1.1)

$$x(t) \longrightarrow \otimes\, g[n] \longrightarrow x[n]$$

a) Nyquist Frequency : $10kHz \times 2 = 20kHz$

Downsampling Ratio = $\dfrac{50kHz}{20kHz} = \underline{2.5}$

b) (i) Upsample by a factor of 2, upping the sampling frequency to 100kHz

(ii) Pass through the ideal low pass filter to remove spectral images, cutoff frequency would be 10k Hz.

In normalized digital it would be $\dfrac{10kHz}{100kHz} \times 2\pi = 0.2\pi$

(iii) Downsample by a factor of 5 with a final sampling frequency of 20kHz

1.2) A] $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

a) $X(z)$ of $x[n-n_0] \Rightarrow \mathcal{Z}\{x[n-n_0]\} = \sum_{n=-\infty}^{\infty} x[n-n_0] z^{-n}$, let $m = n-n_0$

$$\mathcal{Z}\{x[n-n_0]\} = \sum_{n=-\infty}^{\infty} x[m] z^{-(m+n_0)} = z^{-n_0} \underbrace{\sum_{m=-\infty}^{\infty} x[m] z^{-m}}_{X(z)}$$

$$\Rightarrow \mathcal{Z}(x[n-n_0]) = \underline{z^{-n_0} X(z)}$$

b) let $-n = m$

$$\mathcal{Z}\{x[-n]\} = \sum_{n=-\infty}^{\infty} x[-n] z^{-n} \Rightarrow \mathcal{Z}\{x[m]\} = \sum_{m=-\infty}^{\infty} x[m] z^{m} = \underbrace{\sum_{m=-\infty}^{\infty} x[m] (z^{-1})^{-m}}_{X(z^{-1})}$$

$$\Rightarrow \mathcal{Z}\{x[-n]\} = X(z^{-1})$$

c) $X^*(z^*) = \left(\sum_{n=-\infty}^{\infty} x[n] (z^*)^{-n} \right)^*$

$= \sum_{n=-\infty}^{\infty} \left(x[n] (z^*)^{-n} \right)^* = \sum_{n=-\infty}^{\infty} x^*[n] ((z^*)^{-n})^* = \sum_{n=-\infty}^{\infty} x^*[n] z^{-n} \rightarrow \mathcal{Z}\{x^*[n]\} = X^*(z^*)$

d) $Y(z) = \mathcal{Z}\{x[n] * h[n]\} = \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x[k] h[n-k] \right) z^{-n}$

$= \sum_{k=-\infty}^{\infty} x[k] \left(\sum_{n=-\infty}^{\infty} h[n-k] z^{-n} \right) = \sum_{k=-\infty}^{\infty} x[k] \left(z^{-k} H(z) \right)$

$= H(z) \sum_{k=-\infty}^{\infty} x[k] z^{-k}$

$Y(z) = X(z) H(z)$

B] $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$

a) $X(z) = \sum_{n=-\infty}^{\infty} \delta[n-3] z^{-n}$, $x[n] \neq 0$ only when $n=3$

$\Rightarrow X(z) = z^{-3}$, ROC : $|z| > 0$

b) $x[n] = 1$ for $0 \leq n \leq 4$, 0 otherwise

$X(z) = \sum_{n=0}^{4} (1) z^{-n}$, $\sum_{n} = 0^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$, $\alpha = z^{-1}$, $N=5$

$X(z) = \frac{1-(z^{-1})^5}{1-z^{-1}} = \frac{1-z^{-5}}{1-z^{-1}} \cdot \frac{z^5}{z^5} = \frac{z^5-1}{z^5-z^4} = \frac{z^5-1}{z^4(z-1)}$

ROC: $\forall z, z \neq 0$

c) $X(z) = \sum_{n=0}^{\infty} (0.25)^n z^{-n} = \sum_{n=0}^{\infty} (0.25 z^{-1})^n$, infinite sum: $\frac{1}{1-\alpha}$, $\alpha = 0.25 z^{-1}$

