7.5 Bit Error Probability

We now know how to design rules for deciding which of M signals (or symbols) has been sent, and how to estimate the performance of these decision rules. Sending one of M signals conveys $m = \log_2 M$ bits, so that a hard decision on one of these signals actually corresponds to hard decisions on m bits. In this section, we discuss how to estimate the bit error probability, or the bit error rate (BER), as it is often called.

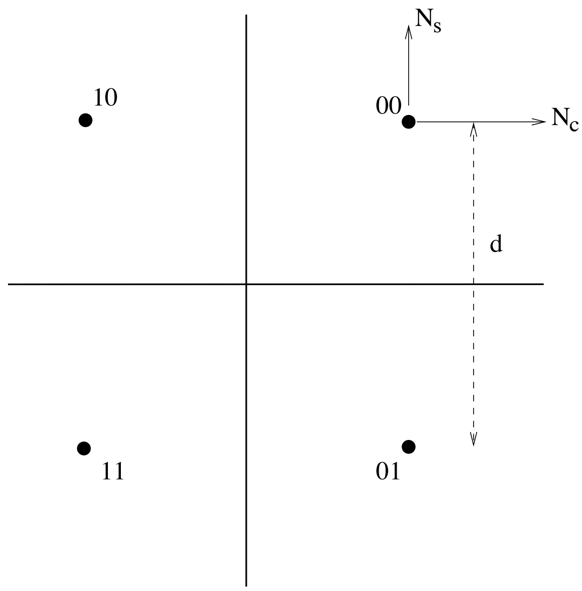

Figure 7.32: QPSK with Gray coding.

QPSK with Gray coding: We begin with the example of QPSK, with the bit mapping shown in Figure 7.32. This bit mapping is an example of a Gray code, in which the bits corresponding to neighboring symbols differ by exactly one bit (since symbol errors are most likely going to occur by decoding into neighboring decision regions, this reduces the number of bit errors). Let us denote the symbol labels as $b[1]b[2]$ for the transmitted symbol, where $b[1]$ and $b[2]$ each take values 0 and 1. Letting $\hat{b}[1]\hat{b}[2]$ denote the label for the ML symbol decision, the probabilities of bit error are given by $p_1 = P[\hat{b}[1] \neq b[1]]$ and $p_2 = P[\hat{b}[2] \neq b[2]]$. The average probability of bit error, which we wish to estimate, is given by $p_b = \frac{1}{2}(p_1 + p_2)$. Conditioned on 00 being sent, the probability of making an error on $b[1]$ is as follows:

$$P[\hat{b}[1] = 1|00 \text{ sent}] = P[\text{ML decision is 10 or 11}|00 \text{ sent}] = P[N_c < -\frac{d}{2}] = Q(\frac{d}{2\sigma}) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$$

where, as before, we have expressed the result in terms of E_b/N_0 using the power efficiency $\frac{d^2}{E_b} = 4$. We also note, by the symmetry of the constellation and the bit map, that the conditional probability of error of $b[1]$ is the same, regardless of which symbol we condition on. Moreover, exactly the same analysis holds for $b[2]$, except that errors are caused by the noise random variable N_s. We therefore obtain that

$$p_b = p_1 = p_2 = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \tag{7.90}$$

The fact that this expression is identical to the bit error probability for binary antipodal signaling is not a coincidence. QPSK with Gray coding can be thought of as two independent BPSK systems, one signaling along the I component, and the other along the Q component.

Gray coding is particularly useful at low SNR (e.g., for heavily coded systems), where symbol errors happen more often. For example, in a coded system, we would pass up fewer bit errors to the decoder for the same number of symbol errors. We define it in general as follows.

Gray Coding: Consider a 2^n-ary constellation in which each point is represented by a binary string $\mathbf{b} = (b_1, ..., b_n)$. The bit assigment is said to be *Gray coded* if, for any two constellation points $\mathbf{b}$ and $\mathbf{b}'$ which are nearest neighbors, the bit representations $\mathbf{b}$ and $\mathbf{b}'$ differ in exactly one bit location.

Nearest neighbors approximation for BER with Gray coded constellation: Consider the ith bit b_i in an n-bit Gray code for a regular constellation with minimum distance d_{min}. For a Gray code, there is at most one nearest neighbor which differs in the ith bit, and the pairwise error probability of decoding to that neighbor is $Q\left(\frac{d_{min}}{2\sigma}\right)$. We therefore have

$$P(\text{bit error}) \approx Q\left(\sqrt{\frac{\eta_P E_b}{2N_0}}\right) \quad \text{with Gray coding} \tag{7.91}$$

where $\eta_P = \frac{d_{min}^2}{E_b}$ is the power efficiency.

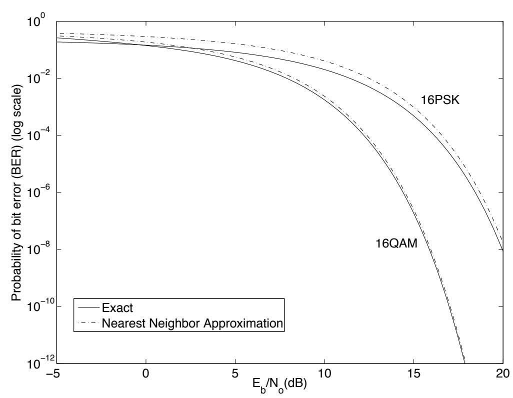

Figure 7.33: BER for 16QAM and 16PSK with Gray coding.

Figure 7.33 shows the BER of 16QAM and 16PSK with Gray coding, comparing the nearest neighbors approximation with exact results (obtained analytically for 16QAM, and by simulation for 16PSK). The slight pessimism and ease of computation of the nearest neighbors approximation implies that it is an excellent tool for link design.

Gray coding may not always be possible. Indeed, for an arbitrary set of $M = 2^n$ signals, we may not understand the geometry well enough to assign a Gray code. In general, a necessary (but not sufficient) condition for an n-bit Gray code to exist is that the number of nearest neighbors for any signal point should be at most n.

BER for orthogonal modulation: For $M = 2^m$-ary equal energy, orthogonal modulation, each of the m bits split the signal set into half. By the symmetric geometry of the signal set, any of the $M-1$ wrong symbols are equally likely to be chosen, given a symbol error, and $\frac{M}{2}$ of these will correspond to error in a given bit. We therefore have

$$P(\text{bit error}) = \frac{\frac{M}{2}}{M-1}\, P(\text{symbol error}), \quad \textbf{BER for M} - \textbf{ary orthogonal signaling} \tag{7.92}$$

Note that Gray coding is out of the question here, since there are only m bits and $2^m - 1$ neighbors, all at the same distance.

