Analysis and Verification of Software

Abstract Interpretation: an Overview

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Overview

The goal: Define and prove automatically a property of the possible behaviors of a complex computer program, e.g. a security property.

The key-idea: The calculus can be done on an abstraction of these behaviors dealing only with those elements of the behaviors related to the considered property.

The theory: Abstract Interpretation [P.Cousot]

Abstract Interpretation

The central idea in abstract interpretation is to construct two different meanings of a programming language where the first gives the usual meaning of programs in the language, and the other can be used to answer certain questions about the runtime behavior of programs in the language.

The **standard meaning** of programs can typically be described by their input-output function, and the standard interpretation will then be a function which maps programs to their input-output functions.

The **abstract meaning** will be described by a function on to mathematical objects that represent the property be analyzed.

The correctness (or soundness) of this approach to program analysis can be established by proving a relationship between these two interpretations.

Abstract Interpretation

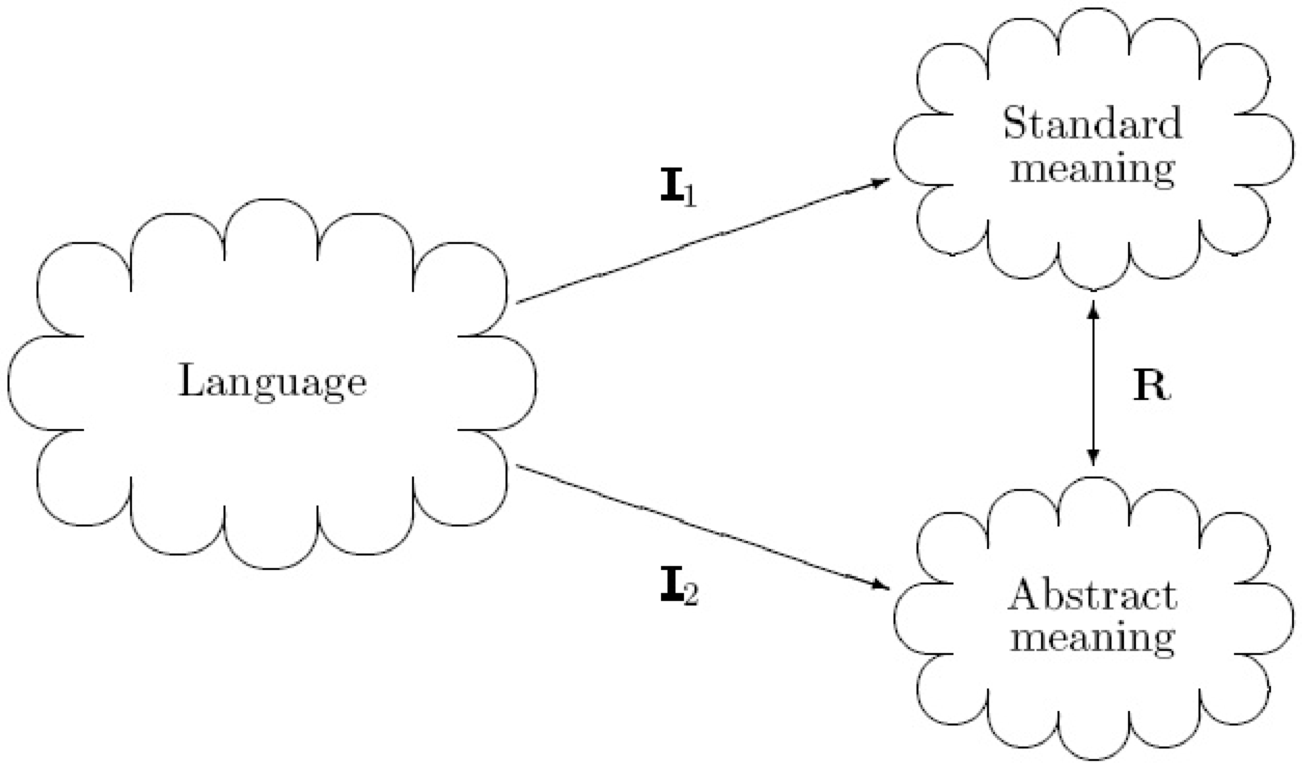

Figure by M. Rosendahl

Abstract Interpretation

- A theory of semantics approximation for computing conservative over-approximations of dynamic properties of programs.
- Successfully applied to infer run-time properties useful for a variety of tasks:
  - debugging (e.g. type inference)
  - code optimization (e.g., compile-time garbage collection)
  - program transformation (e.g., partial evaluation, parallelization)
  - program correctness proofs (e.g., safety, termination, proof of absence of run-time errors, semantic tattooing/watermarking).
- Still a large variety of tasks in the software engineering process that could greatly benefit from it, because of its firm theoretical foundations and mechanical nature.

Trajectories

The concrete semantics of a program formalizes the set of all its possible executions in all possible execution environments.

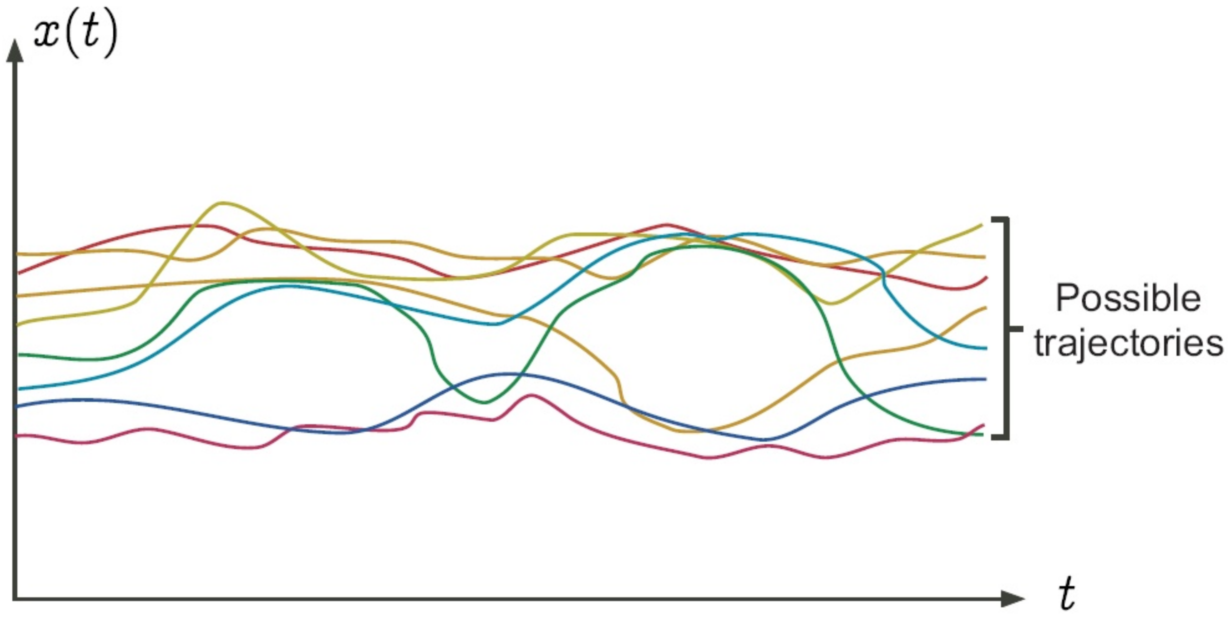

Figure by P.Cousot

Undecidability

- The concrete semantics of a program is not computable (unless the program is trivial);
- All non trivial properties on the concrete program semantics are either undecidable or ”very hard” to solve (either NP complete or co-NP complete).

Safety properties

Safety properties of a program express that no possible execution in any possible execution environment can reach an erroneous state.

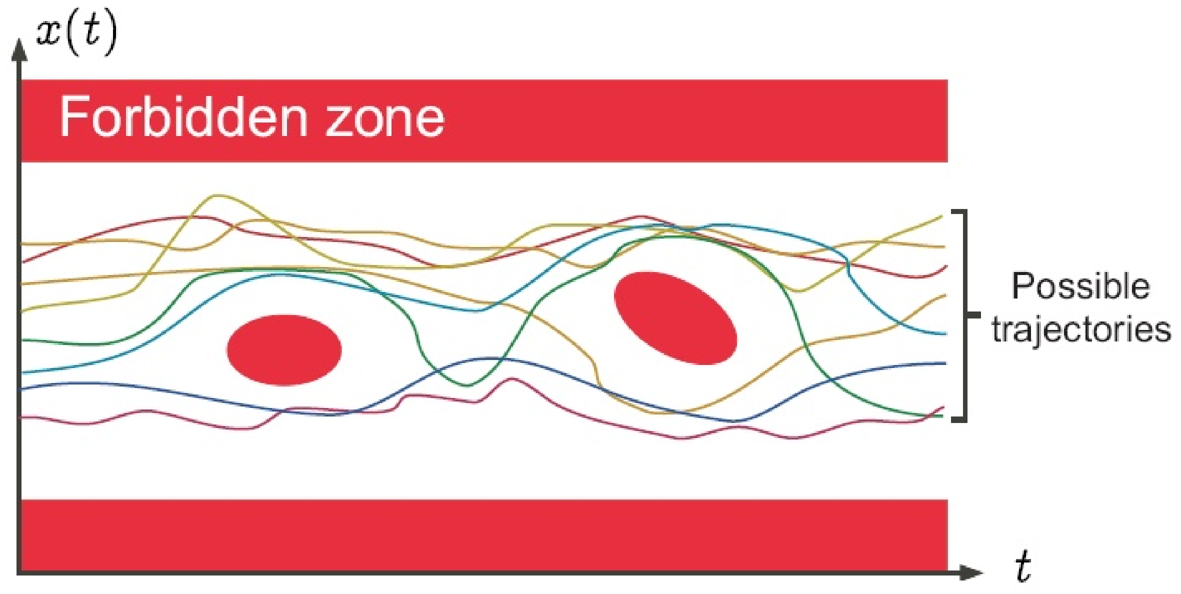

Figure by P.Cousot

We should prove that the intersection of the program concrete semantics and the forbidden zone is empty. This is an undecidable problem, as the concrete semantics is not computable.

What about testing?

Testing is not a good solution:

- it considers only a subset of the possible executions;
- it does not provide a correctness proof;
- the issue is ”coverage”!