$X(z) = \frac{1}{1-0.25 z^{-1}}$

ROC = $|0.25 z^{-1}| < 1 \Rightarrow 0.25 |z^{-1}| < 1 \rightarrow \frac{0.25}{|z|} < 1$

$\hookrightarrow$ $|z| > 0.25$,

d) $x[n] = (0.25)^{n-1} u[n-1] = z^{-1} \frac{1}{0.25} (0.25)^n u[n-1]$

let $m = n-1$

$$X(z) = \sum_{m=0}^{\infty} (0.25)^m z^{-(m+1)} = z^{-1} \sum_{m=0}^{\infty} (0.25z^{-1})^m = z^{-1} \cdot (1-0.25z^{-1})^{-1}$$

$$= \frac{z^{-1}}{1-0.25z^{-1}}$$

ROC : $|z| > 0.25$

e) $x[n] : 0.25^n u[n] - 0.25^0 \delta[n] = (0.25)^n u[n] - \delta[n]$

$$X(z) : \mathcal{Z}\{(0.25)^n u[n]\} - \mathcal{Z}\{\delta[n]\}$$

$$= \frac{1}{1-0.25z^{-1}} - 1 = \frac{1-(1-0.25z^{-1})}{1-0.25z^{-1}} : \frac{0.25z^{-1}}{1-0.25z^{-1}}$$

ROC : $|z| > 0.25$

f) $X(z) = \mathcal{Z}\{(0.25)^n u[n]\} + \mathcal{Z}\{(0.5)^n u[n]\}$

w/ infinite sum: $X(z) = \frac{1}{1-0.25z^{-1}} + \frac{1}{1-0.5z^{-1}}$

ROC: $0.25 + 0.25$

$\Rightarrow |z| > 0.5$

g) $\cos(\theta) = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$

$$x[n] = (0.25)^n \cdot \frac{1}{2}(e^{j0.25\pi n} + e^{-j0.25\pi n}) u[n]$$

$$= \frac{1}{2}((0.25e^{j0.25\pi})^n u[n] + (0.25e^{-j0.25\pi})^n u[n])$$

$$X(z) = \frac{1}{2}\left[(1-(0.25e^{j0.25\pi})z^{-1})^{-1} + ((1-0.25e^{-j0.25\pi})z^{-1})^{-1}\right]$$

ROC : $|z| > 0.25$

2.3] $y[n] = 0.25\, y[n-2] + x[n]$; $h[n] = 0, \; n<0$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 1, \; n \geq 0$

$h[n] = 0.25\, h[n-2] + \delta[n]$

$h[0] = 0.25\, \cancel{h[-2]}^{\,0} + \delta[0]^{\,1} = 1 \qquad h[1] = 0.25\, \cancel{h[-1]}^{\,0} + \cancel{\delta[1]}^{\,0} = 0 \qquad h[2] = 0.25\, \underset{1}{h[0]} + \cancel{\delta[2]}^{\,0} = 0.25$

$h[3] = 0.25\, \cancel{h[1]}^{\,0} + \cancel{\delta[3]}^{\,0} = 0 \longrightarrow h[1001] = \underline{0}$ //

2.4] $y[n] = x[n] * h[n]$

a) $x[n] = 1$ for $0 \leq n \leq 4$

$h[n] = 0.5$ at $n = 3$

$y[n] = 0.5\, x[n-3] = 0.5\left(u[n-3] - u[n-8]\right)$

b) $x[n] = 1, 2, 3, 4$
$\quad n = 1, 2, 3, 4$ } 0 otherwise

$h[n] = 2$ at $n = -3$

$y[n] = 2x[n+3]$, $n = -2, -1, 0, 1$
$\qquad\quad 0 \qquad\qquad$ otherwise

c) $x[n] = 1$, $0 \leq n \leq 4$

$h[n] = 2$, $0 \leq n \leq 4$

$y[n] = n+1$: $0 \leq n \leq 4$

$9-(n+1) = 8-n$: $5 \leq n \leq 8$

$$y[n] = \begin{cases} 0, & n<0 \\ 1, 2, 3, 4, 5, & n = 0, 1, 2, 3, 4 \\ 4, 3, 2, 1, & n = 5, 6, 7, 8 \\ 0, & n \geq 9 \end{cases}$$

d) $x[n] = 1$, $0 \leq n \leq 4$

$h[n] = 1$, $0 \leq n \leq 2$

$y[n] = \{1, 2, 3, 3, 3, 2, 1\}$, for $n = 0, 1, 2, 3, 4, 5, 6$

e)