7.6 Link Budget Analysis

We have seen now that performance over the AWGN channel depends only on constellation geometry and E_b/N_0. In order to design a communication link, however, we must relate E_b/N_0 to physical parameters such as transmit power, transmit and receive antenna gains, range and the quality of the receiver circuitry. Let us first take stock of what we know:
(a) Given the bit rate R_b and the signal constellation, we know the symbol rate (or more generally, the number of modulation degrees of freedom required per unit time), and hence the minimum Nyquist bandwidth B_{min}. We can then factor in the excess bandwidth a dictated by implementation considerations to find the bandwidth $B = (1+a)B_{min}$ required.
(b) Given the constellation and a desired bit error probability, we can infer the E_b/N_0 we need to operate at. Since the SNR satisfies $SNR = \frac{E_b R_b}{N_0 B}$, we have

$$SNR_{reqd} = \left(\frac{E_b}{N_0}\right)_{reqd} \frac{R_b}{B} \tag{7.93}$$

(c) Given the receiver noise figure F (dB), we can infer the noise power $P_n = N_0 B = kT_0 10^{F/10} B$, and hence the minimum required received signal power is given by

$$P_{RX}(\min) = SNR_{reqd} P_n = \left(\frac{E_b}{N_0}\right)_{reqd} \frac{R_b}{B}\, P_n \tag{7.94}$$

This is called the required *receiver sensitivity,* and is usually quoted in dBm, as $P_{RX,\mathrm{dBm}}(\min) = 10 \log_{10} P_{RX}(\min)(\mathrm{mW})$.

Once we know the receiver sensitivity, we need to determine the link parameters such that the receiver actually gets at least that much power, plus a link margin (typically expressed in dB). Our starting point for this is the Friis formula for propagation loss in free space, which we can think of as modeling a line-of-sight wireless link:

Friis formula for free space propagation

$$P_{RX} = P_{TX}\, G_{TX}\, G_{RX}\, \frac{\lambda^2}{16\pi^2 R^2} \tag{7.95}$$

where • G_{TX} is the gain of the transmit antenna,
• G_{RX} is the gain of the receive antenna,
• $\lambda = \frac{c}{f_c}$ is the carrier wavelength ($c = 3 \times 10^8$ meters/sec, is the speed of light, f_c the carrier frequency),
• R is the range (line-of-sight distance between transmitter and receiver).
The antenna gains are with respect to an isotropic radiator. As with most measures related to

power in communication systems, antenna gains are typically expressed on the logarithmic scale, in dBi, where $G_{\text{dBi}} = 10 \log_{10} G$ for an antenna with raw gain G.

It is often convenient to take logarithms in the Friis formula (7.95), converting the multiplications into addition. For example, expressing the powers in dBm and the gains in dB, we have

$$P_{RX,\text{dBm}} = P_{TX,\text{dBm}} + G_{TX,\text{dBi}} + G_{RX,\text{dBi}} + 10 \log_{10} \frac{\lambda^2}{16\pi^2 R^2} \tag{7.96}$$

where the antenna gains are expressed in dBi (referenced to the 0 dBi gain of an isotropic antenna). More generally, we have the link budget equation

$$P_{RX,\text{dBm}} = P_{TX,\text{dBm}} + G_{TX,\text{dB}} + G_{RX,\text{dB}} - L_{pathloss,\text{dB}}(R) \tag{7.97}$$

where $L_{pathloss,\text{dB}}(R)$ is the path loss in dB. For free space propagation, we have from the Friis formula (7.96) that

$$L_{pathloss,\text{dB}}(R) = 10 \log_{10} \frac{16\pi^2 R^2}{\lambda^2} \quad \text{path loss in dB for free space propagation} \tag{7.98}$$

While the Friis formula is our starting point, the link budget equation (7.97) applies more generally, in that we can substitute other expressions for path loss, depending on the propagation environment. For example, for wireless communication in a cluttered environment, the signal power may decay as $\frac{1}{R^4}$ rather than the free space decay of $\frac{1}{R^2}$. A mixture of empirical measurements and statistical modeling is typically used to characterize path loss as a function of range for the environments of interest. For example, the design of wireless cellular systems was accompanied by extensive "measurement campaigns" and modeling. Once we decide on the path loss formula ($L_{pathloss,\text{dB}}(R)$) to be used in the design, the transmit power required to attain a given receiver sensitivity can be determined as a function of range R. Such a path loss formula typically characterizes an "average" operating environment, around which there might be significant statistical variations that are not captured by the model used to arrive at the receiver sensitivity For example, the receiver sensitivity for a wireless link may be calculated based on the AWGN channel model, whereas the link may exhibit rapid amplitude variations due to multipath fading, and slower variations due to shadowing (e.g., due to buildings and other obstacles). Even if fading/shadowing effects are factored into the channel model used to compute BER, and the model for path loss, the actual environment encountered may be worse than that assumed in the model. In general, therefore, we add a link margin $L_{margin,\text{dB}}$, again expressed in dB, in an attempt to budget for potential performance losses due to unmodeled or unforeseen impairments. The size of the link margin depends, of course, on the confidence of the system designer in the models used to arrive at the rest of the link budget.

Putting all this together, if $P_{RX,\text{dBm}}(\text{min})$ is the desired receiver sensitivity (i.e., the minimum required received power), then we compute the transmit power for the link to be

Required transmit power

$$P_{TX,\text{dBm}} = P_{RX,\text{dBm}}(\text{min}) - G_{TX,\text{dB}} - G_{RX,\text{dB}} + L_{pathloss,\text{dB}}(R) + L_{margin,\text{dB}} \tag{7.99}$$

Let us illustrate these concepts using an example.

Example 7.6.1 Consider again the 5 GHz WLAN link of Example 7.1.8. We wish to utilize a 20 MHz channel, using QPSK and an excess bandwidth of 33 %. The receiver has a noise figure

of 6 dB.
(a) What is the bit rate?
(b) What is the receiver sensitivity required to achieve a bit error rate (BER) of 10^{-6}?
(c) Assuming transmit and receive antenna gains of 2 dBi each, what is the range achieved for 100 mW transmit power, using a link margin of 20 dB? Use link budget analysis based on free space path loss.

Solution to (a): For bandwidth B and fractional excess bandwidth a, the symbol rate

$$R_s = \frac{1}{T} = \frac{B}{1+a} = \frac{20}{1+0.33} = 15 \text{ Msymbols/sec}$$

and the bit rate for an M-ary constellation is

$$R_b = R_s \log_2 M = 15 \text{ Msymbols/sec} \times 2 \text{ bits/symbol} = 30 \text{ Mbits/sec}$$

Solution to (b): BER for QPSK with Gray coding is $Q\left(\sqrt{\frac{2E_b}{N_0}}\right)$. For a desired BER of 10^{-6}, we obtain that $\left(\frac{E_b}{N_0}\right)_{reqd}$ is about 10.2 dB. From (7.93), we obtain that

$$SNR_{reqd} = 10.2 + 10\log_{10}\frac{30}{20} \approx 12 \text{ dB}$$

We know from Example 7.1.8 that the noise power is -95 dBm. Thus, the desired receiver sensitivity is

$$P_{RX,\text{dBm}}(\text{min}) = P_{n,\text{dBm}} + SNR_{reqd,\text{dB}} = -95 + 12 = -83 \text{ dBm}$$

Solution to (c): The transmit power is 100 mW, or 20 dBm. Rewriting (7.99), the allowed path loss to attain the desired sensitivity at the desired link margin is

$$\begin{aligned} L_{pathloss,\text{dB}}(R) &= P_{TX,\text{dBm}} - P_{RX,\text{dBm}}(\text{min}) + G_{TX,\text{dBi}} + G_{RX,\text{dBi}} - L_{margin,\text{dB}} \\ &= 20 - (-83) + 2 + 2 - 20 = 87 \text{ dB} \end{aligned} \tag{7.100}$$

We can now invert the formula for free space loss, (7.98), to get a range R of 107 meters, which is of the order of the advertised ranges for WLANs under nominal operating conditions. The range decreases, of course, for higher bit rates using larger constellations. What happens, for example, when we use 16QAM or 64-QAM?