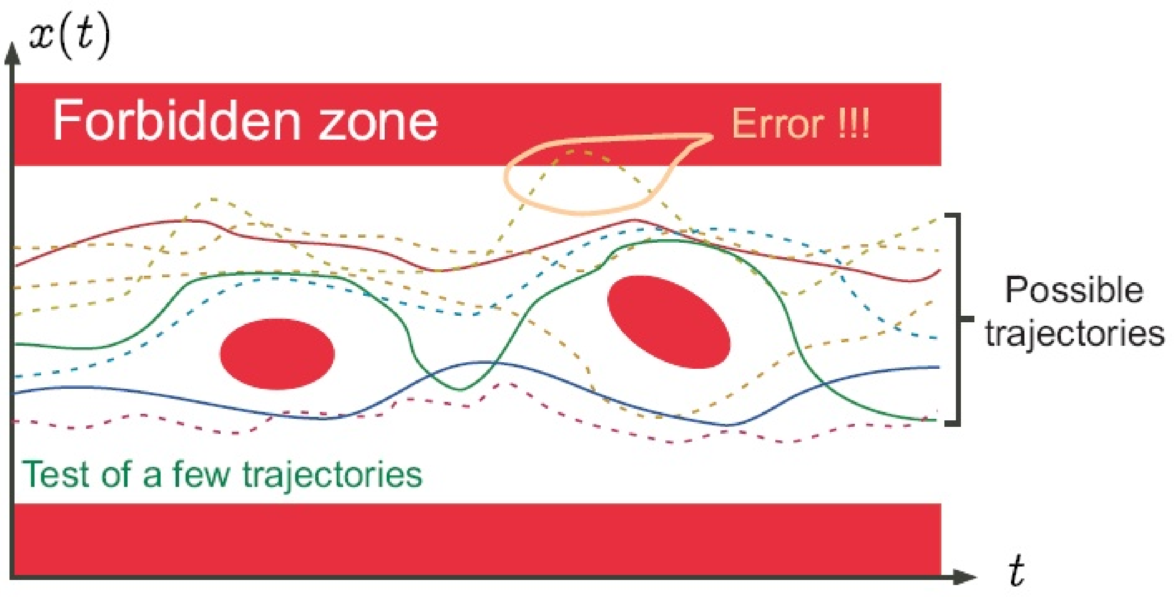

Figure by P.Cousot

Over-Approximation of the Concrete Semantics

Abstract Interpretation consists in considering an abstract semantics, that is an over-approximation of the concrete semantics of the program;

- The abstract semantics covers all possible concrete cases;
- It guarantees correctness: if it is safe (it does not intersect the forbidden zone) then so is the concrete semantics.

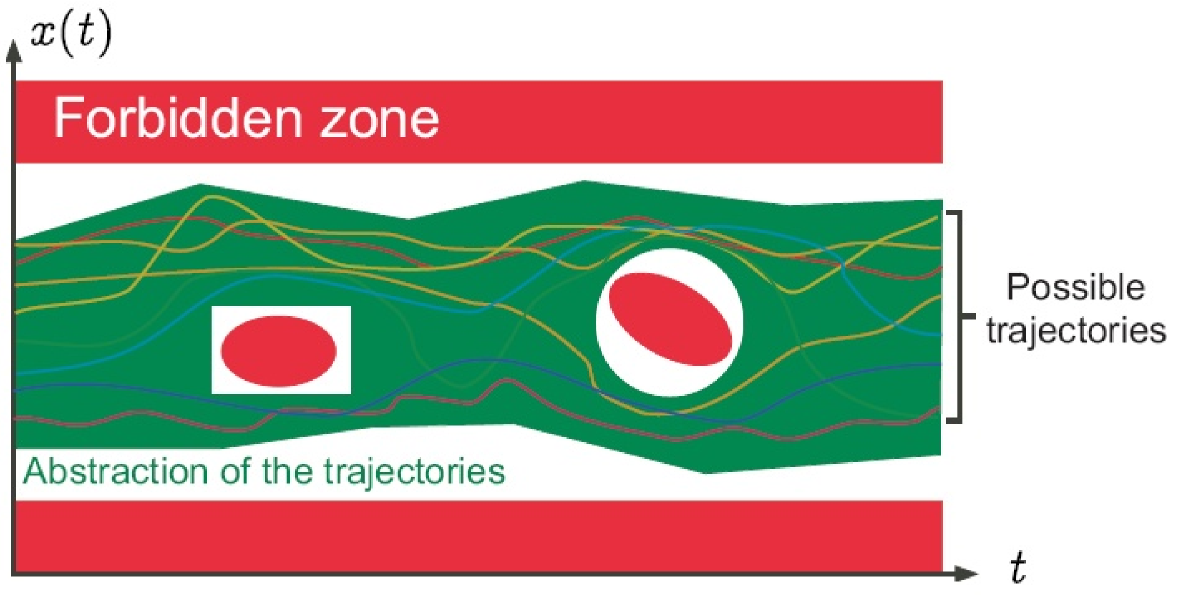

Figure by P.Cousot

Correctness

Correctness in this approach is a MUST. The abstract semantics should not disregard any trajectory!

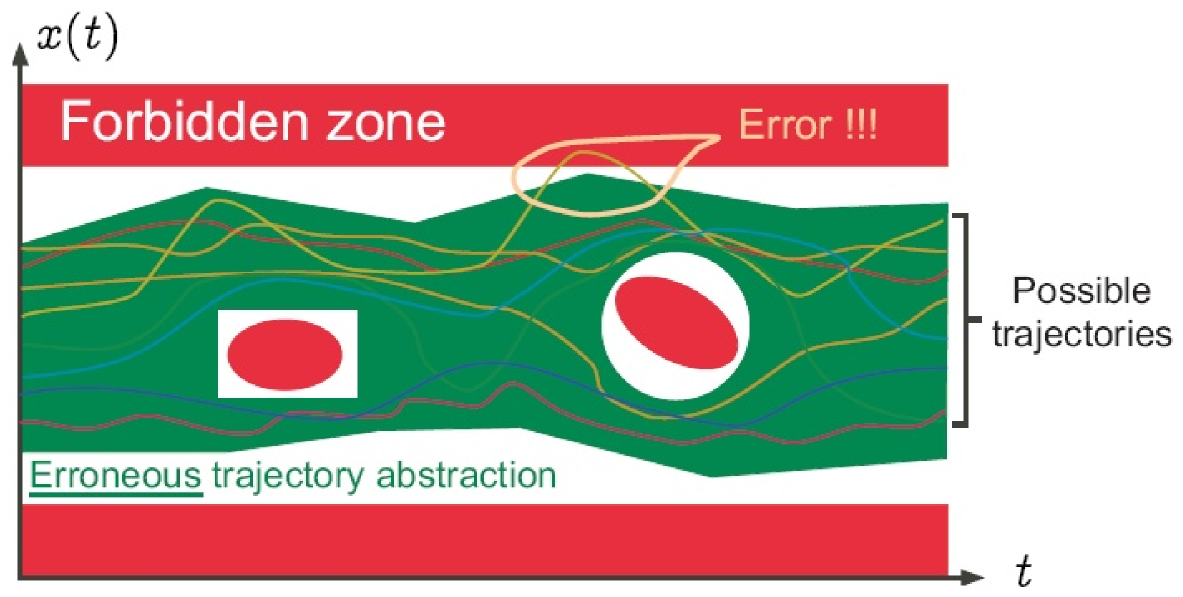

Figure by P.Cousot

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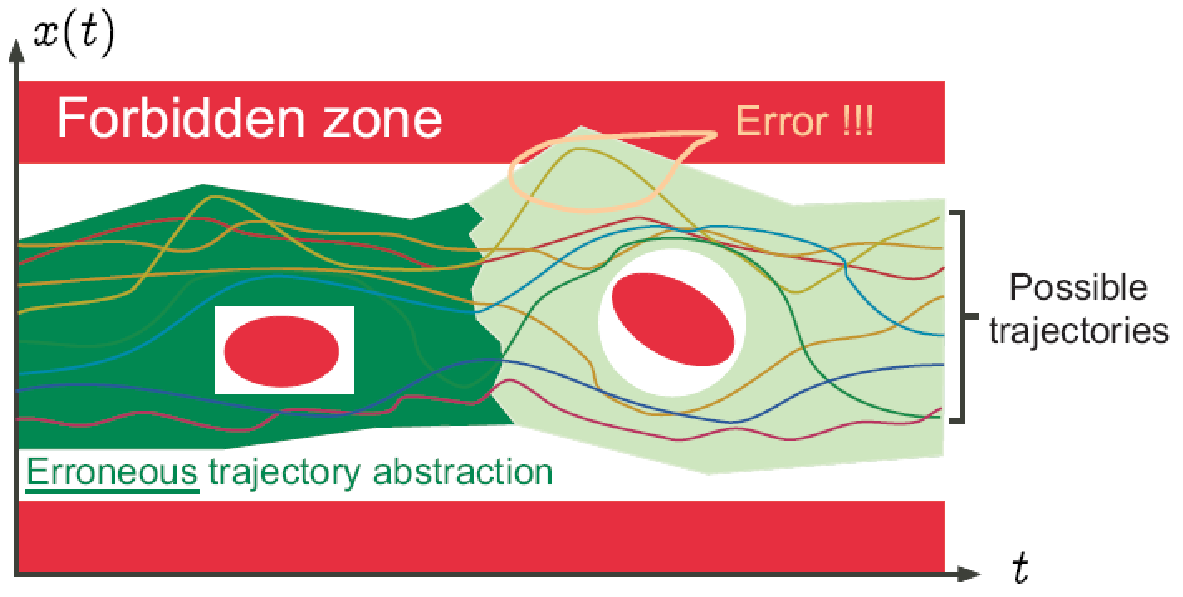


Figure by P.Cousot

Imprecision

The over-approximation may result into a lack of precision, i.e. on generating false alarms.

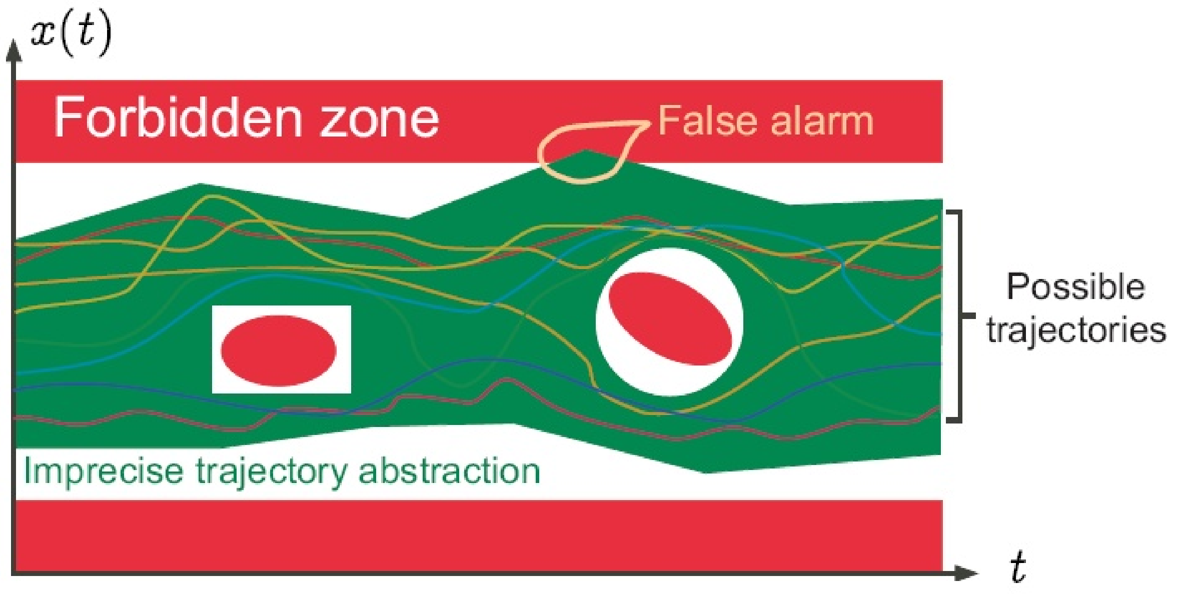


Figure by P.Cousot

The starting point is a formal description of an (operational) concrete semantics

Start from a standard operational semantics that describes formally:

- states: data values of program variables,
- transitions: elementary computation steps;
- traces: sequences (possibly infinite) of states corresponding to executions described by transitions

Concrete Semantics: small-steps transitions

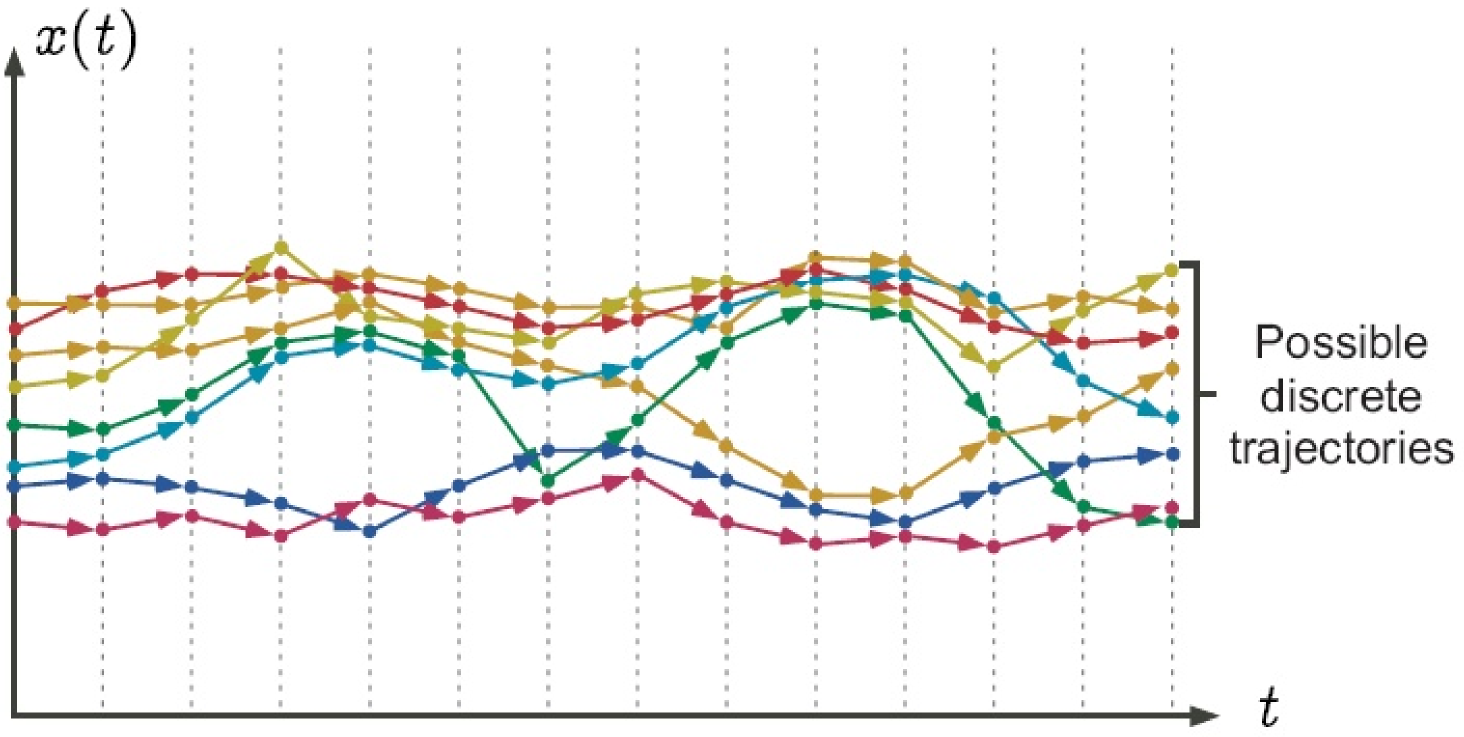


Figure by P.Cousot

Collecting Semantics

The collecting semantics is the set of observable behaviors in the operational semantics. It is the starting point of any analysis design.

- The set of all descendants of the initial state
- The set of all descendants of the initial state that can reach a final state
- The set of all finite traces from the initial state
- The set of all finite and infinite traces from the initial state etc.

Collecting semantics: trajectories of sets of states

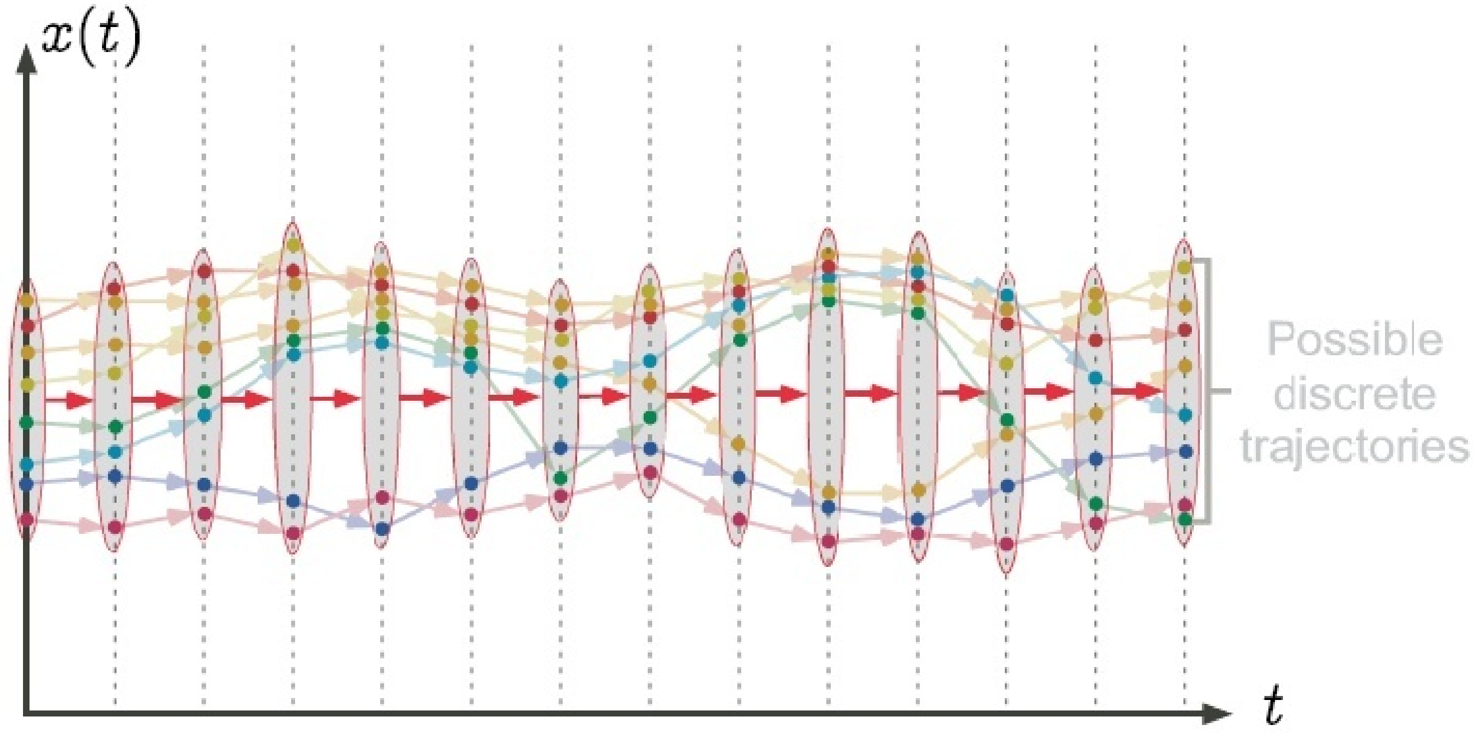


Figure by P.Cousot

Concrete Semantics: trajectories of sets of states

$$F(X) = \mathcal{I} \cup \{s' \mid \exists s \in X : s \xrightarrow{t} s'\}$$

$$\mathcal{R} = \mathsf{lfp}_{\emptyset}^{\subseteq} F$$

$$= \bigcup_{n=0}^{+\infty} F^n(\emptyset) \qquad \text{where} \qquad \begin{aligned} f^0(x) &= x \\ f^{n+1}(x) &= f(f^n(x)) \end{aligned}$$

Abstracting Data and Operations

In order to get an abstract semantics we need:

- abstract program data,
- abstract program basic operations;
- abstract program control (iteration, procedure, concurrency,. . . ;

The abstraction can be parameterized, in order to tune it with respect to different application domains.

Abstracting properties

- Choose an abstract domain, a complete lattice whose elements capture the information to be analyzed.
- An abstraction function α which maps a set of concrete objects to itsmost accurate representation in the abstract domain;
- The inverse concretization function γwhich maps an abstract object to the concrete ones it represents;
- Forget no concrete objects: (abstraction from above) $S \subseteq \gamma(\alpha(S))$

Lattices

The idea behind a partial order is ”a $\subseteq$ b” means ”a is more informative than b”.

A lattice L is a partially ordered set $(L, \sqsubseteq)$ with:

- Least upper bounds $\sqcup$ and greatest lower bounds $\sqcap$ operators
- A least element $\perp$
- A greatest element $\top$

L is complete if for each subset of L there exist a (unique) least upper bound.

Abstracting by Sign

We may in some situations be able to predict whether the result of an expression is positive or negative by only using the sign of the constants in the expression.

As an example consider the expression $-413 \times (2571 + 879)$. We can deduce that the result is negative without actually performing the addition and the multiplication since we know that adding two positive numbers gives a positive result and that multiplying a negative and a positive number gives a negative result.

Abstracting by Sign

In the sign interpretation we will operate with four different values: `zero, pos, neg, num`, where `zero` indicates that the number is zero, `pos` that the number is positive, `neg` that the number is negative, and `num` that we don't know. We will call the set of these values `Sign`.

The rule-of-sign interpretation of addition and multiplication can then be specified in two tables. These interpretations may be viewed as ”abstract” additions and multiplications.

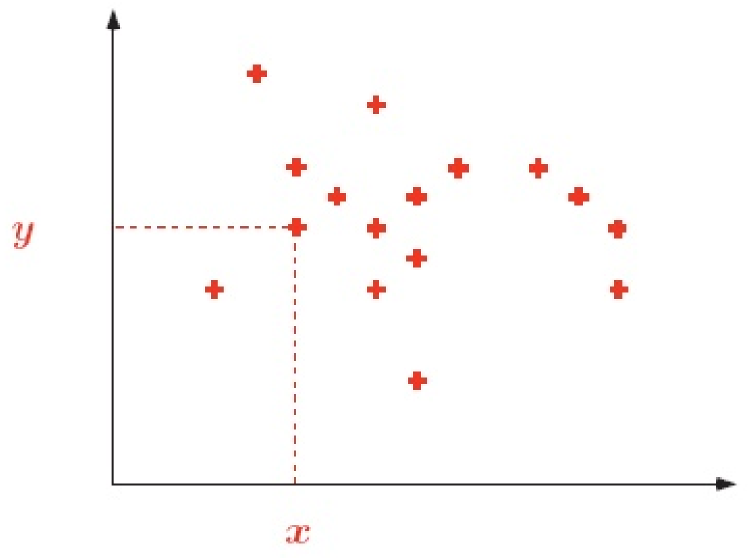
$\oplus : \mathbf{Sign} \times \mathbf{Sign} \rightarrow \mathbf{Sign}$

$\oplus$	zero	pos	neg	num
zero	zero	pos	neg	num
pos	pos	pos	num	num
neg	neg	num	neg	num
num	num	num	num	num

$\otimes : \mathbf{Sign} \times \mathbf{Sign} \rightarrow \mathbf{Sign}$

$\otimes$	zero	pos	neg	num
zero	zero	zero	zero	zero
pos	zero	pos	neg	num
neg	zero	neg	pos	num
num	zero	num	num	num

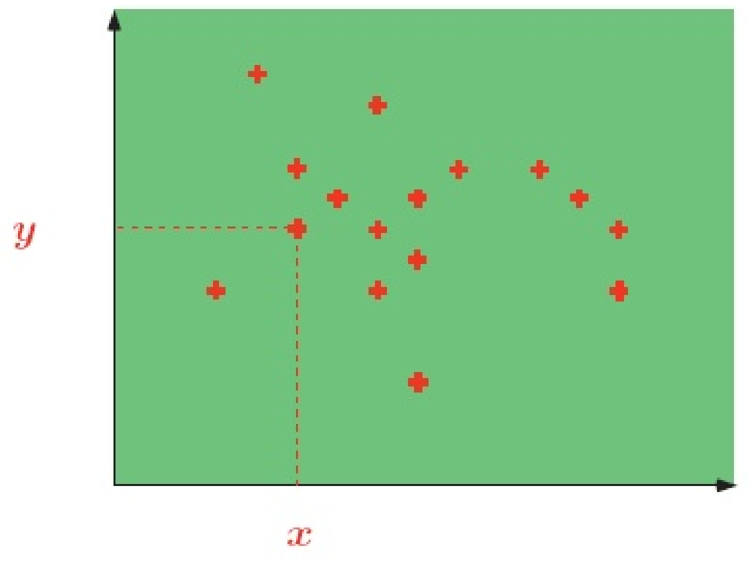
Abstract domains for approximating an (infinite) set of values of two variables



$$\{\ldots, \langle 19,\ 77\rangle, \ldots, \langle 20,\ 03\rangle, \ldots\}$$

Figure by P.Cousot

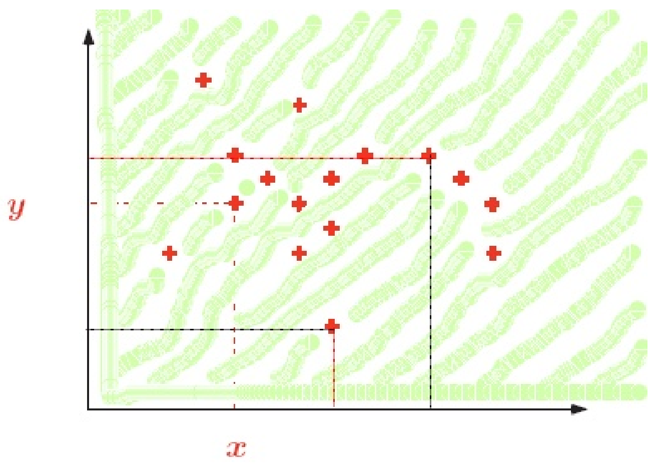
The Domain of Signs



$$\begin{cases} x \geq 0 \\ y \geq 0 \end{cases}$$

Figure by P.Cousot

Operations on Sign



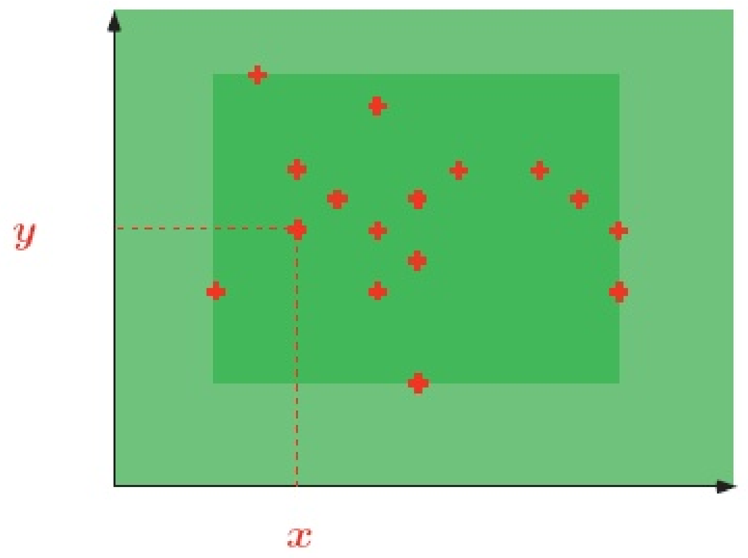
$$(3,1) + (2,3) = (5,4)$$

$$(\mathtt{pos},\mathtt{pos}) \oplus (\mathtt{pos},\mathtt{pos}) = (\mathtt{pos},\mathtt{pos})$$

$$(0,3) * (-2,-4) = (0,-12)$$

$$(\mathtt{zero},\mathtt{pos}) \otimes (\mathtt{neg},\mathtt{neg}) = (\mathtt{zero},\mathtt{neg})$$

