$< 5, \sqrt{6.7}i >$ - Stand back, I know iVectors!

Convergence and Infinite Series

Series

- Geometric series

  $\sum ar^n$
  if $|r| < 1$ then $\sum ar^n$ converges
  The sum can be found with $\frac{a}{1-r}$

  eg:
  $\sum\limits_{n=0}^{\infty} \frac{4}{2^n} = \sum\limits_{n=0}^{\infty} 4 \times (\frac{1}{2})^n$ where $a = 4$ and $r = \frac{1}{2} < 1 \therefore \sum\limits_{n=0}^{\infty} \frac{4}{2^n}$ converges

- P-Series

  $\sum \frac{1}{n^p}$ where $p > 0$
  if $p > 1$ then the series converges
  if $p <= 1$ then the series diverges
  if $p = 1$ then the series is harmonic and diverges (which is useful for the comparison test)

- Alternating Series

  $\sum(-1)^n a_n$
  converges if $\lim_{n\to\infty} a_n = 0$ and $a_{n+1} < a_n$

Tests

- n^{th} term test - Only for divergence

  if $\lim_{n\to\infty} a_n \neq 0$ then $\sum a_n$ diverges

- Integral Test

  if $\int\limits_1^{\infty} f(x)\,dx$ converges then $\sum\limits_{n=1}^{\infty} a_n$ converges where $f(x) = a_n$. The converse of this is also true.

- Limit Comparison

  if $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ where L is both positive and finite, then the two series both either converge or diverge.

  eg:
  $\sum\limits_{n=3}^{\infty} \frac{3}{\sqrt{n^2-4}}$ compared to $\sum\limits_{n=1}^{\infty} \frac{1}{n}$

$\lim_{n\to\infty}\left(\frac{3}{\sqrt{n^2-4}}\times\frac{n}{1}\right)=\frac{\infty}{\infty}$
$L'H\to\lim_{n\to\infty}\left(\frac{3\sqrt{n^2-4}}{n}\right)=\frac{\infty}{\infty}$
$\lim_{n\to\infty}\left(\frac{3\sqrt{n^2-4}}{\sqrt{n^2}}\right)=3\lim_{n\to\infty}\sqrt{\frac{n^2-4}{n^2}}=3\sqrt{\lim_{n\to\infty}\frac{n^2-4}{n^2}}=3\times 1=3\therefore\sum_{n=0}^{\infty}\frac{3}{\sqrt{n^2-4}}$ diverges since $\frac{1}{n}$ diverges and the limit is finite and positive.

- Ratio Test

  $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$
  if <1 a_n converges
  if >1 a_n diverges
  if $=1$ the test is inconclusive

  eg:
  Find the range of x where $\sum_{n=3}^{\infty}\frac{(-1)^n n!(x-4)^n}{3^n}$ converges.
  $\lim_{n\to\infty}\left|\frac{(n+1)!(x-4)^{n+1}}{3^{n+1}}\times\frac{3^n}{n!(x-4)^n}\right|<1$
  $\lim_{n\to\infty}\left|\frac{(x-4)(n+1)}{3}\right|<1$
  $\left|\frac{x-4}{3}\right|\lim_{n\to\infty}|n+1|<1\therefore$ no range of x makes $\sum_{n=0}^{\infty}\frac{(-1)^n n!(x-4)^n}{3^n}$ converge.

- Condition Convergence

  if $\sum_{n=1}^{\infty}a_n$ converges and $\sum_{n=1}^{\infty}|a_n|$ diverges

Taylor Series

- $f(x)=\sum_{n=0}^{\infty}\frac{f^n(c)}{n!}(x-c)^n$
  if $c=0$ then the series is a MacLaurin series

  $f(x)=\sum_{n=0}^{\infty}\frac{f^n(0)}{n!}x^n$

- $\sin(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$

- $\cos(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$

- $e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$

  $e^{-x}=\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}$

- $\ln(x+1)=\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n}$

- Example of finding the Taylor series centered at 0 (aka MacLaurin) of e^x

  $f(x) = e^x$

  $f(0) = 1$
  $f'(x) = e^x$ and $f'(0) = 1$
  $f''(x) = e^x$ and $f''(0) = 1$
  $f'''(x) = e^x$ and $f'''(0) = 1$
  $\frac{1}{0!}x^0 + \frac{1}{1!}x^1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n$
  $\therefore f(x) = e^x = \sum\limits_{n=0}^{\infty} \frac{x^n}{n!}$

Parametric and Polar Equations

Parametric

- $\dot{x} = \frac{dx}{dt}$

- $\dot{y} = \frac{dy}{dt}$

- $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\dot{y}}{\dot{x}}$ and $\dfrac{d^2y}{(dx)^2} = \dfrac{\frac{d}{dt}\frac{dy}{dx}}{\frac{dx}{dt}}$

- $Arc\ length = \int \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$

- $Area/integration = \int y(t) \frac{dx}{dt}\, dt = \int y\dot{x}\, dt$

- $x = r\cos(\theta)$ and $y = r\sin(\theta)$

Polar

- $r = \sqrt{x^2 + y^2}$

- $r^2 = x^2 + y^2$

- If going from parametric to polar, you have to convert t to θ: $\tan(\theta) = \frac{y}{x}$

- $Arc\ length = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$

- $Area/integration = \frac{1}{2}\int r^2\, d\theta$