Problems

Gaussian Models

Problem 7.1 Two random variables X and Y have joint density

$$p_{X,Y}(x,y) = \begin{cases} Ke^{-\frac{2x^2+y^2}{2}} & xy \geq 0 \\ 0 & xy < 0 \end{cases}$$

(a) Find K.
(b) Show that X and Y are each Gaussian random variables.

(c) Express the probability $P[X^2 + X > 2]$ in terms of the Q function.
(d) Are X and Y jointly Gaussian?
(e) Are X and Y independent?
(f) Are X and Y uncorrelated?
(g) Find the conditional density $p_{X|Y}(x|y)$. Is it Gaussian?

Problem 7.2 (computations for Gaussian random vectors) The random vector $\mathbf{X} = (X_1 X_2)^T$ is Gaussian with mean vector $\mathbf{m} = (2, 1)^T$ and covariance matrix $\mathbf{C}$ given by

$$\mathbf{C} = \begin{pmatrix} 1 & -1 \\ -1 & 4 \end{pmatrix}$$

(a) Let $Y_1 = X_1 + 2X_2$, $Y_2 = -X_1 + X_2$. Find $cov(Y_1, Y_2)$.
(b) Write down the joint density of Y_1 and Y_2.
(c) Express the probability $P[Y_1 > 2Y_2 + 1]$ in terms of the Q function.

Problem 7.3 (computations for Gaussian random vectors) The random vector $\mathbf{X} = (X_1 X_2)^T$ is Gaussian with mean vector $\mathbf{m} = (-3, 2)^T$ and covariance matrix $\mathbf{C}$ given by

$$\mathbf{C} = \begin{pmatrix} 4 & -2 \\ -2 & 9 \end{pmatrix}$$

(a) Let $Y_1 = 2X_1 - X_2$, $Y_2 = -X_1 + 3X_2$. Find $\mathrm{cov}(Y_1, Y_2)$.
(b) Write down the joint density of Y_1 and Y_2.
(c) Express the probability $P[Y_2 > 2Y_1 - 1]$ in terms of the Q function with positive arguments.
(d) Express the probability $P[Y_1^2 > 3Y_1 + 10]$ in terms of the Q function with positive arguments.

Problem 7.4 Consider the noisy received signal

$$y(t) = s(t) + n(t)$$

where $s(t) = I_{[0,3]}(t)$ and $n(t)$ is WGN with PSD $\sigma^2 = N_0/2 = 1/4$. The receiver computes the following statistics:

$$Y_1 = \int_0^2 y(t)dt \ , \quad Y_2 = \int_1^3 y(t)dt$$

(a) Specify the joint distribution of Y_1 and Y_2.
(b) Compute the probability $P[Y_1 + Y_2 < 2]$, expressing it in terms of the Q function with positive arguments.

Problem 7.5 Let $n(t)$ denote WGN with PSD $S_n(f) \equiv \sigma^2$. We pass $n(t)$ through a filter with impulse response $h(t) = I_{[0,T]}(t)$ to obtain $z(t) = (n * h)(t)$.
(a) Find and sketch the autocorrelation function of $z(t)$.
(b) Specify the joint distribution of $z(T)$ and $z(T/2)$.
(c) Specify the joint distribution of $z(T)$ and $z(0)$.
(d) Evaluate the probability $P[2z(T/2) > z(0) + z(T)]$. Assume $\sigma^2 = 1$ and $T = 1$.
(e) Evaluate the probability $P[2z(T/2) > z(0) + z(T) + 2]$. Assume $\sigma^2 = 1$ and $T = 1$.

Problem 7.6 (From Gaussian to Rayleigh, Rician, and Exponential Random Variables) Let X_1, X_2 be iid Gaussian random variables, each with mean zero and variance v^2. Define (R, Φ) as the polar representation of the point (X_1, X_2), i.e.,

$$X_1 = R\cos\Phi X_2 = R\sin\Phi$$

where $R \geq 0$ and $\Phi \in [0, 2\pi]$.
(a) Find the joint density of R and Φ.
(b) Observe from (a) that R, Φ are independent. Show that Φ is uniformly distributed in $[0, 2\pi]$, and find the marginal density of R.
(c) Find the marginal density of R^2.
(d) What is the probability that R^2 is at least 20 dB below its mean value? Does your answer depend on the value of v^2?
Remark: The random variable R is said to have a Rayleigh distribution. Further, you should recognize that R^2 has an exponential distribution. We use these results when we discuss noncoherent detection and Rayleigh fading in Chapter 3.
(e) Now, assume that $X_1 \sim N(m_1, v^2)$, $X_2 \sim N(m_2, v^2)$ are independent, where m_1 and m_2 may be nonzero. Find the joint density of R and Φ, and the marginal density of R. Express the latter in terms of the modified Bessel function

$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \exp(x\cos\theta)\, d\theta$$

Remark: The random variable R is said to have a Rician distribution in this case. This specializes to a Rayleigh distribution when $m_1 = m_2 = 0$.

Hypothesis Testing

Problem 7.7 The received signal in a digital communication system is given by

$$y(t) = \begin{cases} s(t) + n(t) & 1 \textit{ sent} \\ n(t) & 0 \textit{ sent} \end{cases}$$

where n is AWGN with PSD $\sigma^2 = N_0/2$ and $s(t)$ is as shown below. The received signal is passed through a filter, and the output is sampled to yield a decision statistic. An ML decision rule is employed based on the decision statistic. The set-up is shown in Figure 7.34. (a) For

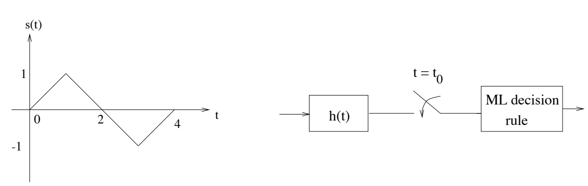

Figure 7.34: Set-up for Problem 7.7

$h(t) = s(-t)$, find the error probability as a function of E_b/N_0 if $t_0 = 1$.
(b) Can the error probability in (a) be improved by choosing the sampling time t_0 differently?
(c) Now, find the error probability as a function of E_b/N_0 for $h(t) = I_{[0,2]}$ and the best possible choice of sampling time.

(d) Finally, comment on whether you can improve the performance in (c) by using a linear combination of two samples as a decision statistic, rather than just using one sample.

Problem 7.8 Consider binary hypothesis testing based on the decision statistic Y, where $Y \sim N(2,9)$ under H_1 and $Y \sim N(-2,4)$ under H_0.
(a) Show that the optimal (ML or MPE) decision rule is equivalent to comparing a function of the form $ay^2 + by$ to a threshold.
(b) Specify the MPE rule explicitly (i.e., specify a, b and the threshold) when $\pi_0 = \frac{1}{4}$.
(c) Express the conditional error probability $P_{e|0}$ for the decision rule in (b) in terms of the Q function with positive arguments. Also provide a numerical value for this probability.

Problem 7.9 Find and sketch the decision regions for a binary hypothesis testing problem with observation Z, where the hypotheses are equally likely, and the conditional distributions are given by
H_0: Z is uniform over $[-2,2]$
H_1: Z is Gaussian with mean 0 and variance 1.

Problem 7.10 The receiver in a binary communication system employs a decision statistic Z which behaves as follows:
$Z = N$ if 0 is sent
$Z = 4 + N$ if 1 is sent
where N is modeled as Laplacian with density

$$p_N(x) = \frac{1}{2} e^{-|x|}, \quad -\infty < x < \infty$$

Note: Parts (a) and (b) can be done independently.
(a) Find and sketch, as a function of z, the log likelihood ratio

$$K(z) = \log L(z) = \log \frac{p(z|1)}{p(z|0)}$$

where $p(z|i)$ denotes the conditional density of Z given that i is sent ($i = 0, 1$).
(b) Find $P_{e|1}$, the conditional error probability given that 1 is sent, for the decision rule

$$\delta(z) = \begin{cases} 0, & z < 1 \\ 1, & z \geq 1 \end{cases}$$

(c) Is the rule in (b) the MPE rule for any choice of prior probabilities? If so, specify the prior probability $\pi_0 = P[\text{ 0 sent}]$ for which it is the MPE rule. If not, say why not.

Problem 7.11 Consider binary hypothesis testing in which the observation Y is modeled as uniformly distributed over $[-2,2]$ under H_0, and has conditional density $p(y|1) = c(1-|y|/3)I_{[-3,3]}(y)$ under H_1, where $c > 0$ is a constant to be determined.
(a) Find c.
(b) Find and sketch the decision regions Γ_0 and Γ_1 corresponding to the ML decision rule.
(c) Find the conditional error probabilities.

Problem 7.12 Consider binary hypothesis testing with scalar observation Y. Under hypothesis H_0, Y is modeled as uniformly distributed over $[-5, 5]$. Under H_1, Y has conditional density $p(y|1) = \frac{1}{8}e^{-|y|/4}, \quad -\infty < y < \infty$.
(a) Specify the ML rule and clearly draw the decision regions Γ_0 and Γ_1 on the real line.
(b) Find the conditional probabilities of error for the ML rule under each hypothesis.

Problem 7.13 For the setting of Problem 7.12, suppose that the prior probability of H_0 is 1/3.
(a) Specify the MPE rule and draw the decision regions.
(b) Find the conditional error probabilities and the average error probability. Compare with the corresponding quantities for the ML rule considered in Problem 7.12.

Problem 7.14 The receiver output Z in an on-off keyed optical communication system is modeled as a Poisson random variable with mean $m_0 = 1$ if 0 is sent, and mean $m_1 = 10$ if 1 is sent.
(a) Show that the ML rule consists of comparing Z to a threshold, and specify the numerical value of the threshold. Note that Z can only take nonnegative integer values.
(b) Compute the conditional error probabilities for the ML rule (specify numerical values in addition to deriving formulas).
(c) Find the MPE rule if the prior probability of sending 1 is 0.1.
(d) Compute the average error probability for the MPE rule.

Problem 7.15 Consider binary signaling in AWGN, with $s_1(t) = (1 - |t|)I_{[-1,1]}(t)$ and $s_0(t) = -s_1(t)$. The received signal is given by $y(t) = s_i(t) + n(t)$, $i = 0, 1,$, where the noise n has PSD $\sigma^2 = \frac{N_0}{2} = 0.1$. For all of the error probabilities computed in this problem, specify in terms of the Q function with positive arguments and also give numerical values.
(a) How would you implement the ML receiver using the received signal $y(t)$? What is its conditional error probability given that s_0 is sent?
Now, consider a suboptimal receiver, where the receiver generates the following decision statistics:

$$y_0 = \int_{-1}^{-0.5} y(t)dt, \quad y_1 = \int_{-0.5}^{0} y(t)dt, \quad y_2 = \int_{0}^{0.5} y(t)dt, \quad y_0 = \int_{0.5}^{1} y(t)dt$$

(b) Specify the conditional distribution of $\mathbf{y} = (y_0, y_1, y_2, y_3)^T$, conditioned on s_0 being sent.
(c) Specify the ML rule when the observation is $\mathbf{y}$. What is its conditional error probability given that s_0 is sent?
(d) Specify the ML rule when the observation is $y_0 + y_1 + y_2 + y_3$. What is its conditional error probability, given that s_0 is sent?
(e) Among the error probabilities in (a), (c) and (d), which is the smallest? Which is the biggest? Could you have rank ordered these error probabilities without actually computing them?

Problem 7.16 The received sample Y in a binary communication system is modeled as follows: $Y = A + N$ if 0 is sent, and $Y = -A + N$ if 1 is sent, where N is *Laplacian* noise with density

$$p_N(x) = \frac{\lambda}{2}e^{-\lambda|x|}, \quad -\infty < x < \infty$$

(a) Find the ML decision rule. Simplify as much as possible.
(b) Find the conditional error probabilities for the ML rule.
(c) Now, suppose that the prior probability of sending 0 is 1/3. Find the MPE rule, simplifying as much as possible.
(d) In the setting of (c), find the LLR $\log \frac{P[0|Y=A/2]}{P[1|Y=A/2]}$.

Receiver design and performance analysis for the AWGN channel

Problem 7.17 Let $p_1(t) = I_{[0,1]}(t)$ denote a rectangular pulse of unit duration. Consider two 4-ary signal sets as follows:
Signal Set A: $s_i(t) = p_1(t-i)$, $i = 0, 1, 2, 3$.
Signal Set B: $s_0(t) = p_1(t) + p_1(t-3)$, $s_1(t) = p_1(t-1) + p_1(t-2)$, $s_2(t) = p_1(t) + p_1(t-2)$, $s_3(t) = p_1(t-1) + p_1(t-3)$.
(a) Find signal space representations for each signal set with respect to the orthonormal basis $\{p_1(t-i), i = 0, 1, 2, 3\}$.
(b) Find union bounds on the average error probabilities for both signal sets as a function of E_b/N_0. At high SNR, what is the penalty in dB for using signal set B?
(c) Find an exact expression for the average error probability for signal set B as a function of E_b/N_0.

Problem 7.18 Three 8-ary signal constellations are shown in Figure 7.35.
(a) Express R and $d_{min}^{(2)}$ in terms of $d_{min}^{(1)}$ so that all three constellations have the same E_b.

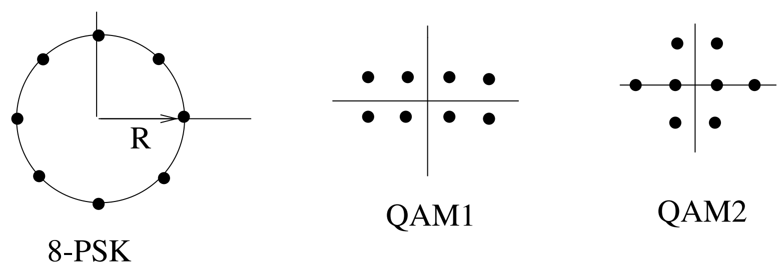

Figure 7.35: Signal constellations for Problem 7.18

(b) For a given E_b/N_0, which constellation do you expect to have the smallest bit error probability over a high SNR AWGN channel?
(c) For each constellation, determine whether you can label signal points using 3 bits so that the label for nearest neighbors differs by at most one bit. If so, find such a labeling. If not, say why not and find some "good" labeling.
(d) For the labelings found in part (c), compute nearest neighbors approximations for the average bit error probability as a function of E_b/N_0 for each constellation. Evaluate these approximations for $E_b/N_0 = 15dB$.

Problem 7.19 Consider the signal constellation shown in Figure 7.36, which consists of two QPSK constellations of different radii, offset from each other by $\frac{\pi}{4}$. The constellation is to be used to communicate over a passband AWGN channel.

(a) Carefully redraw the constellation (roughly to scale, to the extent possible) for $r = 1$ and $R = \sqrt{2}$. Sketch the ML decision regions.
(b) For $r = 1$ and $R = \sqrt{2}$, find an intelligent union bound for the conditional error probability, given that a sign al point from the inner circle is sent, as a function of E_b/N_0.
(c) How would you choose the parameters r and R so as to optimize the power efficiency of the constellation (at high SNR)?

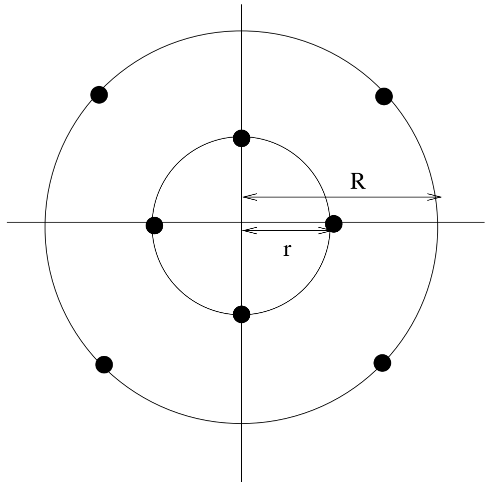

Figure 7.36: Constellation for Problem 7.19

Problem 7.20 (Exact symbol error probabilities for rectangular constellations) Assuming each symbol is equally likely, derive the following expressions for the average error probability for 4PAM and 16QAM:

$$P_e = \frac{3}{2}Q\left(\sqrt{\frac{4E_b}{5N_0}}\right), \quad \textbf{symbol error probability for 4PAM} \tag{7.101}$$

$$P_e = 3Q\left(\sqrt{\frac{4E_b}{5N_0}}\right) - \frac{9}{4}Q^2\left(\sqrt{\frac{4E_b}{5N_0}}\right), \quad \textbf{symbol error probability for 16QAM} \tag{7.102}$$

(Assume 4PAM with equally spaced levels symmetric about the origin, and rectangular 16QAM equivalent to two 4PAM constellations independently modulating the I and Q components.)

Problem 7.21 The signal constellation shown in Figure 7.37 is obtained by moving the outer corner points in rectangular 16QAM to the I and Q axes.
(a) Sketch the ML decision regions.
(b) Is the constellation more or less power-efficient than rectangular 16QAM?

Problem 7.22 The V.29 standard for 9.6 Kbits/sec modems uses a 16-ary signal constellation with 4 signals with coordinates $(\pm-1, \pm-1)$, four others with coordinates $(\pm 3, \pm 3)$, and two each having coordinates $(\pm 3, 0)$, $(\pm 5, 0)$, $(0, \pm 3)$, and $(0, \pm 5)$, respectively.
(a) Sketch the signal constellation and indicate the ML decision regions.
(b) Find an intelligent union bound on the average symbol error probability as a function of E_b/N_0.
(c) Find the nearest neighbors approximation to the average symbol error probability as a function of E_b/N_0.
(d) Find the nearest neighbors approximation to the average symbol error probability for 16QAM as a function of E_b/N_0.
(e) Comparing (c) and (d) (i.e., comparing the performance at high SNR), which signal set is more power efficient?

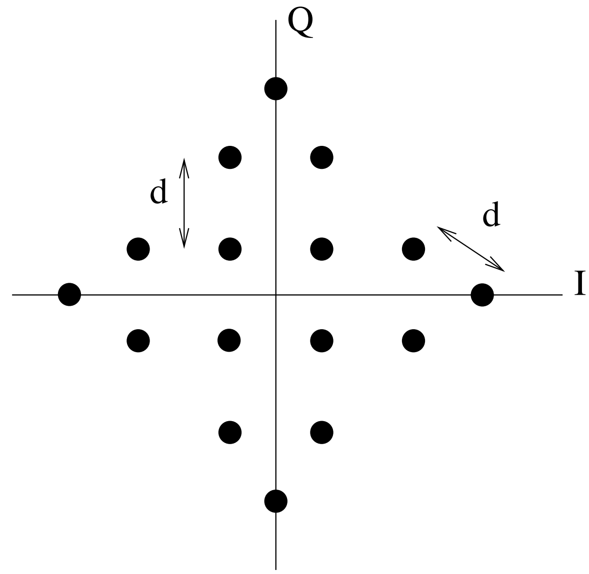

Figure 7.37: Constellation for Problem 7.21

Problem 7.23 A QPSK demodulator is designed to put out an *erasure* when the decision is ambivalent. Thus, the decision regions are modified as shown in Figure 7.38, where the cross-hatched region corresponds to an erasure. Set $\alpha = \frac{d_1}{d}$, where $0 \leq \alpha \leq 1$.

(a) Use the intelligent union bound to find approximations to the probability p of symbol error

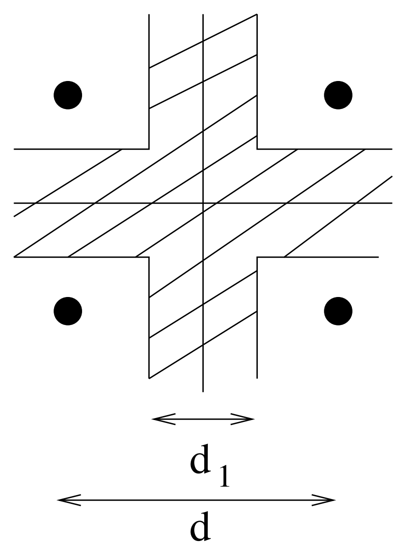

Figure 7.38: QPSK with erasures

and the probability q of symbol erasure in terms of E_b/N_0 and α.
(b) Find exact expressions for p and q as functions of E_b/N_0 and α.
(c) Using the approximations in (a), find an approximate value for α such that $q = 2p$ for $E_b/N_0 = 4dB$.
Remark: The motivation for (c) is that a typical error-correcting code can correct twice as many erasures as errors.

Problem 7.24 Consider the constant modulus constellation shown in Figure 7.39. where $\theta \leq$

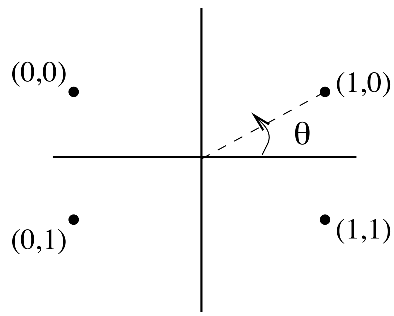

Figure 7.39: Signal constellation with unequal error protection (Problem 7.24).

$\pi/4$. Each symbol is labeled by 2 bits (b_1, b_2) as shown. Assume that the constellation is used over a complex baseband AWGN channel with noise Power Spectral Density (PSD) $N_0/2$ in each dimension. Let $(\hat{b_1}, \hat{b_2})$ denote the maximum likelihood (ML) estimates of (b_1, b_2).
(a) Find $P_{e1} = P[\hat{b_1} \neq b_1]$ and $P_{e2} = P[\hat{b_2} \neq b_2]$ as a function of E_s/N_0, where E_s denotes the signal energy.
(b) Assume now that the transmitter is being heard by two receivers, $R1$ and $R2$, and that $R2$ is twice as far away from the transmitter as $R1$. Assume that the received signal energy falls off as $1/r^4$, where r is the distance from the transmitter, and that the noise PSD for both receivers is identical. Suppose that $R1$ can demodulate both bits $b1$ and $b2$ with error probability at least as good as 10^{-3}, i.e., so that $\max\{P_{e1}(R1), P_{e2}(R1)\} = 10^{-3}$. Design the signal constellation (i.e., specify θ) so that $R2$ can demodulate at least one of the bits with the same error probability, i.e., such that $\min\{P_{e1}(R2), P_{e2}(R2)\} = 10^{-3}$.
Remark: You have designed an unequal error protection scheme in which the receiver that sees a poorer channel can still extract part of the information sent.