The Domain of Intervals



$$\begin{cases} x \in [19,\ 77] \\ y \in [20,\ 03] \end{cases}$$

Figure by P.Cousot

Abstracting by Intervals

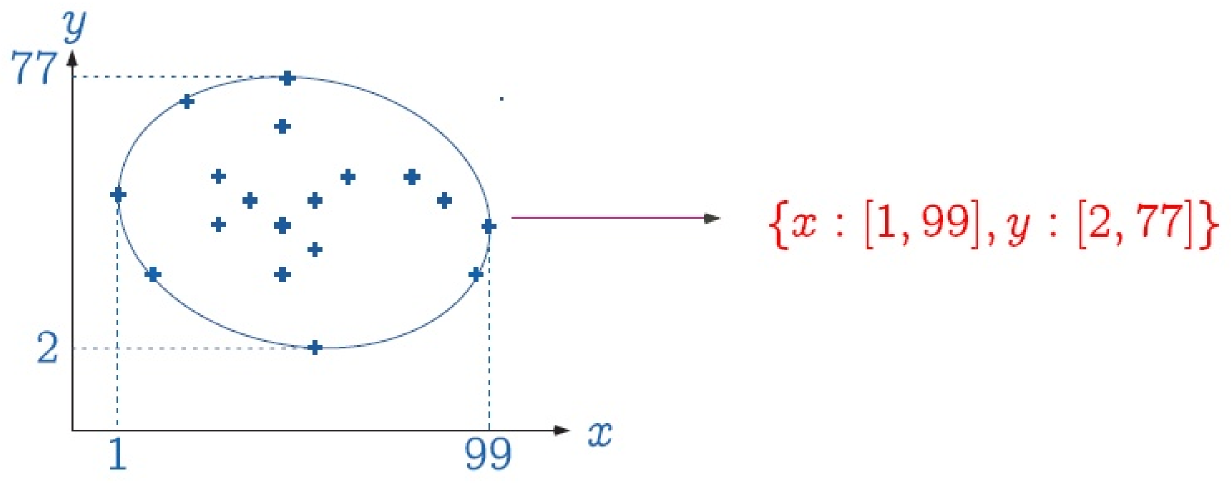


Figure by P.Cousot

Concretization of Intervals

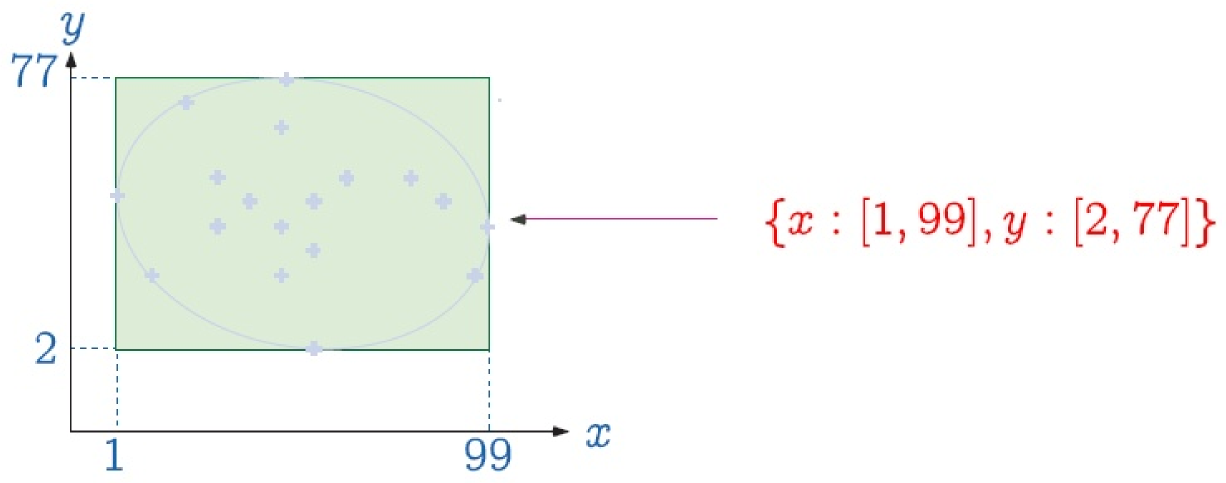
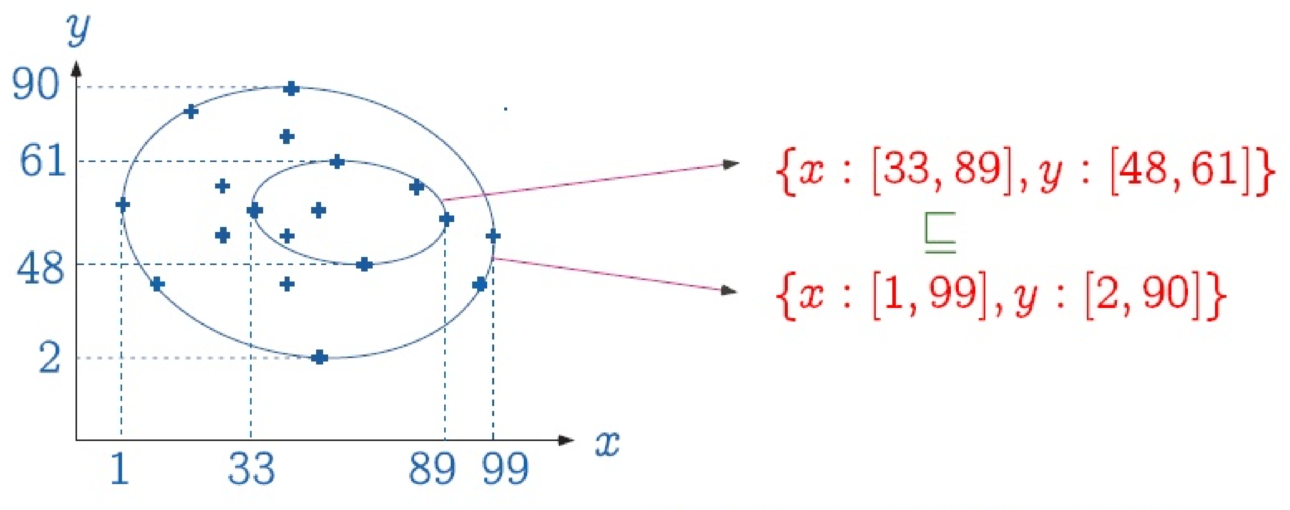


Figure by P.Cousot

Monotonicity of the Abstraction

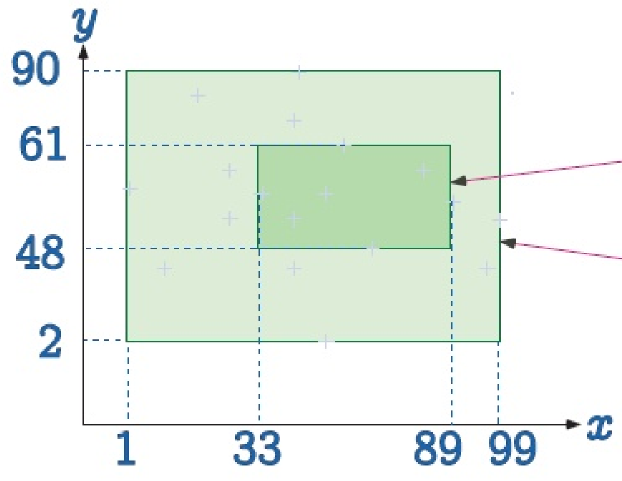


$$\{x : [33, 89], y : [48, 61]\}$$
$$\sqsubseteq$$
$$\{x : [1, 99], y : [2, 90]\}$$

$$X \subseteq Y \Rightarrow \alpha(X) \sqsubseteq \alpha(Y)$$

Figure by P.Cousot

Monotonicity of the Concretization



$$\{x : [33, 89], y : [48, 61]\}$$
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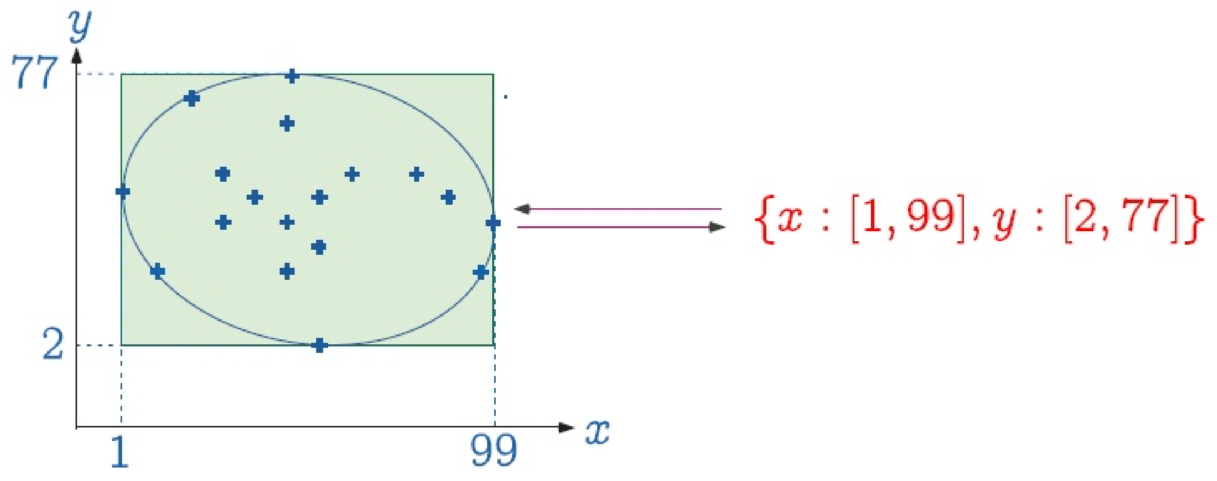
$$\{x : [1, 99], y : [2, 90]\}$$

$$X \sqsubseteq Y \Rightarrow \gamma(X) \subseteq \gamma(Y)$$

Figure by P.Cousot

The composition $\gamma \circ \alpha$ is extensive

If we apply α first, and then γ, we over-approximate the initial set.

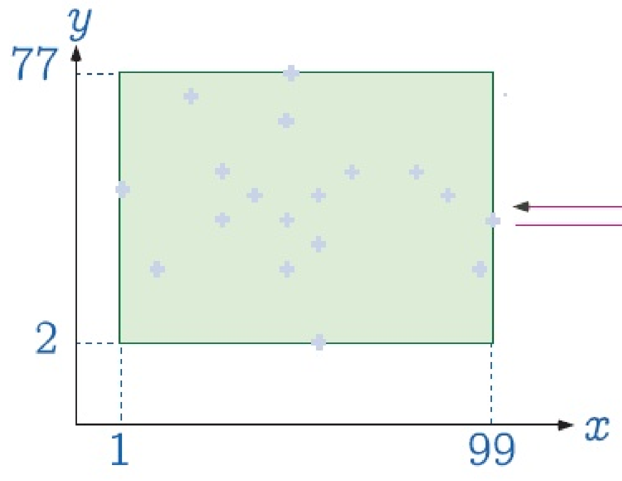


$$X \subseteq \gamma \circ \alpha(X)$$

Figure by P.Cousot

The composition $\alpha \circ \gamma$ is reductive

If we apply γ first, and then α, we don't loose precision.



$$\{x : [1, 99], y : [2, 77]\}$$
$$=/\sqsubseteq$$
$$\{x : [1, 99], y : [2, 77]\}$$

