phi := P \wedge \neg P$

P	Q	Φ
t	t	f
t	f	f
f	t	t
f	f	f

$\Phi := \neg P \wedge Q$

P	Q	Φ
t	t	f
t	f	t
f	t	f
f	f	f

$\Phi := P \wedge \neg Q$

P	Q	Φ
t	t	f
t	f	t
f	t	t
f	f	f

$\Phi := (P \wedge \neg Q) \vee (\neg P \wedge Q)$

P	Q	Φ
t	t	t
t	f	f
f	t	f
f	f	f

$\Phi := P \wedge Q$

P	Q	Φ
t	t	t
t	f	f
f	t	t
f	f	f

$\Phi := Q$

P	Q	Φ
t	t	t
t	f	t
f	t	f
f	f	f

$\Phi := P$

P	Q	Φ
t	t	t
t	f	t
f	t	t
f	f	f

$\Phi := P \vee Q$

P	Q	Φ
t	t	f
t	f	f
f	t	f
f	f	t

$\Phi := \neg(P \vee Q)$

P	Q	Φ
t	t	f
t	f	f
f	t	t
f	f	t

$\Phi := \neg P$

P	Q	Φ
t	t	f
t	f	t
f	t	f
f	f	t

$\Phi := \neg Q$

P	Q	Φ
t	t	f
t	f	t
f	t	t
f	f	t

$\Phi := \neg(P \wedge Q)$

P	Q	Φ
t	t	t
t	f	f
f	t	f
f	f	t

$\Phi := (P \wedge Q) \vee \neg(P \vee Q)$

P	Q	Φ
t	t	t
t	f	f
f	t	t
f	f	t

$\Phi := \neg P \vee Q$

P	Q	Φ
t	t	t
t	f	t
f	t	f
f	f	t

$\Phi := P \vee \neg Q$

P	Q	Φ
t	t	t
t	f	t
f	t	t
f	f	t

$\Phi := P \vee \neg P$

Table 1: The 16 truth tables for 2-letter formulas

3. The number 13 is prime ($d|n$ means 'd divides n'):

$$\forall d\ (d|13 \Rightarrow d = 1 \vee d = 13)$$

4. The number n is prime:

$$\forall d\ (n \neq 1 \wedge (d|n \Rightarrow d = 1 \vee d = n))$$

5. There are infinitely many primes:

$$\forall x\ (\exists n \forall d\ (n \neq 1 \wedge (d|n \Rightarrow d = 1 \vee d = n) \wedge x < n))$$

Exercise 2.32

1. Everyone loved Diana ($L(x, y)$ means 'x loved y', d is Diana):

$$\forall x\ L(x, d)$$

2. Diana loved everyone:

$$\forall x\ L(d, x)$$

3. Man is mortal ($M(x)$ means 'x is a man', $M'(x)$ means 'x is mortal'):

$$\forall x\ (M(x) \Rightarrow M'(x))$$

4. Some birds do not fly ($B(x)$ means 'x is a bird', $F(x)$ means 'x can fly'):

$$\exists x\ (B(x) \wedge \neg F(x))$$

Exercise 2.33

1. Dogs that bark do not bite:

$$\forall x\ (\mathrm{Dog}(x) \wedge \mathrm{Bark}(x) \Rightarrow \neg \mathrm{Bite}(x))$$

2. All that glitters is not gold:

$$\exists x\ (\mathrm{Glitters}(x) \wedge \neg \mathrm{Gold}(x))$$

3. Friends of Diana's friends are her friends:

$$\forall x, y\ (\mathrm{Friend}(d, x) \wedge \mathrm{Friend}(x, y) \Rightarrow \mathrm{Friend}(d, y))$$

4. The limit of $\frac{1}{n}$ as n approaches infinity is zero:

$$\lim_{n \to \infty} \frac{1}{n} = 0$$

Exercise 2.34

1. Everyone loved Diana except Charles: