

$\langle 5, \sqrt{6.7}i \rangle$ - Stand back, I know *i*Vectors!

Infinite Series, Convergence tests, and Taylor Series

Infinite Series

- Geometric series

$$\sum ar^n$$

if $|r| < 1$ then $\sum ar^n$ converges
The sum can be found with $\frac{a}{1-r}$

eg:

$$\sum_{n=0}^{\infty} \frac{4}{2^n} = \sum_{n=0}^{\infty} 4 \times \left(\frac{1}{2}\right)^n \text{ where } a = 4 \text{ and}$$

$$r = \frac{1}{2} < 1 \therefore \sum_{n=0}^{\infty} \frac{4}{2^n} \text{ converges}$$

- P-Series

$$\sum \frac{1}{n^p} \text{ where } p > 0$$

if $p > 1$ then the series converges
if $p \leq 1$ then the series diverges
if $p = 1$ then the series is harmonic and diverges (which is useful for the comparison test)

- Alternating Series

$$\sum (-1)^n a_n$$

converges if $\lim_{n \rightarrow \infty} a_n = 0$ and $a_{n+1} < a_n$

Convergence Tests

- n^{th} term test - Only for divergence

$$\text{if } \lim_{n \rightarrow \infty} a_n \neq 0 \text{ then } \sum a_n \text{ diverges}$$

- Integral Test

$$\text{if } \int_1^{\infty} f(x) dx \text{ converges then } \sum_{n=1}^{\infty} a_n \text{ converges where } f(x) = a_n. \text{ The converse of this is also true.}$$

- Limit Comparison

$$\text{if } \lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L \text{ where } L \text{ is both positive and finite, then the two series both either converge or diverge.}$$

eg:

$$\sum_{n=3}^{\infty} \frac{3}{\sqrt{n^2-4}} \text{ compared to } \sum_{n=1}^{\infty} \frac{1}{n}$$

$$\lim_{n \rightarrow \infty} \left(\frac{3}{\sqrt{n^2-4}} \times \frac{n}{1} \right) = \frac{\infty}{\infty}$$

$$L'H \rightarrow \lim_{n \rightarrow \infty} \left(\frac{3\sqrt{n^2-4}}{n} \right) = \frac{\infty}{\infty}$$

$\lim_{n \rightarrow \infty} \left(\frac{3\sqrt{n^2 - 4}}{\sqrt{n^2}} \right) = 3 \lim_{n \rightarrow \infty} \sqrt{\frac{n^2 - 4}{n^2}} = 3 \sqrt{\lim_{n \rightarrow \infty} \frac{n^2 - 4}{n^2}} = 3 \times 1 = 3 \therefore \sum_{n=0}^{\infty} \frac{3}{\sqrt{n^2 - 4}}$ diverges since $\frac{1}{n}$ diverges and the limit is finite and positive.

- Ratio Test

$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$
 if < 1 a_n converges
 if > 1 a_n diverges
 if $= 1$ the test is inconclusive

eg:

Find the range of x where $\sum_{n=3}^{\infty} \frac{(-1)^n n! (x-4)^n}{3^n}$ converges.

$$\lim_{n \rightarrow \infty} \left| \frac{(n+1)! (x-4)^{n+1}}{3^{n+1}} \times \frac{3^n}{n! (x-4)^n} \right| < 1$$

$$\lim_{n \rightarrow \infty} \left| \frac{(x-4)(n+1)}{3} \right| < 1$$

$$\left| \frac{x-4}{3} \right| \lim_{n \rightarrow \infty} |n+1| < 1 \therefore \text{no range of } x$$

makes $\sum_{n=0}^{\infty} \frac{(-1)^n n! (x-4)^n}{3^n}$ converge.