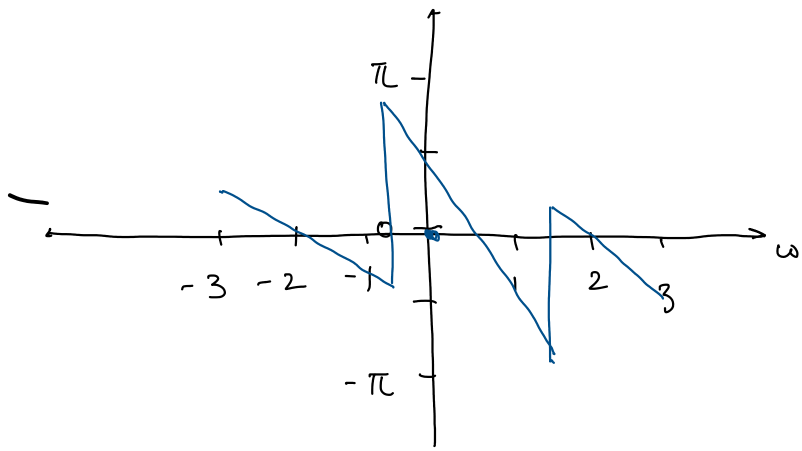
$$x[n] = u[n] - u[n-5] = 1, \quad 0 \le n \le 4$$

$$h[n] = u[n] : \text{ impulse train}$$

$$y[n] = \sum_{k=-\infty}^{n} x[k] \rightarrow y[n] = \begin{cases} 0, & n < 0 \\ n+1, & 0 \le n \le 4 \\ 5, & n \ge 5 \end{cases}$$

2.4 B] $x[n] : \begin{matrix} 1, & n = 0,1,2,3 \\ 0, & \text{otherwise} \end{matrix}$

$$X(e^{j\omega}) = \sum_{n=0}^{3} e^{-j\omega n} = \frac{1-e^{-j4\omega}}{1-e^{-j\omega}} = e^{-j\frac{3\omega}{2}} \frac{\sin(2\omega)}{\sin(\omega/2)}, \quad \angle X(e^{j\omega}) = -\frac{3\omega}{2} + \begin{cases} 0, & \frac{\sin(2\omega)}{\sin(\omega/2)} > 0 \\ \pi, -\pi, & \frac{\sin(2\omega)}{\sin(\omega/2)} < 0 \end{cases}$$



2.4 C]

$$r[n] = x[n] * x[n] = \sum_{k=-\infty}^{\infty} x[k]\,x[n-k]$$

$$R(e^{j\omega}) = \left[X(e^{j\omega})\right]^2, \quad x[n] = 1, \quad 0 \le n \le 5$$

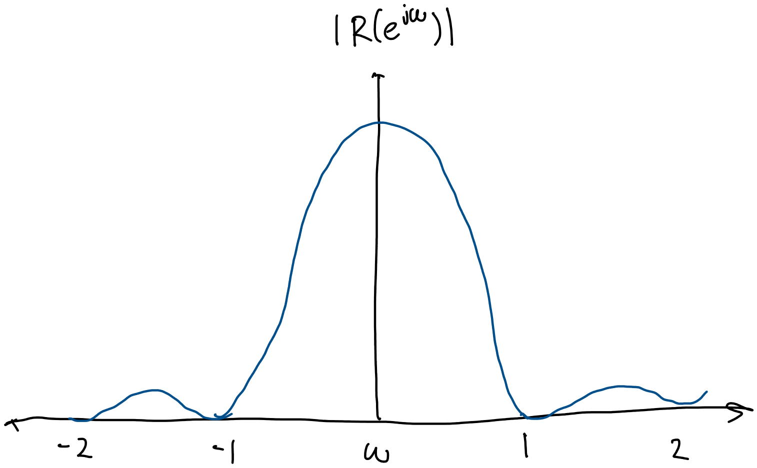
$$X(e^{j\omega}) = \sum_{n=0}^{5} e^{-j\omega n} = \frac{1-e^{-j6\omega}}{1-e^{-j\omega}} = \left(\frac{\sin(3\omega)}{\sin(\omega/2)}\right) e^{-j(5\omega/2)}$$