Problem 7.25 (Demodulation with amplitude mismatch) Consider a 4PAM system using the constellation points $\{\pm 1, \pm 3\}$. The receiver has an accurate estimate of its noise level. An automatic gain control (AGC) circuit is supposed to scale the decision statistics so that the noiseless constellation points are in $\{\pm 1, \pm 3\}$. ML decision boundaries are set according to this nominal scaling.
(a) Suppose that the AGC scaling is faulty, and the *actual* noiseless signal points are at $\{\pm 0.9, \pm 2.7\}$. Sketch the points and the mismatched decision regions. Find an intelligent union bound for the symbol error probability in terms of the Q function and E_b/N_0.
(b) Repeat (a), assuming that faulty AGC scaling puts the noiseless signal points at $\{\pm 1.1, \pm 3.3\}$.
(c) AGC circuits try to maintain a constant output power as the input power varies, and can be viewed as imposing a scale factor on the input inversely proportional to the square root of the input power. In (a), does the AGC circuit overestimate or underestimate the input power?

Problem 7.26 (Demodulation with phase mismatch) Consider a BPSK system in which the receiver's estimate of the carrier phase is off by θ.
(a) Sketch the I and Q components of the decision statistic, showing the noiseless signal points and the decision region.
(b) Derive the BER as a function of θ and E_b/N_0 (assume that $\theta < \frac{\pi}{2}$).
(c) Assuming now that θ is a random variable taking values uniformly in $[-\frac{\pi}{4}, \frac{\pi}{4}]$, numerically

compute the BER averaged over θ, and plot it as a function of E_b/N_0. Plot the BER without phase mismatch as well, and estimate the dB degradation due to the phase mismatch.

Problem 7.27 (Soft decisions for BPSK) Consider a BPSK system in which 0 and 1 are equally likely to be sent, with 0 mapped to +1 and 1 to -1 as usual. Thus, the decision statistic $Y = A + N$ if 0 is sent, and $Y = -A + N$ if 1 is sent, where $A > 0$ and $N \sim N(0, \sigma^2)$.
(a) Show that the LLR is conditionally Gaussian given the transmitted bit, and that the conditional distribution is scale-invariant, depending only on E_b/N_0.
(b) If the BER for hard decisions is 10%, specify the conditional distribution of the LLR, given that 0 is sent.

Problem 7.28 (Soft decisions for PAM) Consider soft decisions for 4PAM signaling as in Example 7.2.3. Assume that the signals have been scaled to $\pm 1, \pm 3$ (i.e., set $A = 1$ in Example 7.2.3. The system is operating at E_b/N_0 of 6 dB. Bits $b_1, b_2 \in \{0, 1\}$ are mapped to the symbols using Gray coding. Assume that $(b_1, b_2) = (0, 0)$ for symbol -3, and $(1, 0)$ for symbol +3.
(a) Sketch the constellation, along with the bit maps. Indicate the ML hard decision boundaries.
(b) Find the posterior symbol probability $P[-3|y]$ as a function of the noisy observation y. Plot it as a function of y.
Hint: The noise variance σ^2 can be inferred from the signal levels and SNR.
(c) Find $P[b_1 = 1|y]$ and $P[b_2 = 1|y]$, and plot as a function of y.
Remark: The posterior probability of $b_1 = 1$ equals the sum of the posterior probabilities of all symbols which have $b_1 = 1$ in their labels.
(d) Display the results of part (c) in terms of LLRs.

$$LLR_1(y) = \log \frac{P[b_1 = 0|y]}{P[b_1 = 1|y]}, \qquad LLR_2(y) = \log \frac{P[b_2 = 0|y]}{P[b_2 = 1|y]}$$

Plot the LLRs as a function of y, saturating the values as ± 50.
(e) Try other values of E_b/N_0 (e.g., 0 dB, 10 dB). Comment on any trends you notice. How do the LLRs vary as a function of distance from the noiseless signal points? How do they vary as you change E_b/N_0.
(f) In order to characterize the conditional distribution of the LLRs, simulate the system over multiple symbols at E_b/N_0 such that the BER is about 5%. Plot the histograms of the LLRs for each of the two bits, and comment on whether they look Gaussian. What happens as you increase or decrease E_b/N_0?

Problem 7.29 (Effect of Rayleigh fading) Constructive and destructive interference between multiple paths in wireless systems lead to large fluctuations in received amplitude, modeled as a Rayleigh random variable A (see Problem 7.6). The energy per bit is therefore proportional to A^2. Thus, using Problem 7.6(c), we can model E_b as an exponential random variable with mean $\bar{E}_b$, where $\bar{E}_b$ is the *average* energy per bit.
(a) Show that the BER for BPSK over a Rayleigh fading channel is given by

$$P_e = \frac{1}{2}\left(1 - (1 + \frac{N_0}{\bar{E}_b})^{-\frac{1}{2}}\right)$$

How does the BER decay with E_b/N_0 at high SNR?
Hint: Compute $\mathbb{E}\left[Q\left(\sqrt{\frac{2E_b}{N_0}}\right)\right]$ using the distribution of E_b/N_0. Integrate by parts to evaluate.

(b) Plot BER versus $\frac{\bar{E}_b}{N_0}$ for BPSK over the AWGN and Rayleigh fading channels (BER on log scale, $\frac{\bar{E}}{N_0}$ in dB). Note that $\bar{E}_b = E_b$ for the AWGN channel. At BER of 10^{-4}, what is the degradation in dB due to Rayleigh fading?

Link budget analysis

Problem 7.30 You are given an AWGN channel of bandwidth 3 MHz. Assume that implementation constraints dictate an excess bandwidth of 50%. Find the achievable bit rate, the E_b/N_0 required for a BER of 10^{-8}, and the receiver sensitivity (assuming a receiver noise figure of 7 dB) for the following modulation schemes, assuming that the bit-to-symbol map is optimized to minimize the BER whenever possible:
(a) QPSK, (b) 8PSK, (c) 64QAM (d) Coherent 16-ary orthogonal signaling.
Remark: Use nearest neighbors approximations for the BER.

Problem 7.31 Consider the setting of Example 7.6.1.
(a) For all parameters remaining the same, find the range and bit rate when using a 64QAM constellation.
(b) Suppose now that the channel model is changed from AWGN to Rayleigh fading (see Problem 7.29). Find the receiver sensitivity required for QPSK at BER of 10^{-6}. What is the range, assuming all other parameters are as in Example 7.6.1? How does the range change if you reduce the link margin to 10 dB (now that fading is being accounted for, there are fewer remaining uncertainties).

