Problem 1. $f(x, y, z) = (x - C_x)^2 - y^2 + z^2 - r^2 = 0$

$= x^2 - 2xC_x + C_x^2 - y^2 + z^2 - r^2$

$= (1)x^2 + (-1)y^2 + (1)z^2 - 2x \cdot C_x + (C_x^2 - r^2)$

a) $Q = \begin{bmatrix} A & B & C & D \\ B & E & F & G \\ C & F & H & I \\ D & G & I & J \end{bmatrix}$, $f(x,y,z) = Ax^2 + 2Bxy + 2Cxz + 2Dx + Ey^2 + 2Fyz + 2Gy + Hz^2 + 2Iz + J$

$A = 1$ $\quad H = 1$

$D = -C_x$ $\quad J = (C_x^2 - r^2)$

$E = (-1)$

$$Q = \begin{bmatrix} 1 & 0 & 0 & -C_x \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -C_x & 0 & 0 & C_x^2 - r^2 \end{bmatrix}$$

b) $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \begin{bmatrix} 2(x - C_x) \\ -2y \\ 2z \end{bmatrix}$

$\|\nabla f\| = \sqrt{[2(x - C_x)]^2 + (-2y)^2 + (2z)^2}$

$= \sqrt{4(x - C_x)^2 + 4y^2 + 4z^2}$

$= 2\sqrt{(x - C_x)^2 + y^2 + z^2}$

$$\hat{n} = \frac{\nabla f}{\|\nabla f\|} = \begin{bmatrix} x - C_x \\ -y \\ z \end{bmatrix} \Big/ \sqrt{(x - C_x)^2 + y^2 + z^2}$$

c) $(x - C_x)^2 - y^2 + z^2 - r^2 = 0 \rightarrow (x - C_x)^2 + z^2 = y^2 + r^2$

let $y = u$, $R(u) = \sqrt{u^2 + r^2}$, $\quad x - C_x = R(u)\cos\theta \rightarrow x = C_x + \sqrt{u^2 + r^2}\cos\theta$

$z = \sqrt{u^2 + r^2}\sin\theta$

$$P(\theta, u) = \begin{bmatrix} C_x + \sqrt{u^2 + r^2}\cos\theta \\ u \\ \sqrt{u^2 + r^2}\sin\theta \end{bmatrix}$$

Problem 2. $y = \frac{1}{x}$, $x \geq 1$

a) let $x = u$, $R(u) = \frac{1}{u}$

$$y = R(u)\cos\theta = \frac{\cos\theta}{u}$$

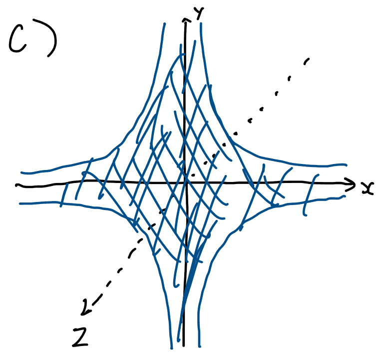
$$z = R(u)\sin\theta = \frac{\sin\theta}{u}$$

$$P(\theta, u) = \begin{bmatrix} u \\ \frac{\cos\theta}{u} \\ \frac{\sin\theta}{u} \end{bmatrix}$$

b) $N(\theta, u) = \frac{\partial P}{\partial \theta} \times \frac{\partial P}{\partial u}$, $n(\theta, u) = \frac{\partial P}{\partial \theta} \times \frac{\partial P}{\partial u} = \begin{bmatrix} 0 \\ -\frac{\sin\theta}{u} \\ \frac{\cos\theta}{u} \end{bmatrix} \times \begin{bmatrix} 1 \\ -\frac{\cos\theta}{u^2} \\ -\frac{\sin\theta}{u^2} \end{bmatrix}$