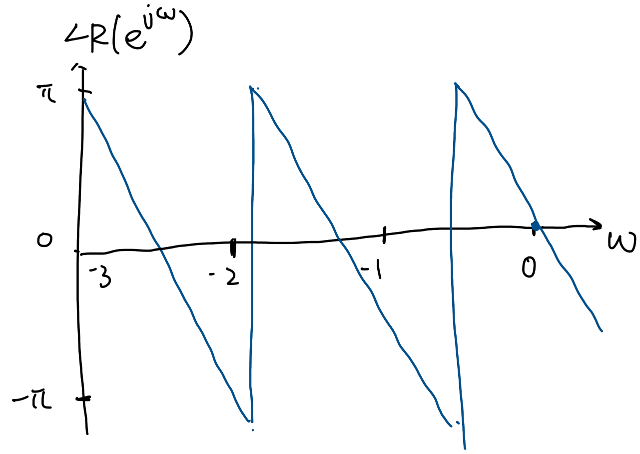
$$|R(e^{j\omega})| = |X(e^{j\omega})|^2 = \left|\frac{\sin(3\omega)}{\sin(\omega/2)}\right|^2 = \left(\frac{\sin(3\omega)}{\sin(\omega/2)}\right)^2$$

$$\angle X(e^{j\omega}) = 2 \cdot \angle X(e^{j\omega})$$

$$= 2\left(\frac{-5\omega}{2}\right) = -5\omega$$

$$= 2\left(\frac{-5w}{2}\right) = -5w$$





d)

a) $\mathcal{F}\{x[n-n_0]\} = \sum_{n=-\infty}^{\infty} \underbrace{x[n-n_0]}_{\text{let this be } m} e^{-j\omega n} = \sum_{m=-\infty}^{\infty} x[m]\, \underbrace{e^{-j\omega(m+n_0)}}_{e^{-j\omega m}\, e^{-j\omega n_0}}$

$\rightarrow \mathcal{F}\{x[n-n_0]\} = e^{-j\omega n_0} X(e^{j\omega})$

b) $x[n] \in \mathbb{R}$, even if $x[n] = x^*[n] = x[-n]$ $\quad X(e^{j\omega}) = \underbrace{X(e^{-j\omega})}_{= \sum_{n=-\infty}^{\infty} x[n] e^{j\omega n}} = X^*(e^{j\omega})$

$\Rightarrow \sum_{m=-\infty}^{\infty} x[-m]^{-j\omega m}$

let $m = -n$

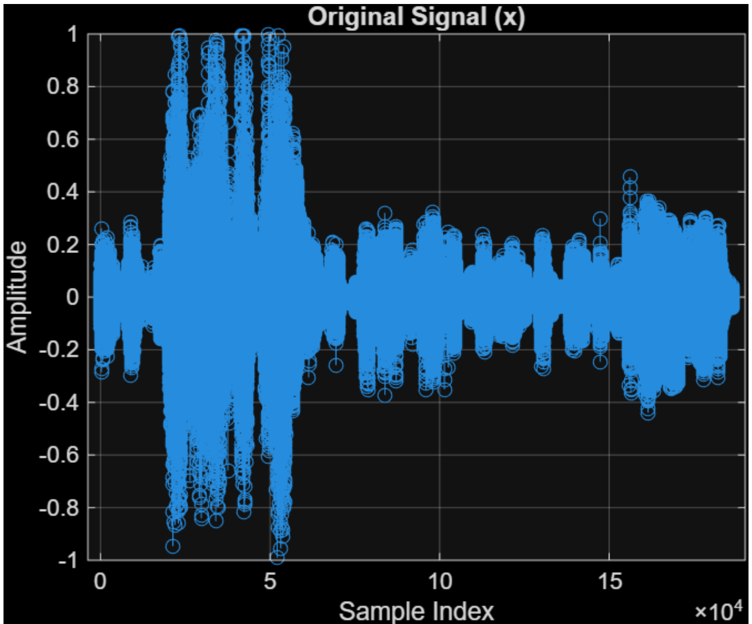
$X(e^{-j\omega}) = \sum_{m=-\infty}^{\infty} x[m] e^{-j\omega m} = X(e^{j\omega})$