Problem 7.32 Consider a line-of-sight communication link operating in the 60 GHz band (where large amounts of unlicensed bandwidth have been set aside by regulators). From the Friis formula, the received power scales as λ^2, and hence as the inverse square of the carrier frequency, so that 60 GHz links have much worse propagation than, say, 5 GHz links when antenna gains are fixed. However, we can get much better antenna gains at small carrier wavelengths, since the gains scale as $1/\lambda^2$ for fixed antenna aperture. Suppose that we wish to design a 2 Gbps link using QPSK with an excess bandwidth of 50%. The receiver noise figure is 8 dB. The transmit and receive antenna gains are each 10 dBi.
(a) What is the transmit power in dBm required to attain a range of 10 meters (e.g., for in-room communication) with a link margin of 12 dB?
(b) For a transmit power of 20 dBm, what are the antenna gains required at the transmitter and receiver (assume that the gains at both ends are equal) to attain a range of 200 meters (e.g., for an outdoor last-hop link)?
(c) For the antenna gains found in (b), what happens to the attainable range if you account for additional path loss due to oxygen absorption (typical in the 60 GHz band) of 16 dB/km?
(d) In (c), what happens to the attainable range if there is a further path loss of 30 dB/km due to heavy rain (on top of the loss due to oxygen absorption)?

Some mathematical derivations

Problem 7.33 (Bounds on the Q function) We derive the bounds (7.6) and (7.5) for

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt \qquad (7.103)$$

(a) Show that, for $x \geq 0$, the following upper bound holds:

$$Q(x) \leq \frac{1}{2} e^{-x^2/2}$$

Hint: Try pulling out a factor of e^{-x^2} from (7.103), and then bounding the resulting integrand. Observe that $t \geq x \geq 0$ in the integration interval.
(b) For $x \geq 0$, derive the following upper and lower bounds for the Q function:

$$(1 - \frac{1}{x^2}) \frac{e^{-x^2/2}}{\sqrt{2\pi} x} \leq Q(x) \leq \frac{e^{-x^2/2}}{\sqrt{2\pi} x}$$

Hint: Write the integrand in (7.103) as a product of $1/t$ and $te^{-t^2/2}$ and then integrate by parts to get the upper bound. Integrate by parts once more using a similar trick to get the lower bound. Note that you can keep integrating by parts to get increasingly refined upper and lower bounds.

Problem 7.34 (Geometric derivation of Q function bound) Let X_1 and X_2 denote independent standard Gaussian random variables.
(a) For $a > 0$, express $P[|X_1| > a, |X_2| > a]$ in terms of the Q function.
(b) Find $P[X_1^2 + X_2^2 > a^2]$.
Hint: Transform to polar coordinates. Or use the results of Problem 7.6.
(c) Sketch the regions in the (x_1, x_2) plane corresponding to the events considered in (a) and (b).
(d) Use (a)-(c) to obtain an alternative derivation of the bound $Q(x) \leq \frac{1}{2} e^{-x^2/2}$ for $x \geq 0$ (i.e., the bound in Problem 7.33(a)).

Problem 7.35 (Properties of covariance matrices) Let $\mathbf{C}$ denote the covariance matrix of a random vector $\mathbf{X}$ of dimension m. Let $\{\lambda_i, i = 1, ..., m\}$ denotes its eigenvalues, and let $\{\mathbf{v}_i, i = 1, ..., m\}$ denote the corresponding eigenvectors, chosen to form an orthonormal basis for $\mathbb{R}^m$ (let us take it for granted that this can always be done). That is, we have $\mathbf{C}\mathbf{v}_i = \lambda_i \mathbf{v}_i$ and $\mathbf{v}_i^T \mathbf{v}_j = \delta_{ij}$.
(a) Show that $\mathbf{C}$ is nonnegative definite. That is, for any vector $\mathbf{a}$, we have $\mathbf{a}^T \mathbf{C} \mathbf{a} \geq 0$.
Hint: Show that you can write $\mathbf{a}^T \mathbf{C} \mathbf{a} = E[Y^2]$ for some random variable Y.
(b) Show that any eigenvalue $\lambda_i \geq 0$.
(c) Show that $\mathbf{C}$ can be written in terms of its eigenvectors as follows:

$$\mathbf{C} = \sum_{i=1}^m \lambda_i \mathbf{v}_i \mathbf{v}_i^T \qquad (7.104)$$

Hint: The matrix equality $\mathbf{A} = \mathbf{B}$ is equivalent to saying that $\mathbf{A}\mathbf{x} = \mathbf{B}\mathbf{x}$ for any vector $\mathbf{x}$. We use this to show that the two sides of (7.104) are equal. For any vector $\mathbf{x}$, consider its expansion

$\mathbf{x} = x_i \mathbf{v}_i$ with respect to the basis $\{\mathbf{v}_i\}$. Now, show that applying the matrices on each side of (7.104) gives the same result.
The expression (7.104) is called the *spectral factorization* of $\mathbf{C}$, with the eigenvalues $\{\lambda_i\}$ playing the role of a discrete spectrum. The advantage of this view is that, as shown in the succeeding parts of this problem, algebraic operations on the eigenvalues, such as taking their inverse or square root, correspond to analogous operations on the matrix $\mathbf{C}$.
(d) Show that, for $\mathbf{C}$ invertible, the inverse is given by

$$\mathbf{C}^{-1} = \sum_{i=1}^{m} \frac{1}{\lambda_i} \mathbf{v}_i \mathbf{v}_i^T \tag{7.105}$$

Hint: Check this by directly multiplying the right-hand sides of (7.104) and (7.105), and using the orthonormality of the eigenvectors.
(e) Show that the matrix

$$\mathbf{A} = \sum_{i=1}^{m} \sqrt{\lambda_i} \mathbf{v}_i \mathbf{v}_i^T \tag{7.106}$$

can be thought of as a square root of $\mathbf{C}$, in that $\mathbf{A}^2 = \mathbf{C}$. We denote this as $\mathbf{C}^{\frac{1}{2}}$.
(f) Suppose now that $\mathbf{C}$ is not invertible. Show that there is a nonzero vector $\mathbf{a}$ such that the entire probability mass of $\mathbf{X}$ lies along the $(m-1)$-dimensional plane $\mathbf{a}^T(\mathbf{X} - \mathbf{m}) = 0$. That is, the m-dimensional joint density of $\mathbf{X}$ does not exist.
Hint: If $\mathbf{C}$ is not invertible, then there is a nonzero $\mathbf{a}$ such that $\mathbf{C}\mathbf{a} = 0$. Now left multiply by $\mathbf{a}^T$ and write out $\mathbf{C}$ as an expectation.
Remark: In this case, it is possible to write one of the components of $\mathbf{X}$ as a linear combination of the others, and work in a lower-dimensional space for computing probabilities involving $\mathbf{X}$. Note that this result does not require Gaussianity.

