Calculus

Various notes, formulae, equations and examples from Calculus (2)

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$$\int_a^b f(x)\,dx = F(b) - F(a)$$

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What and Where

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[(720) - 663 -1279]

$< 5, \sqrt{6.7}i >$ - Stand back, I know iVectors!

Infinite Series, Convergence tests, and Taylor Series

Infinite Series

- Geometric series

$$\sum ar^n$$

if $|r| < 1$ then $\sum ar^n$ converges

The sum can be found with $\frac{a}{1-r}$

eg:

$\sum_{n=0}^{\infty} \frac{4}{2^n} = \sum_{n=0}^{\infty} 4 \times (\frac{1}{2})^n$ where $a = 4$ and

$r = \frac{1}{2} < 1 \therefore \sum_{n=0}^{\infty} \frac{4}{2^n}$ converges

- P-Series

$$\sum \frac{1}{n^p} \text{ where } p > 0$$

if $p > 1$ then the series converges
if $p <= 1$ then the series diverges
if $p = 1$ then the series is harmonic and diverges (which is useful for the comparison test)

- Alternating Series

$$\sum (-1)^n a_n$$

converges if $\lim_{n\to\infty} a_n = 0$ and $a_{n+1} < a_n$

Convergence Tests

- n^{th} term test - Only for divergence

if $\lim_{n\to\infty} a_n \neq 0$ then $\sum a_n$ diverges

- Integral Test

if $\int_1^{\infty} f(x)\,dx$ converges then $\sum_{n=1}^{\infty} a_n$ converges where $f(x) = a_n$. The converse of this is also true.

- Limit Comparison

if $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ where L is both positive and finite, then the two series both either converge or diverge.

eg:

$\sum_{n=3}^{\infty} \frac{3}{\sqrt{n^2-4}}$ compared to $\sum_{n=1}^{\infty} \frac{1}{n}$

$$\lim_{n\to\infty} \left(\frac{3}{\sqrt{n^2-4}} \times \frac{n}{1} \right) = \frac{\infty}{\infty}$$

$$L'H \to \lim_{n\to\infty} \left(\frac{3\sqrt{n^2-4}}{n} \right) = \frac{\infty}{\infty}$$

$\lim_{n\to\infty}\left(\frac{3\sqrt{n^2-4}}{\sqrt{n^2}}\right) = 3\lim_{n\to\infty}\sqrt{\frac{n^2-4}{n^2}} = 3\sqrt{\lim_{n\to\infty}\frac{n^2-4}{n^2}} = 3\times 1 = 3 \therefore \sum_{n=0}^{\infty}\frac{3}{\sqrt{n^2-4}}$ diverges since $\frac{1}{n}$ diverges and the limit is finite and positive.

- Ratio Test

  $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$
  if < 1 a_n converges
  if > 1 a_n diverges
  if $= 1$ the test is inconclusive

  eg:

  Find the range of x where $\sum_{n=3}^{\infty}\frac{(-1)^n n!(x-4)^n}{3^n}$ converges.

  $\lim_{n\to\infty}\left|\frac{(n+1)!(x-4)^{n+1}}{3^{n+1}}\times\frac{3^n}{n!(x-4)^n}\right| < 1$

  $\lim_{n\to\infty}\left|\frac{(x-4)(n+1)}{3}\right| < 1$

  $\left|\frac{x-4}{3}\right|\lim_{n\to\infty}|n+1| < 1$ $\therefore$ no range of x makes $\sum_{n=0}^{\infty}\frac{(-1)^n n!(x-4)^n}{3^n}$ converge.

- Condition Convergence

  if $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty}|a_n|$ diverges

Taylor Series

- $f(x) = \sum_{n=0}^{\infty}\frac{f^n(c)}{n!}(x-c)^n$
  if $c = 0$ then the series is a MacLaurin series

  $$f(x) = \sum_{n=0}^{\infty}\frac{f^n(0)}{n!}x^n$$

- $\sin(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$

- $\cos(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$

- $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$

  $$e^{-x} = \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}$$

- $\ln(x+1) = \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n}$

- Example of finding the Taylor series centered at 0 (aka MacLaurin) of e^x

  $f(x) = e^x$

  $f(0) = 1$
  $f'(x) = e^x$ and $f'(0) = 1$
  $f''(x) = e^x$ and $f''(0) = 1$
  $f'''(x) = e^x$ and $f'''(0) = 1$
  $\frac{1}{0!}x^0 + \frac{1}{1!}x^1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n$
  $\therefore f(x) = e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$

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Parametric, Polar and Vectors

Parametric

- $\dot{x} = \frac{dx}{dt}$
- $\dot{y} = \frac{dy}{dt}$
- $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\dot{y}}{\dot{x}}$
- $\frac{d^2y}{(dx)^2} = \frac{\frac{d}{dt}\frac{dy}{dx}}{\frac{dx}{dt}}$
- $Arc\ length = \int \sqrt{\dot{x}^2 + \dot{y}^2}\, dt$
- $Area/integration = \int y(t)\frac{dx}{dt}\, dt = \int y\dot{x}\, dt$
- $x = r\cos(\theta)$ and $y = r\sin(\theta)$

Polar

- $r = \sqrt{x^2 + y^2}$
- $r^2 = x^2 + y^2$
- If going from parametric to polar, you have to convert t to θ: $\tan(\theta) = \frac{y}{x}$
- $Arc\ length = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$
- $Area/integration = \frac{1}{2}\int r^2\, d\theta$

Vectors

- You can't divide a vector by another vector, only by a scalar...
- Vectors are really just parametric equations in disguise, they just have an x and a y component represented by:

  $\vec{i} = x$ the x component of a vector

  $\vec{j} = y$ the y component of a vector

- All vectors have Magnitude and Direction.

