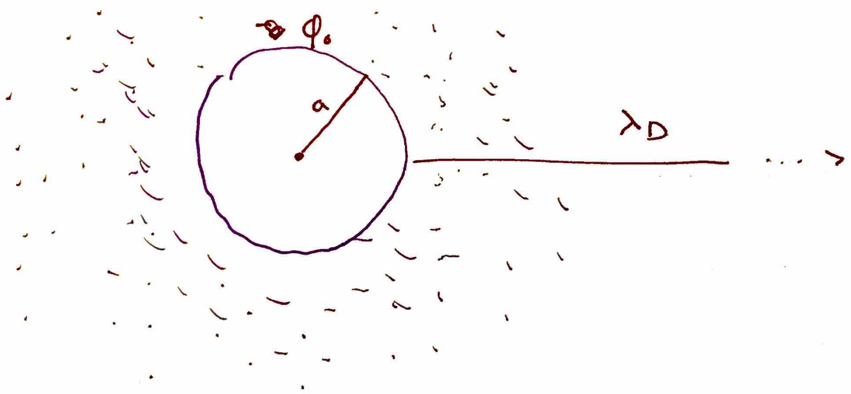
1.10 | A spherical conductor of radius a is immersed in a plasma and ~~an~~ charged to a potential ϕ_0. The electrons remain Maxwellian and move to form a Debye shield, but the ions are stationary during the time frame of the experiment. Assuming $\phi_0 \ll KT_e/e$, derive an expression for the potential as a function of r in terms of a, ϕ_0, + λ_D.
(Hint: Assume a solution of the form e^{-kr}/r.)

For this problem we start the same way as ~~the~~ ~~textbook~~ section 1.4 + problem 1.5. The major difference is we are now in 3-D.



Poisson's equation:

$$-\epsilon_0 \nabla^2 (\phi(\vec{r})) = \rho = \rho_i + \rho_e \; ; \; Z = 1$$

$\rho_i = e n_i = e n_0$ + ions do not have enough time to interact.

$\rho_e = -e n_e$ + Maxwellian => electron distribution follows the Boltzmann relation.

$$n_e = n_0 \exp(+e\phi(r)/KT_e)$$

We are going to assume thermal energies are greater than the sphere - ~~charge~~ electron potential energies; $e\phi \ll kT$

$*$ if $e\phi_0 \ll kT \Rightarrow e\phi \ll kT$

$\therefore$

The Poisson equation:

$\phi = 0$

$\phi = \phi_0$

$$-\cancel{\epsilon_0}\, \nabla^2 (\phi(r)) = -\frac{e n_0}{\epsilon_0}\left[1 - \left(1 + \frac{e\phi(r)}{kTe}\right)\right]$$

$$= \frac{e^2 n_0}{\epsilon_0 kTe}\phi(r) = \frac{1}{\lambda_D^2}\phi(r)$$

~~The next step~~ Now we are going to apply the hint.

$\phi(r) = A e^{-kr}/r$, $\nabla^2(\ ) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial(\ )}{\partial r}\right)$

$$\nabla^2(\phi(r)) = \frac{1}{r^2}\left(2r\frac{\partial(\phi)}{\partial r} + r^2\frac{\partial \phi}{\partial r^2}\right) = \frac{2}{r}\phi'(r) + \phi''(r) \,; \quad \phi'(r) = \frac{\partial \phi}{\partial r}$$

$\vdots$

$$\phi(r) = A e^{-kr}/r$$

$$\phi'(r) = \left(-Ake^{-kr}/r - \frac{Ae^{-kr}}{r^2}\right) = -(k + 1/r)\frac{Ae^{-kr}}{r} = -(k + 1/r)\phi(r)$$

$$\phi''(r) = 1/r^2\,\phi(r) - (k + 1/r)\phi'(r)$$

$\therefore$

$$\nabla^2(\phi(r)) = \phi(r)/r^2 - (k + 1/r)\phi'(r) + \frac{2}{r}\phi'(r)$$

$$\nabla^2(\varphi) = \frac{\varphi(r)}{r^2} - (k - 1/r)\,\varphi'(r) \; ; \quad \varphi'(r) = -(k + 1/r)\,\varphi$$

$$\nabla^2(\varphi) = \frac{\varphi(r)}{r^2} + (k^2 - 1/r^2)\,\varphi(r) = k^2\varphi(r)$$

$$\nabla^2(\varphi) = k^2\varphi(r)$$

Back to the Poisson equation

$$k^2\varphi(r) = \frac{1}{\lambda_D^2}\varphi(r) \Rightarrow k = \frac{1}{\lambda_D} \quad \therefore \quad \varphi(r) = \underbrace{A}_{A}\, e^{-r/\lambda_D}/r$$

~~What~~ We need now the coefficient A.

~~Be~~ B Applying the boundary condition:

$\varphi(r=a) = \varphi_0$

$$\therefore \quad \varphi_0 = A\, e^{-a/\lambda_D}/a \Leftrightarrow A = \varphi_0\, a\, e^{a/\lambda_D}$$

$$\therefore \quad \varphi(r) = \varphi_0\, e^{-r/\lambda_D + a/\lambda_D}/(r/a) = \varphi_0 \left(\frac{e^{-\frac{(r-a)}{\lambda_D}}}{r/a} \right)$$

$$\cancel{= \varphi_0\, e^{+a}\, \frac{e^{-(r/a-1)/\lambda_D}}{r/a}}$$

$$\boxed{\varphi(r) = \frac{\varphi_0\, a}{r}\, e^{-(r-a)/\lambda_D}}$$