Problem 7.36 (Derivation of joint Gaussian density) We wish to derive the density of a real-valued Gaussian random vector $\mathbf{X} = (X_1, ..., X_m)^T \sim N(0, \mathbf{C})$, starting from the assumption that any linear combination of the elements of $\mathbf{X}$ is a Gaussian random variable. This can then be translated by $\mathbf{m}$ to get any desired mean vector. To this end, we employ the characteristic function of $\mathbf{X}$, defined as

$$\phi_{\mathbf{X}}(\mathbf{w}) = E[e^{j\mathbf{w}^T\mathbf{X}}] = E[e^{j(w_1 X_1 + ... + w_m X_m)}] = \int e^{j\mathbf{w}^T\mathbf{X}} p_{\mathbf{X}}(\mathbf{x}) \, d\mathbf{x} \tag{7.107}$$

as a multidimensional Fourier transform of $\mathbf{X}$. The density $p_{\mathbf{X}}(\mathbf{x})$ is therefore given by a multidimensional inverse Fourier transform

$$p_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^m} \int e^{-j\mathbf{w}^T\mathbf{x}} \phi_{\mathbf{X}}(\mathbf{w}) \, d\mathbf{w} \tag{7.108}$$

(a) Show that the characteristic function of a standard Gaussian random variable $Z \sim N(0,1)$ is given by $\phi_Z(w) = e^{-w^2/2}$.
(b) Set $Y = \mathbf{w}^T\mathbf{X}$. Show that $Y \sim N(0, v^2)$, where $v^2 = \mathbf{w}^T\mathbf{C}\mathbf{w}$.
(c) Use (a) and (b) to show that

$$\phi_{\mathbf{X}}(\mathbf{w}) = e^{-\frac{1}{2}\mathbf{w}^T\mathbf{C}\mathbf{w}} \tag{7.109}$$

(d) To obtain the density using the integral in (7.108), make the change of variable $\mathbf{u} = \mathbf{C}^{\frac{1}{2}}\mathbf{w}$. Show that you get

$$p_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^m} \frac{1}{|\mathbf{C}|^{\frac{1}{2}}} \int e^{-j\mathbf{u}^T\mathbf{C}^{-\frac{1}{2}}\mathbf{x}} e^{-\frac{1}{2}\mathbf{u}^T\mathbf{u}} \, d\mathbf{u}$$

where $|\mathbf{A}|$ denotes the determinant of a matrix $\mathbf{A}$.
Hint: We have $\phi_{\mathbf{X}}(\mathbf{w}) = e^{-\frac{1}{2}\mathbf{u}^T\mathbf{u}}$ and $d\mathbf{u} = |\mathbf{C}^{\frac{1}{2}}|d\mathbf{w}$.
(d) Now, set $\mathbf{C}^{-\frac{1}{2}}\mathbf{x} = \mathbf{z}$, with $p_{\mathbf{X}}(\mathbf{x}) = f(\mathbf{z})$. Show that

$$f(\mathbf{z}) = \frac{1}{(2\pi)^m}\frac{1}{|\mathbf{C}|^{\frac{1}{2}}}\int e^{-j\mathbf{u}^T\mathbf{z}}e^{-\frac{1}{2}\mathbf{u}^T\mathbf{u}}\ d\mathbf{u} = \frac{1}{|\mathbf{C}|^{\frac{1}{2}}}\prod_{i=1}^{m}\left(\frac{1}{2\pi}\int e^{-ju_iz_i}e^{-u_i^2/2}\ du_i\right)$$

(e) Using (a) to evaluate $f(\mathbf{z})$ in (d), show that

$$f(\mathbf{z}) = \frac{1}{|\mathbf{C}|^{\frac{1}{2}}}\frac{1}{(2\pi)^{\frac{m}{2}}}e^{-\frac{1}{2}\mathbf{z}^T\mathbf{z}}$$

Now substitute $\mathbf{C}^{-\frac{1}{2}}\mathbf{x} = \mathbf{z}$ to get the density $p_{\mathbf{X}}(\mathbf{x})$.

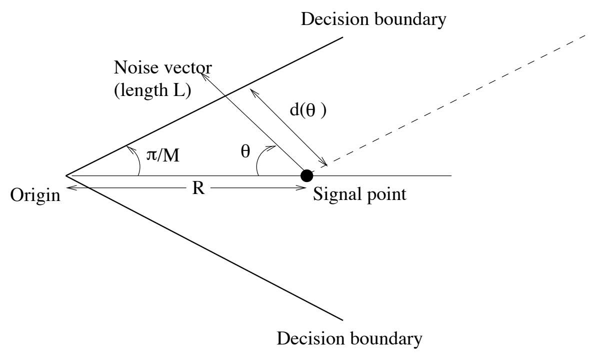

Figure 7.40: Figure for Problem 7.37

Problem 7.37 (Symbol error probability for PSK) In this problem, we derive an expression for the symbol error probability for M-ary PSK that requires numerical evaluation of a single integral over a finite interval. Figure 7.40 shows the decision boundaries corresponding to a point in a PSK constellation. ls psk*A two-dimensional noise vector originating from the signal point must reach beyond the boundaries to cause an error. A direct approach to evaluating error probability requires integration of the two-dimensional Gaussian density over an infinite region. We avoid this by switching to polar coordinates, with the noise vector having radius L and angle θ as shown.
(a) Due to symmetry, the error probability equals twice the probability of the noise vector crossing the top decision boundary. Argue that this happens if $L > d(\theta)$ for some $\theta \in (0, \pi - \frac{\pi}{M})$.
(b) Show that the probability of error is given by

$$P_e = 2\int_0^{\pi-\frac{\pi}{M}} P[L > d(\theta)|\theta]p(\theta)\ d\theta$$

(c) Use Problem 7.6 to show that $P[L > d] = e^{-\frac{d^2}{2\sigma^2}}$, that L is independent of θ, and that θ is uniform over $[0, 2\pi]$.
(d) Show that $d(\theta) = \frac{R\sin\frac{\pi}{M}}{\sin(\theta+\frac{\pi}{M})}$.
(e) Conclude that the error probability is given by

$$P_e = \frac{1}{\pi}\int_0^{\pi-\frac{\pi}{M}} e^{-\frac{R^2\sin^2\frac{\pi}{M}}{2\sigma^2\sin^2(\theta+\frac{\pi}{M})}}d\theta$$

(f) Use the change of variable $\phi = \pi - (\theta + \frac{\pi}{M})$ (or alternatively, realize that $\theta + \frac{\pi}{M} \pmod{2\pi}$ is also uniform over $[0, 2\pi]$) to conclude that

$$P_e = \frac{1}{\pi} \int_0^{\pi - \frac{\pi}{M}} e^{-\frac{R^2 \sin^2 \frac{\pi}{M}}{2\sigma^2 \sin^2 \phi}} \, d\phi = \frac{1}{\pi} \int_0^{\pi - \frac{\pi}{M}} e^{-\frac{E_b \log_2 M \sin^2 \frac{\pi}{M}}{N_0 \sin^2 \phi}} \, d\phi, \quad \textbf{symbol error probability for M} - \textbf{ary PSK} \tag{7.110}$$

Problem 7.38 (M-ary orthogonal signaling performance as $M \to \infty$) We wish to derive the result that

$$\lim_{M \to \infty} P(\text{correct}) = \begin{cases} 1 & \frac{E_b}{N_0} > \ln 2 \\ 0 & \frac{E_b}{N_0} < \ln 2 \end{cases} \tag{7.111}$$

(a) Show that

$$P(\text{correct}) = \int_{-\infty}^{\infty} \left[\Phi \left(x + \sqrt{\frac{2E_b \log_2 M}{N_0}} \right) \right]^{M-1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx$$

(b) Show that, for any x,

$$\lim_{M \to \infty} \left[\Phi \left(x + \sqrt{\frac{2E_b \log_2 M}{N_0}} \right) \right]^{M-1} = \begin{cases} 0 & \frac{E_b}{N_0} < \ln 2 \\ 1 & \frac{E_b}{N_0} > \ln 2 \end{cases}$$

Hint: Use L'Hospital's rule on the log of the expression whose limit is to be evaluated.
(c) Substitute (b) into the integral in (a) to infer the desired result.