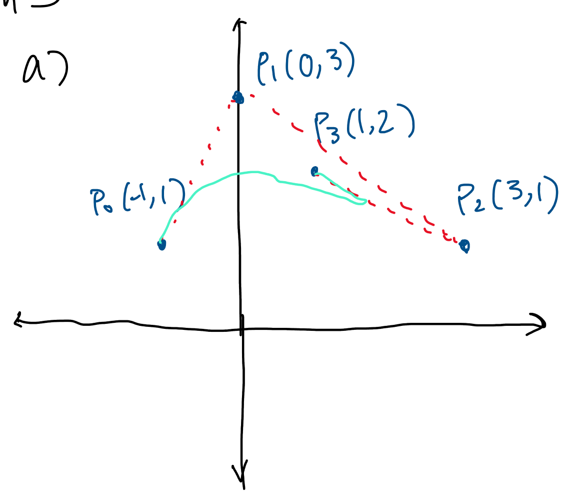
$$\frac{\partial P}{\partial \theta} \times \frac{\partial P}{\partial u} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -\frac{\sin\theta}{u} & \frac{\cos\theta}{u} \\ 1 & -\frac{\cos\theta}{u^2} & -\frac{\sin\theta}{u^2} \end{bmatrix} = \underline{\begin{bmatrix} \frac{1}{u^3} \\ \frac{\cos\theta}{u} \\ \frac{\sin\theta}{u} \end{bmatrix}} //$$

c)



Problem 3

a)



b) $P'(1) = 3(P_3 - P_2)$

$= 3[(1,2) - (3,1)] = 3(-2,1)$

$= (-6,3)$

$P'(1)$ is the vector of tangent at $u=1$

c) Continuity: $P'(1) = P'_{New}(0)$; $P'_{New}(0) = 3(P_1 - P_0) = 3[(3,-1) - (1,2)]$

$= 3(2,-3) = (6,-9)$

$\rightarrow P'(1) \neq P'_{New}(0)$, no C_1 continuity

d) Change P_1 from $(3,-1)$ to $(2,-1)$ for the second curve

Problem 4

a)
$$\begin{bmatrix} P(0) \\ P(1) \\ P'(0) \\ P'(1) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -a(1+b) & b & a(1-b) & 0 \\ 0 & -a(1+b) & b & a(1-b) \end{bmatrix}}_{M_{boundary}} \begin{bmatrix} P_{k-1} \\ P_k \\ P_{k+1} \\ P_{k+2} \end{bmatrix}$$

$a = \frac{1}{2}$

$a(1+b) - a(1-b) = a(2b) = b$

$a(1+b) - a(1-b) = b$

$$M_C = M_H \cdot M_{boundary}$$

$$M_H = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$M_C =$
$$\begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -a(1+b) & b & a(1-b) & 0 \\ 0 & -a(1+b) & b & a(1-b) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}(1+b) & \frac{1}{2}(3+b) & -\frac{1}{2}(3-b) & \frac{1}{2}(1-b) \\ (1+b) & -\frac{1}{2}(5+3b) & 2 & -\frac{1}{2}(1-b) \\ -\frac{1}{2}(1+b) & b & \frac{1}{2}(1-b) & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

b)
$$P(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \cdot M_C \cdot \begin{bmatrix} P_{k-1} \\ P_k \\ P_{k+1} \\ P_{k+2} \end{bmatrix} = P_{k-1}\left[-\frac{(1+b)}{2}u^3 + (1+b)u^2 - \frac{(1+b)}{2}u\right] +$$
$$P_k\left[\frac{(3+b)}{2}u^3 - \frac{(3b+5)}{2}u^2 + bu + 1\right] +$$
$$P_{k+1}\left[-\frac{(3-b)}{2}u^3 + 2u^2 + \frac{(1-b)}{2}u\right] +$$
$$P_{k+2}\left[\frac{(1-b)}{2}u^3 - \frac{(1-b)}{2}u^2\right]$$

c)
$$P'(1) = \frac{1}{2}\left[(1+b)(P_{k+1} - P_k) + (1-b)(P_{k+2} - P_{k+1})\right]$$
$$P'(0) = \frac{1}{2}\left[(1+b)(P_k - P_{k-1}) + (1-b)(P_{k+1} - P_k)\right]$$

$\to$ End of k $\to$ Start of $k+1$ $\to$ $P_k'(0) = \frac{1}{2}\left[(1+b)(P_{k+1} - P_k) + (1-b)(P_{k+2} - P_{k+1})\right]$

C_1 continuity is satisfied

d) let $(P_{k+1} - P_k) = V_{out}$, $(P_k - P_{k-1}) = V_{in}$

$P'(0) = \frac{1}{2}(1+b)V_{in} + \frac{1}{2}(1-b)V_{out}$;

if $b = 1$, tangent $= V_{in}$

$b = -1$, tangent $= V_{out}$

$b = 0$, tangent $=$ average of the two