# CSC410 tutorial: SAT for problem solving

Encoding problems to SAT problems

October 2, 2019

# Motivation

- Boolean satifiasbility problem (SAT):
  Given formula e.g. $(x \vee y \vee z) \wedge (\neg x \vee \neg y) \wedge (\neg y \vee \neg z)$,
  is there an assignment that makes the formula true?
- Efficient algorithm to solve SAT problems.
- Solver, e.g. Z3 (demo).

# Motivation

You have a problem to solve, e.g. need to write a Sudoku solver.

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | 1 | | | 7 | 8 | | | |
| | 8 | | | 4 | | 9 | | |
| | | 5 | 6 | | | | 1 | |
| 1 | | | | 6 | | | | 5 |
| | 4 | | 9 | 1 | 5 | | 7 | 2 |
| | 6 | 7 | | 8 | | 4 | | |
| | | | 3 | | | 1 | | |
| | 7 | | 8 | 9 | | | 2 | 3 |
| | | | | | 4 | | | |

A 9x9 grid, with **81** variables: $0 < i, j \leq 9$, $x_{ij}$ is the value (**digit from 1 to 9**) of cell $i, j$ .

**Constraints:** each row, column an block of 3x3 contains all the digits from 1 to 9. Some digits are already assigned.

- is it a boolean SAT problem? **No**.

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You have a problem to solve, e.g. need to write a Sudoku solver.

- is it a boolean SAT problem? **No**.
- **Solution 1:** design a specialized algorithm and **optimize** it.
- Solution 2: **encode it** into a SAT problem, use **off-the-shelf optimized solver**.

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- **Original problem**: non-boolean variables and constraints.
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- **SAT encoding**: **boolean** variables and **clauses**.
    - **variables**: find an interpretable boolean representation of the original variables.
      Ex: binary representation of integers.
    - **clauses**: express the constraints as a conjunction of clauses.

# Sudoku

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**Constraints:**

- each row contains all the digits from 1 to 9,
- each column contains all the digits from 1 to 9,
- each block of 3x3 contains all the digits from 1 to 9,
- initial grid specifies values of some cells.

# Sudoku : SAT version

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A 9x9 grid, with **9x81** variables:
$0 < i \leq 9, 0 < j \leq 9, 0 < k \leq 9$, $b_{i,j,k}$ is true iff the value of cell $i, j$ is $k$.

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**Constraints:**

- each row contains all the digits:
  $\forall 0 < k \leq 9, 0 < i \leq 9, \sum_{j=1}^{9} b_{i,j,k} = 1$

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**Exactly one**

Exactly one $\{x_i\}_{i=1..n}$ is true: $\sum_{i=1}^{n} x_i = 1$.
$\binom{N}{2} + 1$ clauses:

- At least one: $x_1 \vee x_2 \vee \ldots \vee x_n$.
- At most one (no pair with both $x_i$, $x_j$ true):
  $\neg x_i \vee \neg x_j$, for $i, j = 1..n$ and $i \neq j$.

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**Constraints:**

- each column contains all the digits:
  $\forall 0 < k \leq 9, 0 < j \leq 9, \sum_{i=1}^{9} b_{i,j,k} = 1$

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**Constraints:**

- each block of 3x3:
  $\forall 0 < k \leq 9, 0 < a, b \leq 3,$
  $\sum_{r=1}^{3} \sum_{l=1}^{3} b_{3a+r,3b+l,k} = 1$

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- initial grid specifies values of some cells.