  $\vec{a} = x\vec{i} + y\vec{j}$

  $\vec{a} =< x, y >$

  The magnitude (a unit vector) is the same as the $|\vec{a}| = \sqrt{x^2 + y^2}$

  The direction (another unit vector) is defined by $\frac{\vec{a}}{|\vec{a}|}$

  (An unit vector is simply a vector who has a magnitude of 1)

- Addition & Subtraction

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$(a_x + b_x)\vec{i} + (a_y + b_y)\vec{j} = \vec{c}$

Simply add the x components together, and the y components together.

- Multiplication

  Scalar product

  $6 \times \vec{a} = (6 \times a_x)\vec{i} + (6 \times a_y)\vec{j}$

  Vector multiplied by a scalar. This is also how you divide a vector by a scalar.

  Simply multiply the scalar out to both the x and y components.

  Dot product

  $\vec{a} \cdot \vec{b} = (a_x \times b_x) + (a_y \times b_y)$

  Two vectors multiplied together

  Add the product of the x components to the product of the y components to form a scalar.

  Cross Product

  $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta n$

  where n is a unit vector perpendicular to the plane containing $\vec{a}$ and $\vec{b}$ (think the right hand rule)

  Two vectors multiplied together

  Results in another vector.

- Helpful vectors and other things

  Angle between two vectors

  $$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

  Projection vectors

  $$proj_{\vec{a}}\vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2} \times \vec{b}$$

  Normal vectors

  $$norm_{\vec{a}}\vec{b} = \vec{a} - proj_{\vec{a}}\vec{b}$$

Integration Stuff

Integration by Parts

- $\int u\,dv = uv - \int v\,du$

Must be able to take the derivative of u

- eg:

$$f(x) = \int xe^x\,dx$$

u		dv
x	$\searrow$	e^x
1	$\searrow$	e^x
		$f(x) = xe^x - \int e^x\,dx$

Two Sided Integration by Parts

- Same as above however used when working with parts that repeat such as $\sin$ and $\cos$.

- eg:

$$f(x) = \int e^x \cos 2x\,dx$$

u		dv
e^x	$\overset{+}{\searrow}$	$\cos 2x$
e^x	$\overset{-}{\searrow}$	$\frac{1}{2}\sin 2x$
e^x	$\overset{+}{\leftarrow}$	$-\frac{1}{4}\cos 2x$

Where $\frac{1}{2}\int \sin 2x\,dx$

$u = 2x \therefore du = 2dx$

$$\therefore f(x) = \frac{1}{2}e^x \sin 2x + \frac{1}{4}e^x \cos 2x - \frac{1}{4}\int e^x \cos 2x\,dx$$

Trig Reduction

- Useful for integrating multiple $\sin$ and $\cos$ in one problem. Change the one that is odd in power to its substitute give by $1 = \sin^2 + \cos^2$. ie: if $\sin$ is $\sin^3$ change it to $\sin\left(1 - cos^2\right)$ and visa versa. If they are equal in powers, change $\sin$ into $\cos$.

- eg:

$f(x) = \int \sin^3 x \cos^2 x\,dx$

$= \int \left[\sin^2 x \cos^2 x\right]\,dx$

$= \int \left(1 - \cos^2 x\right)\left(\cos^2 x\right)\,dx$

$= \int \left(\cos^2 x - \cos^4 x\right)\,dx$ where $u = \cos x$

$= -\int u^2 - u^4\,du$

$= -\frac{\cos^3 x}{3} - \frac{\cos^5 x}{5} + c$

Partial Fractions with linear factors

- $\int \frac{1}{(x-p)(x-q)} = \int \frac{A}{x-q} + \int \frac{B}{x-q}$ and solve for A and B.

Partial Fractions with non-linear factors

- $\int \frac{1}{(x^2+2)(x-1)} = \int \frac{Ax+B}{x^2+2} + \frac{C}{x-1}$ and solve for A, B, and C.

Improper Integrals

- A definite integral where one or both sides have an infinite discontinuity.

- eg:
  $\int\limits_1^\infty \frac{1}{x}\,dx = [\ln|x|]_1^\infty$
  $= \lim\limits_{b\to\infty} \ln|b| - \ln|1|$

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TODO:

Stuff that I may add as I get time…

- Conic sections and General Quadratic
- Eulers Method
- Homogenious Differentials
- First order linear equations
- Orthogonal Tra.
- The Shell Method
  Volume of a torus
- Arc Length - functional
- Surface area of a curve
- Centroids
- Theorem of Pappus
- Indeterminate Forms (L'H)
- Lagrane Error Bound for infinite series
- Trig Substitution (just requires a lot of work)
- Hyperbolic Trig (also a lot of work…)

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