EC516 HW03 Solutions

Problem 3.1

a) Downsample by a factor of 5/2

b) Sampling rate increase by 2, filter with cutoff pi/5 & gain 2, then sampling rate decrease by 5

Problem 3.2 PART A:

(a)

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} x[n-n_0]z^{-n} &= \sum_{n+n_0=-\infty}^{\infty} x[n]z^{-(n+n_0)} \\
&= \sum_{n=-\infty}^{\infty} x[n]z^{-(n+n_0)} \\
&= z^{-n_0} \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\
&= z^{-n_0} X(z)
\end{aligned}$$

(b)

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} x[-n]z^{-n} &= \sum_{-n=-\infty}^{\infty} x[n]z^{n} \\
&= \sum_{n=-\infty}^{\infty} x[n]z^{n} \\
&= \sum_{n=-\infty}^{\infty} x[n](z^{-1})^{-n} \\
&= X(z^{-1})
\end{aligned}$$

(c)

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} x^*[n]z^{-n} &= \sum_{n=-\infty}^{\infty} \left(x[n](z^{-n})^*\right)^* \\
&= \left(\sum_{n=-\infty}^{\infty} x[n](z^{-n})^*\right)^* \\
&= \left(\sum_{n=-\infty}^{\infty} x[n](z^*)^{-n}\right)^* \\
&= (X(z^*))^*
\end{aligned}$$

(d)

$$\begin{aligned}
\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x[n-m]h[m]z^{-n} &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} x[n-m]h[m]z^{-(n-m)}z^{-m} \\
&= \sum_{m=-\infty}^{\infty}\sum_{\ell=-\infty-m}^{\infty-m} x[\ell]h[m]z^{-\ell}z^{-m} \\
&= \sum_{m=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty} x[\ell]h[m]z^{-\ell}z^{-m} \\
&= \left(\sum_{\ell=-\infty}^{\infty} x[\ell]z^{-\ell}\right)\cdot\left(\sum_{m=-\infty}^{\infty} h[m]z^{-m}\right) \\
&= X(z)H(z)
\end{aligned}$$

Problem 3.2 PART B

(a)

$$\sum_{n=-\infty}^{\infty} \delta[n-3]z^{-n} = z^{-3} \qquad \text{for } |z| > 0$$

(b)

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \left(u[n]-u[n-5]\right)z^{-n} &= \sum_{n=0}^{4} z^{-n} \\
&= \begin{cases} \frac{1-z^{-5}}{1-z^{-1}} & \text{for } z \neq 1 \\ 5 & \text{for } z = 1 \end{cases}
\end{aligned}$$

(c)

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} (0.25)^n u[n]z^{-n} &= \sum_{n=0}^{\infty}(0.25z^{-1})^n \\
&= \frac{1}{1-0.25z^{-1}} \qquad \text{for } |z| > 0.25
\end{aligned}$$

(d)

$$\sum_{n=-\infty}^{\infty} (0.25)^{n-1}u[n-1]z^{-n} = \frac{z^{-1}}{1-0.25z^{-1}} \qquad \text{for } |z| > 0.25$$

(e)

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} (0.25)^n u[n-1]z^{-n} &= 0.25\cdot\sum_{n=-\infty}^{\infty} (0.25)^{n-1}u[n-1]z^{-n} \\
&= \frac{0.25z^{-1}}{1-0.25z^{-1}} \qquad \text{for } |z| > 0.25
\end{aligned}$$

(f)

$$\sum_{n=-\infty}^{\infty} \left((0.25)^n u[n] + (0.5)^n u[n]\right) z^{-n} = \frac{1}{1-0.25z^{-1}} + \frac{1}{1-0.5z^{-1}} \qquad \text{for } |z| > 0.5$$

(g)

$$\begin{aligned}\sum_{n=-\infty}^{\infty} (0.25)^n \cos(0.25\pi n) u[n] z^{-n} &= \sum_{n=0}^{\infty} (0.25)^n \frac{1}{2}\left(e^{j0.25\pi n} + e^{-j0.25\pi n}\right) z^{-n} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}(0.25)^n e^{j0.25\pi n} z^{-n} + \frac{1}{2}\sum_{n=0}^{\infty}(0.25)^n e^{-j0.25\pi n} z^{-n} \\ &= \frac{1}{2\cdot(1-0.25e^{j0.25\pi}z^{-1})} + \frac{1}{2\cdot(1-0.25e^{-j0.25\pi}z^{-1})} \\ &\qquad \text{for } |z| > 0.25\end{aligned}$$

Problem 3.4 PART A

(a)

$$(u[n]-u[n-5]) * 0.5\delta[n-3] = 0.5(u[n-3]-u[n-8])$$

(b)

$$n(u[n-1]-u[n-5]) * 2\delta[n+3] = 2(n+3)(u[n+2]-u[n-2])$$

(c)

$$\begin{aligned}(u[n]-u[n-5]) * (u[n]-u[n-5]) &= \sum_{m=-\infty}^{\infty} (u[m]-u[m-5])(u[n-m]-u[n-m-5]) \\ &= \sum_{m=0}^{4} (u[n-m]-u[n-m-5]) \\ &= \begin{cases} 1 & n=0 \\ 2 & n=1 \\ \vdots & \\ 5 & n=4 \\ \vdots & \\ 2 & n=7 \\ 1 & n=8 \\ 0 & \text{else} \end{cases}\end{aligned}$$

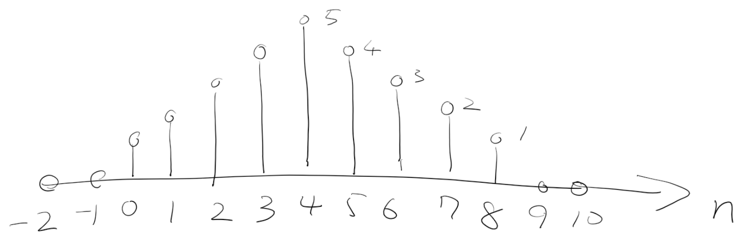


Figure 1: Problem 3.3 (c)

(d)

$$
\begin{aligned}
(u[n]-u[n-5])*(u[n]-u[n-3]) &= \sum_{m=-\infty}^{\infty}(u[m]-u[m-5])(u[n-m]-u[n-m-3]) \\
&= \sum_{m=0}^{4}(u[n-m]-u[n-m-3]) \\
&= \begin{cases} 1 & n=0 \\ 2 & n=1 \\ 3 & n=2 \\ 3 & n=3 \\ 3 & n=4 \\ 2 & n=5 \\ 1 & n=6 \\ 0 & \text{else} \end{cases}
\end{aligned}
$$

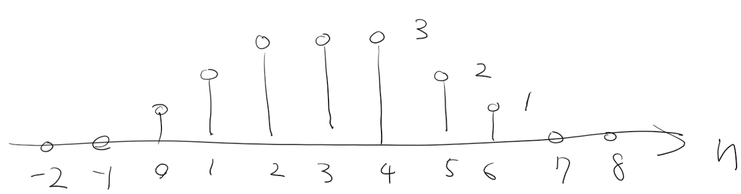


Figure 2: Problem 3.4 A(d)

(e)

$$
\begin{aligned}
(u[n]-u[n-5]) * u[n] &= \sum_{m=-\infty}^{\infty} (u[m]-u[m-5])u[n-m] \\
&= \sum_{m=0}^{4} u[n-m] \\
&= \begin{cases} 0 & n<0 \\ 1 & n=0 \\ 2 & n=1 \\ 3 & n=2 \\ 4 & n=3 \\ 5 & n=4 \\ 5 & n=5 \\ 5 & n=6 \\ \vdots & \end{cases}
\end{aligned}
$$

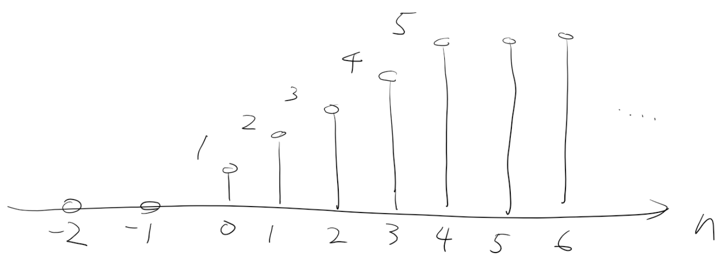


Figure 3: Problem 3.4A (e)

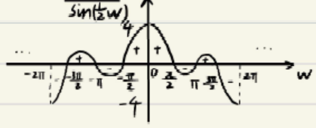
Problem 3.4 Part B

$$X(e^{jw}) = \sum_{n=0}^{3} e^{-jwn}$$

$$= \frac{1-e^{-j4w}}{1-e^{-jw}}$$

$$= e^{-j\frac{3}{2}w} \frac{\sin(2w)}{\sin(\frac{1}{2}w)}$$

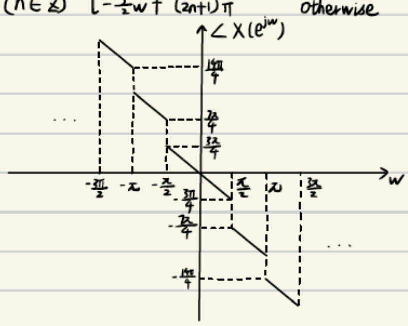
$$\angle X(e^{jw}) = \angle(e^{-j\frac{3}{2}w}) + \angle\left(\frac{\sin(2w)}{\sin(\frac{1}{2}w)}\right)$$



zero crossings: $2w = k\pi \longrightarrow w = \frac{k\pi}{2}$ $(k \in \mathbb{Z}$ and $\frac{k}{4} \notin \mathbb{Z})$

$$\angle X(e^{jw}) = \begin{cases} -\frac{3}{2}w + 2n\pi & 4k\pi - \frac{\pi}{2} \le w \le 4k\pi + \frac{\pi}{2},\ 4k\pi + \pi \le w \le 4k\pi + \frac{3}{2}\pi,\ 4k\pi - \frac{3}{2}\pi \le w \le 4k\pi - \pi\ (k \in \mathbb{Z}) \\ -\frac{3}{2}w + (2n+1)\pi & \text{otherwise} \end{cases}$$