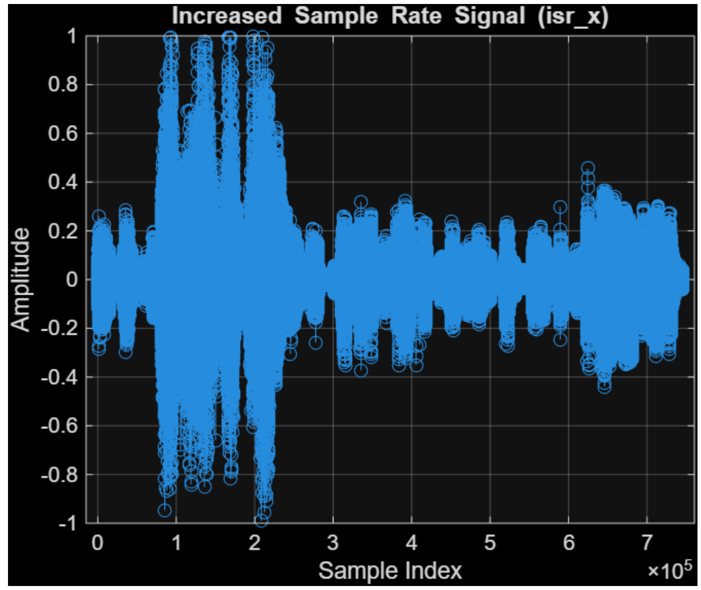
c) Same as b) but odd, $x[n] = -x[-n]$ $\quad X(e^{j\omega}) = -X(e^{-j\omega}) = -X^*(e^{j\omega})$

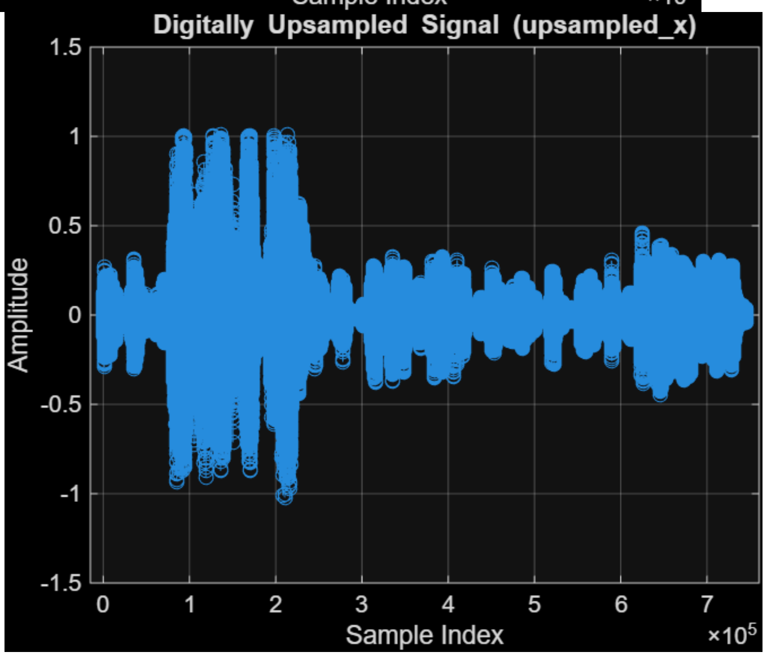
$X(e^{-j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{j\omega n} = \sum_{m=-\infty}^{\infty} x[-m] e^{-j\omega m}$

$= \sum_{m=-\infty}^{\infty} (-x[m]) e^{-j\omega m} = -\sum_{m=-\infty}^{\infty} x[m] e^{-j\omega m} = -X(e^{j\omega})$

Upsampled Plots:







If we were to observe all three of the signals without the scale of the index, all three look generally similar.