- Condition Convergence

if $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} |a_n|$ diverges

Taylor Series

- $f(x) = \sum_{n=0}^{\infty} \frac{f^n(c)}{n!} (x-c)^n$

if $c = 0$ then the series is a MacLaurin series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$$

- $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

- $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$$e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$$

- $\ln(x+1) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n}$

- Example of finding the Taylor series centered at 0 (aka MacLaurin) of e^x

$$f(x) = e^x$$

$$f(0) = 1$$

$$f'(x) = e^x \text{ and } f'(0) = 1$$

$$f''(x) = e^x \text{ and } f''(0) = 1$$

$$f'''(x) = e^x \text{ and } f'''(0) = 1$$

$$\frac{1}{0!}x^0 + \frac{1}{1!}x^1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \dots + \frac{1}{n!}x^n$$

$$\therefore f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Parametric, Polar and Vectors

Parametric

- $\dot{x} = \frac{dx}{dt}$
- $\dot{y} = \frac{dy}{dt}$
- $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\dot{y}}{\dot{x}}$
- $\frac{d^2y}{(dx)^2} = \frac{\frac{d}{dt} \frac{dy}{dx}}{\frac{dx}{dt}}$
- $Arc\ length = \int \sqrt{\dot{x}^2 + \dot{y}^2} dt$
- $Area/integration = \int y(t) \frac{dx}{dt} dt = \int y \dot{x} dt$
- $x = r \cos(\theta)$ and $y = r \sin(\theta)$

Polar

- $r = \sqrt{x^2 + y^2}$
- $r^2 = x^2 + y^2$
- If going from parametric to polar, you have to convert t to θ : $\tan(\theta) = \frac{y}{x}$
- $Arc\ length = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$
- $Area/integration = \frac{1}{2} \int r^2 d\theta$

Vectors

- You can't divide a vector by another vector, only by a scalar...
- Vectors are really just parametric equations in disguise, they just have an x and a y component represented by:

$\vec{i} = x$ the x component of a vector

$\vec{j} = y$ the y component of a vector

- All vectors have Magnitude and Direction.

$$\vec{a} = x\vec{i} + y\vec{j}$$

$$\vec{a} = \langle x, y \rangle$$

The magnitude (a unit vector) is the same as the $|\vec{a}| = \sqrt{x^2 + y^2}$

The direction (another unit vector) is defined by $\frac{\vec{a}}{|\vec{a}|}$

(An unit vector is simply a vector who has a magnitude of 1)

- Addition & Subtraction

$$(a_x + b_x)\vec{i} + (a_y + b_y)\vec{j} = \vec{c}$$

Simply add the x components together, and the y components together.

- Multiplication

Scalar product

$$6 \times \vec{a} = (6 \times a_x)\vec{i} + (6 \times a_y)\vec{j}$$

Vector multiplied by a scalar. This is also how you divide a vector by a scalar.

Simply multiply the scalar out to both the x and y components.

Dot product

$$\vec{a} \cdot \vec{b} = (a_x \times b_x) + (a_y \times b_y)$$

Two vectors multiplied together

Add the product of the x components to the product of the y components to form a scalar.

Cross Product

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin \theta \vec{n}$$

where \vec{n} is a unit vector perpendicular to the plane containing \vec{a} and \vec{b} (think the right hand rule)

Two vectors multiplied together

Results in another vector.

- Helpful vectors and other things

Angle between two vectors

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

Projection vectors

$$proj_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \times \vec{a}$$

Normal vectors

$$norm_{\vec{a}} \vec{b} = \vec{b} - proj_{\vec{a}} \vec{b}$$

Integration Stuff

Integration by Parts

-

Trig Substitution

-

Trig Reduction

-

Partial Fractions with linear factors

-

Improper Integrals

-

Hyperbolic Trig

Parts and Pieces

-

TODO:**Stuff that I may add as I get time...**

- Conic sections and General Quadratic
- Eulers Method
- Homogenous Differentials
- First order linear equations
- Orthogonal Tra.
- The Shell Method
Volume of a torus
- Arc Length - functional
- Surface area of a curve
- Centroids
- Theorem of Pappus
- Indeterminate Forms (L'H)
- Lagrange Error Bound for infinite series