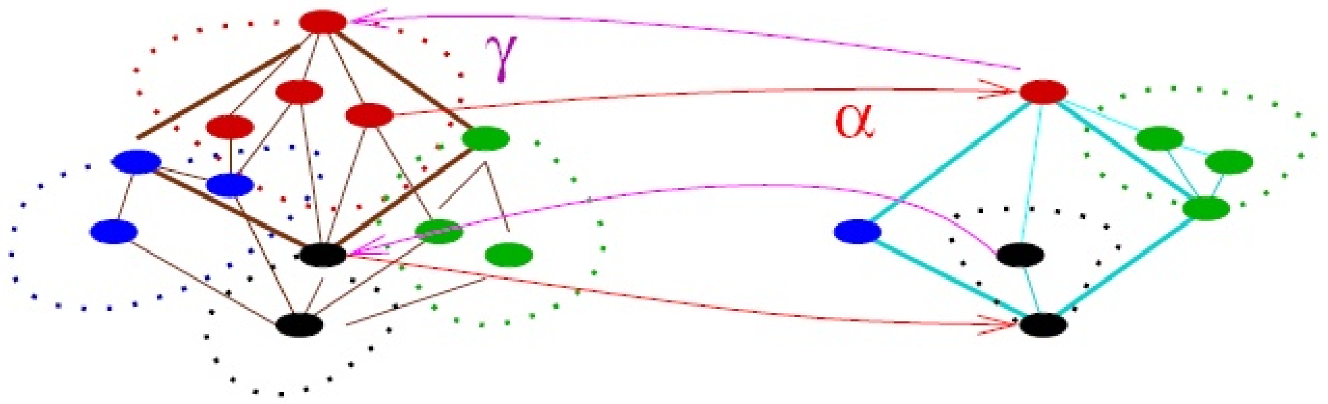
$$\alpha \circ \gamma(Y) =/\sqsubseteq Y$$

Figure by P.Cousot

Galois Connection

A Galois Connection between a concrete p.o. and an abstract p.o arises when:

- The Concretization and Abstraction functions are monotone
- The composition $\gamma \circ \alpha$ is extensive
- The composition $\alpha \circ \gamma$ is reductive



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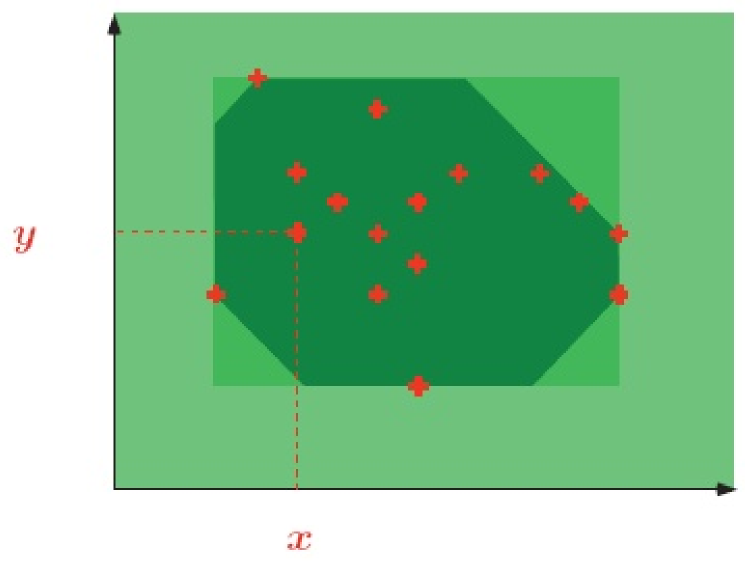
- The composition $\alpha \circ \gamma$ is reductive

$$\langle \mathcal{D}, \subseteq \rangle \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \langle \overline{\mathcal{D}}, \sqsubseteq \rangle$$

iff
$$\begin{aligned}
& \forall x, y \in \mathcal{D} : x \subseteq y \Longrightarrow \alpha(x) \sqsubseteq \alpha(y) \\
\wedge\ & \forall \overline{x}, \overline{y} \in \overline{\mathcal{D}} : \overline{x} \sqsubseteq \overline{y} \Longrightarrow \gamma(\overline{x}) \subseteq \gamma(\overline{y}) \\
\wedge\ & \forall x \in \mathcal{D} : x \subseteq \gamma(\alpha(x)) \\
\wedge\ & \forall \overline{y} \in \overline{\mathcal{D}} : \alpha(\gamma(\overline{y})) \sqsubseteq \overline{x}
\end{aligned}$$

iff
$$\forall x \in \mathcal{D}, \overline{y} \in \overline{\mathcal{D}} : \alpha(x) \sqsubseteq y \Longleftrightarrow x \subseteq \gamma(y)$$

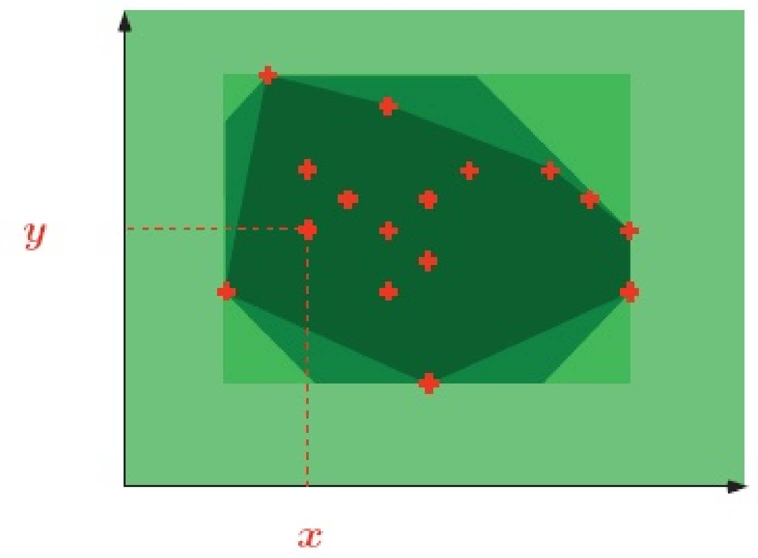
The Domain of Octagons



$$\begin{cases} 1 \le x \le 9 \\ x + y \le 77 \\ 1 \le y \le 9 \\ x - y \le 99 \end{cases}$$

Figure by P.Cousot

The Domain of Polyhedra



$$\begin{cases} 19x + 77y \le 2004 \\ 20x + 03y \ge 0 \end{cases}$$

Figure by P.Cousot

Abstract Functions

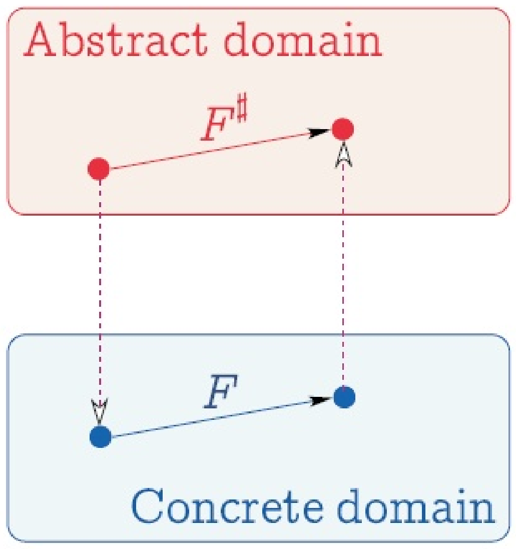


Figure by P.Cousot

There is always an optimal abstraction of the concrete function F, defined by $F^\sharp = \alpha \circ F \circ \gamma$.

However, for the correctness of the analysis it is sufficient that

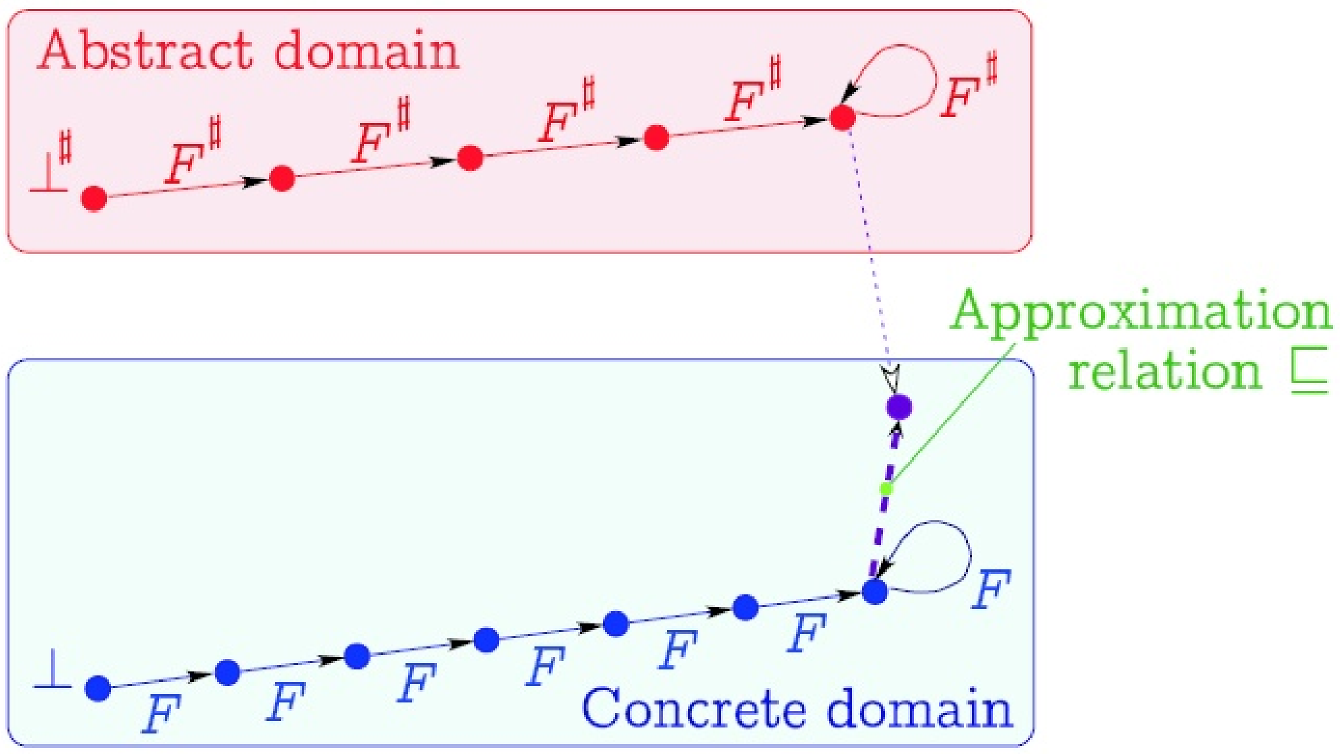
$$\forall a \in \mathcal{D}^\sharp : \gamma(F^\sharp(a)) \supseteq F(\gamma(a)).$$