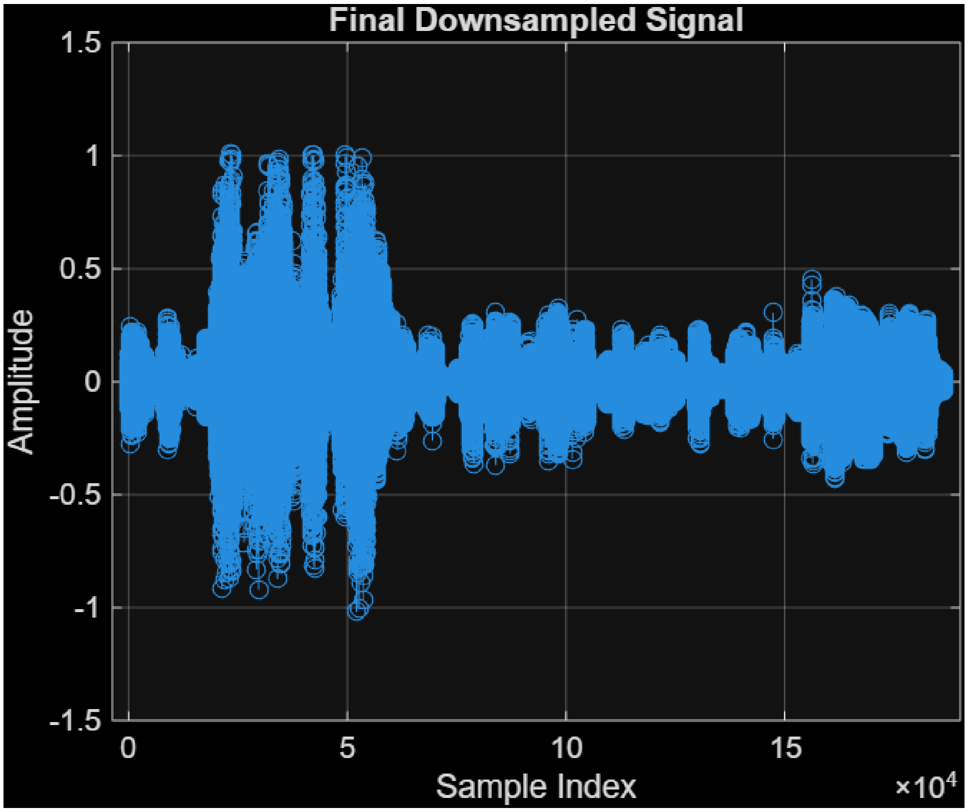
Comparing between the increased sample rate and original signal:

- Increasing the sample rate of the signal seems to expand the signal's range by another factor of 10, seemingly from the multiplied factor of U.
- Data points appear more dense in the original signal while there are interspersed points of 0 in the increased sample rate signal. One in every 4 data points is a zero point while the original signal is non-zero throughout.

Comparing the Digitally Upsampled signal to both signals:

- All of the upsampled data points are non-zero, just like the original, contrary to the increased sample rate signal.
- Since the upsampling filter acts as an anti-imaging filter it has eliminated most of the introduced sparsed zero points, meaning data points appear to be seemingly smoother than the increased sample rate signal.
- Lastly, the amplitude of the upsampled signal actually increased by a bit, approaching [-1.2, 1.2] compared to [-1, 1] of both the original and increased sample rate signal.

Downsampled Plot:



The filter introduces a delay, appearing as though it has time shifted.