$(n \in \mathbb{Z})$

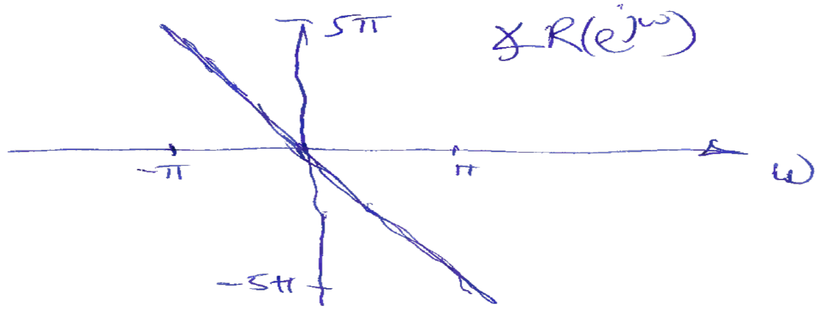


Problem 3.4 PART C

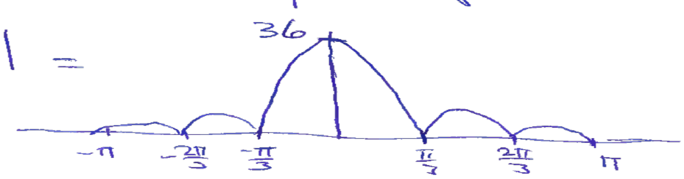
$$r[n] = x[n] * x[n]$$
$$= (6\text{ pt. box}) * (6\text{ pt. box})$$
$$= \text{triangle}$$
$$R(e^{j\omega}) = X(e^{j\omega}) \times X(e^{j\omega})$$
$$= \left(X(e^{j\omega})\right)^2$$

$$\therefore R(e^{j\omega}) = \left(\frac{\sin(3\omega)}{\sin \omega/2} \cdot e^{-j\frac{5}{2}\omega}\right)^2$$
$$= \underbrace{\left(\frac{\sin(3\omega)}{\sin \omega/2}\right)^2}_{\text{never negative}} \cdot e^{-j5\omega}$$

$$\therefore \measuredangle R(e^{j\omega}) = \measuredangle \left(\frac{\sin(3\omega)}{\sin(\omega/2)}\right)^2 - 5\omega$$
$$= -5\omega$$



$|R(e^{j\omega})| =$



Problem 3.4 PART D

(a) $$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

$$\therefore\ x[n-n_0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega(n-n_0)}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{e^{-j\omega n_0}X(e^{j\omega})}_{\text{DTFT of } x[n-n_0]}\, e^{j\omega n}\,d\omega$$

(b) $x[n]$ real $\Leftrightarrow$ $x[n] = x^*[n]$

$x[n]$ even $\Leftrightarrow$ $x[n] = x[-n]$

$x[n] \Leftrightarrow X(e^{j\omega})$, $x[-n] \Leftrightarrow X(e^{-j\omega})$,

$x^*[n] \Leftrightarrow X^*(e^{-j\omega})$

$x[n]$ real $\Leftrightarrow$ $X(e^{j\omega}) = X^*(e^{-j\omega})$ — (A)

$x[n]$ even $\Leftrightarrow$ $X(e^{j\omega}) = X(e^{-j\omega})$ — (B)

Substituting (B) in (A)

$X(e^{j\omega}) = X^*(e^{j\omega}) \Rightarrow X(e^{j\omega})$ real

(B) $\Rightarrow$ $X(e^{j\omega})$ is even.

(c) (A) Real $x[n]$ $\Leftrightarrow$ $X(e^{j\omega}) = X^*(e^{-j\omega})$

(B) Odd $x[n]$ $\Leftrightarrow$ $X(e^{j\omega}) = -X(e^{-j\omega})$

(B) $\Rightarrow$ $X(e^{j\omega})$ is odd

Substituting (B) in (A), we obtain

$$X(e^{j\omega}) = -X^*(e^{j\omega})$$

$\Rightarrow X(e^{j\omega})$ is purely imaginary.

Problem 3.3

$y[n] = 0.25\, y[n-2] + x[n]$

$h[0] = 0.25\, \overset{=0}{h[-2]} + \delta[0] = 1$

$h[1] = 0.25\, \overset{0}{h[-1]} + \underset{0}{\delta[1]} = 0$

$h[2] = 0.25\, h[0] + \delta[2] = 0.25$

$h[3] = 0.25 \cdot h[1] + \delta[3] = 0$

We can imply that

$$h[n] = \begin{cases} (0.25)^{\frac{n}{2}} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

$h[1001] = 0$