Fixpoint Theorems

- **Knaster-Tarski theorem**: If $F : L \to L$ is monotone and L is a complete lattice, the set of fixpoints of F is also a complete lattice.
- **Kleene theorem**: If $F : L \to L$ is monotone, L is a complete lattice and F preserves all least upper bounds then lfp(F) is the limit of the sequence:

$$\begin{cases} F_0 = \bot \\ F_{n+1} = F(F_n) \end{cases}$$

Approximate Semantics



$\alpha(\mathsf{lfp}\ F) \sqsubseteq \mathsf{lfp}\ F^\sharp$

Figure by P.Cousot

Widening and Narrowing

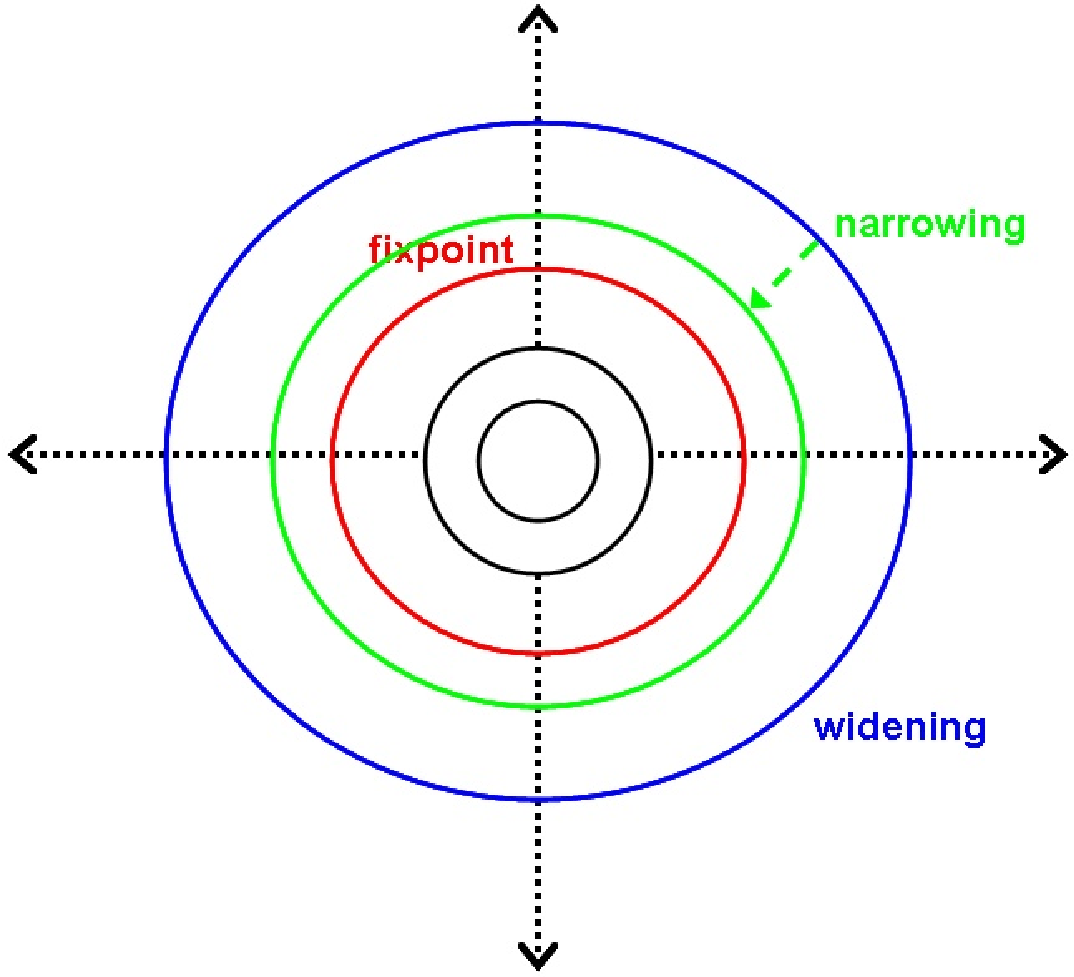


Figure by A.Venet

Convergence Acceleration by Widening

A widening operator on a partial order D is binary operator $\nabla : D \to D$ such that:

- It is an **upper bound operator**:

$$\forall x, y \in D : x \sqsubseteq x \nabla y, y \sqsubseteq x \nabla y$$

- It **enforces convergence**:

  for all increasing chains $x_0, x_1, \ldots$, the chain defined by $y_0 = x_0$, $y_{i+1} = y_i \nabla x_{i+1}$ is not strictly increasing (i.e. it converges after a finite number of steps).

Recovering Accuracy by Narrowing

A narrowing operator on a partial order D is binary operator $\Delta : D \to D$ such that:

- It is an **abstract intersection operator**:

$$\forall x, y \in D : x \sqcap y \sqsubseteq x \Delta y$$

- It **enforces convergence**:

  for all decreasing chains $x_0, x_1, \ldots$, the chain defined by $y_0 = x_0$, $y_{i+1} = y_i \Delta x_{i+1}$ is not strictly decreasing, i.e. it converges after a finite number of steps.

Widening on the interval Domain (threeshold)

Let k be a fixed positive integer constant.

$$\begin{array}{rcl}
\bot \nabla_k x & = & x \\
x \nabla_k \bot & = & x \\
[\ell_0, u_0] \nabla_k [\ell_1, u_1] & = & [\text{min}(\ell_0, \ell_1) \text{ if } \text{min}(\ell_0, \ell_1) > -k, \text{ else } -\infty \\
 & & \text{max}(u_0, u_1) \text{ if } \text{max}(u_0, u_1) < k, \text{ else } +\infty]
\end{array}$$

Observe that for all k, ∇_k is commutative, associative, and order-preserving. However, it is not reflexive. For instance, if $k = 7$ we get:

$$[-8, 4] \nabla [-8, 4] = [-\infty, 4]$$

Example: Convergence Acceleration by Widening

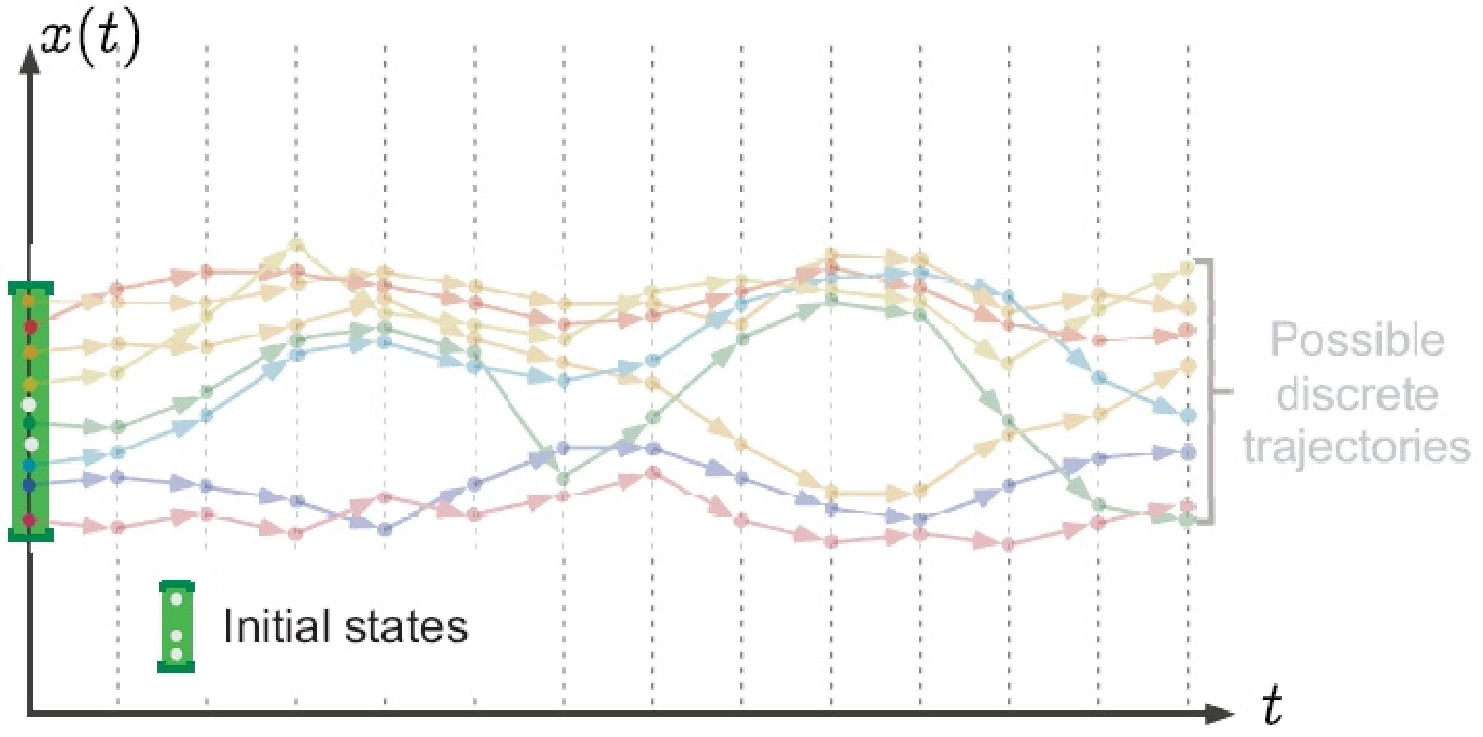


Figure by P.Cousot

Example: Convergence Acceleration by Widening !

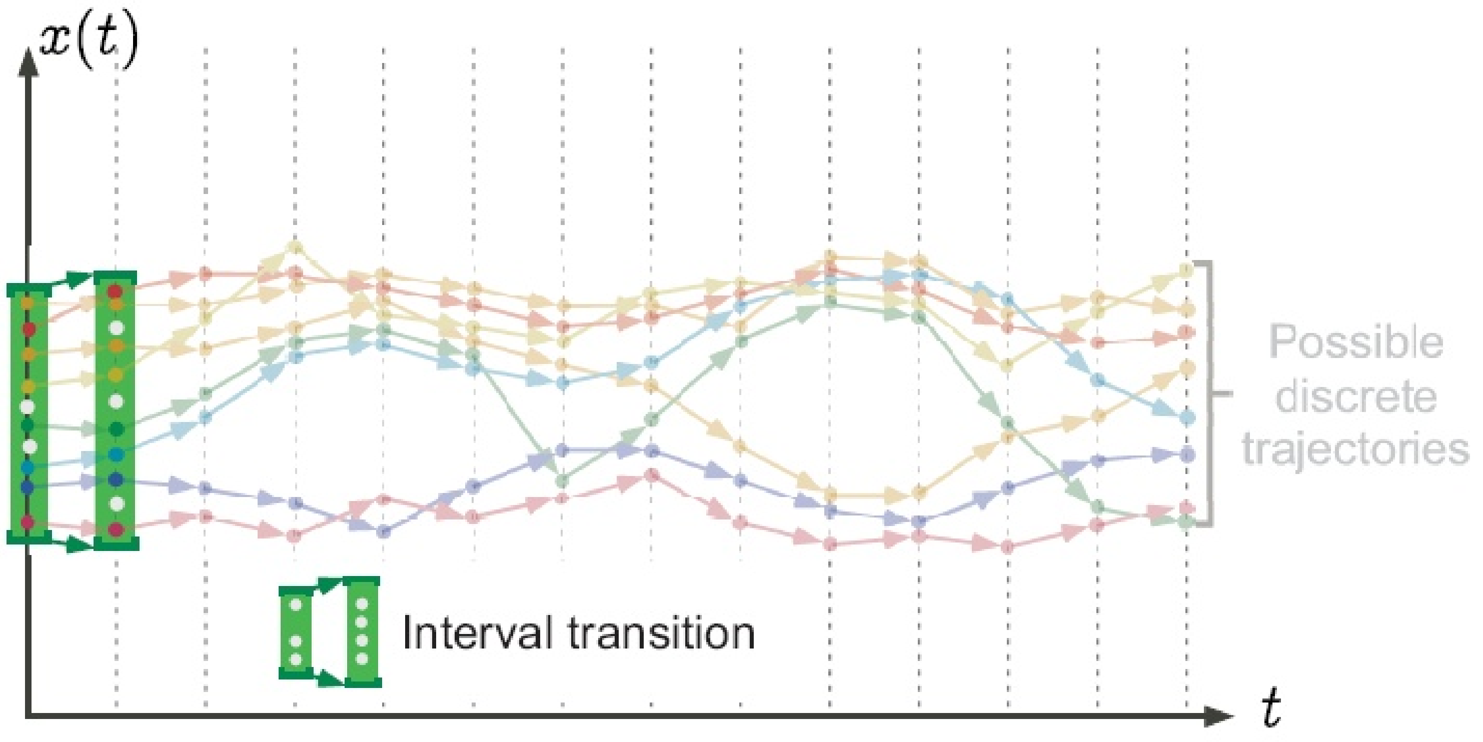


Figure by P.Cousot

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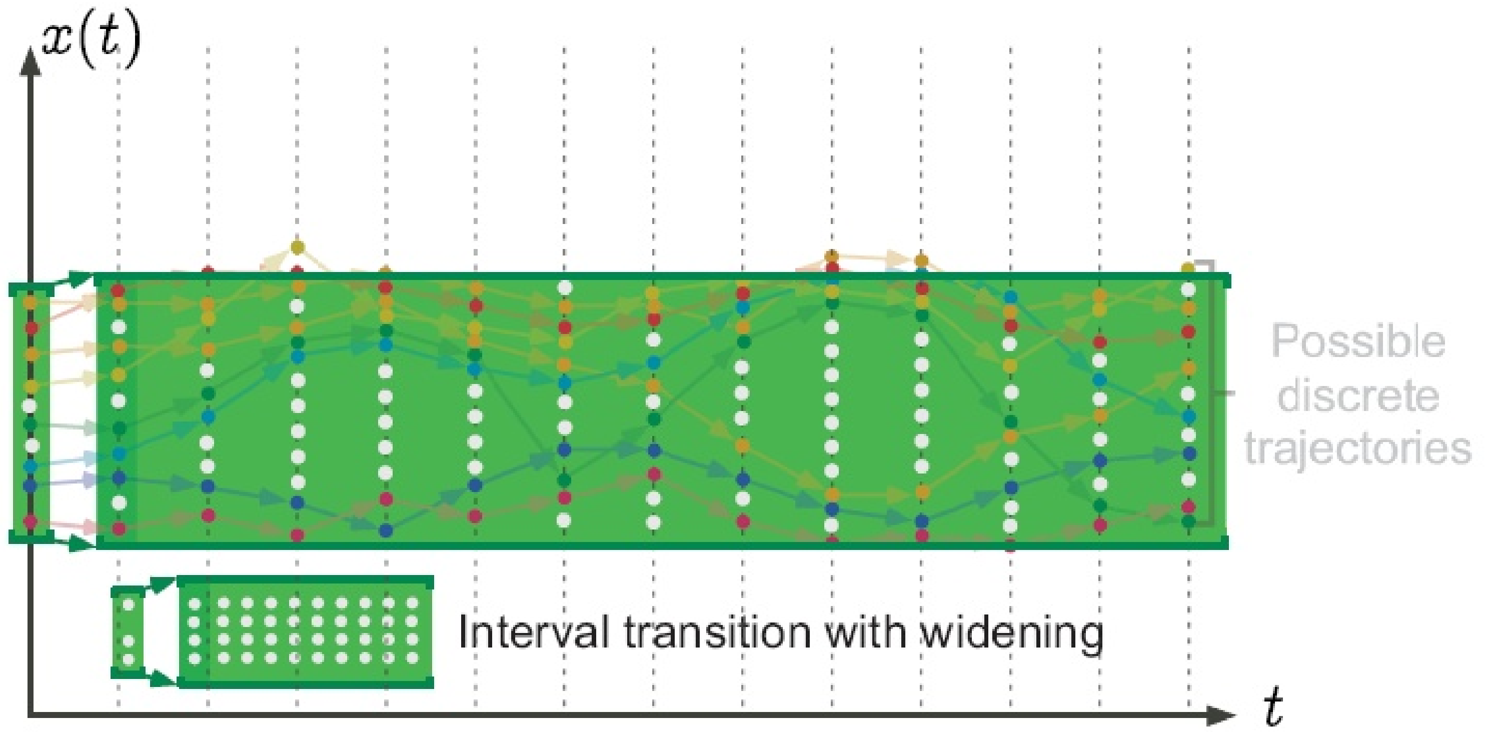
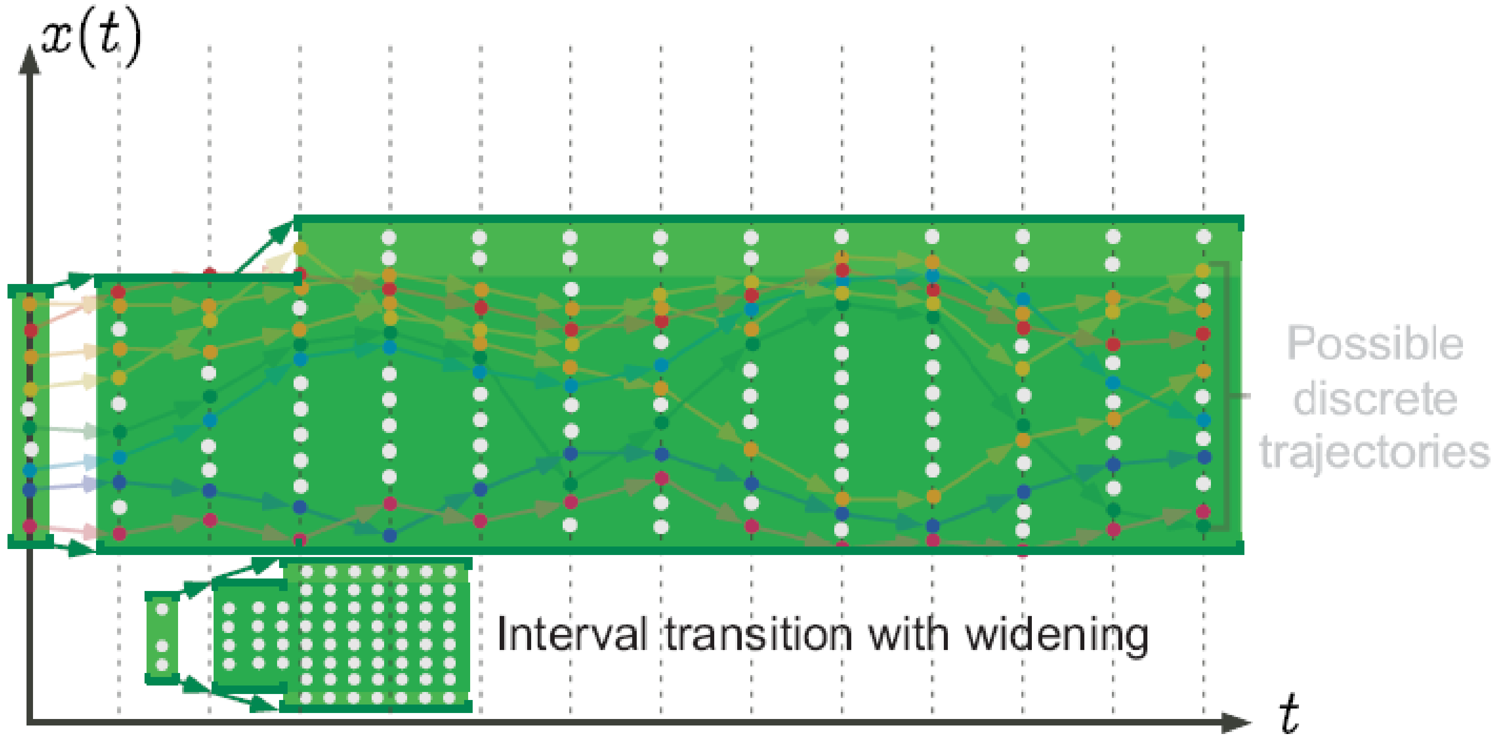

$x(t)$
Possible discrete trajectories
Interval transition with widening
t

Figure by P.Cousot

Example: Convergence Acceleration by Widening !!!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Refining the Abstract Semantics

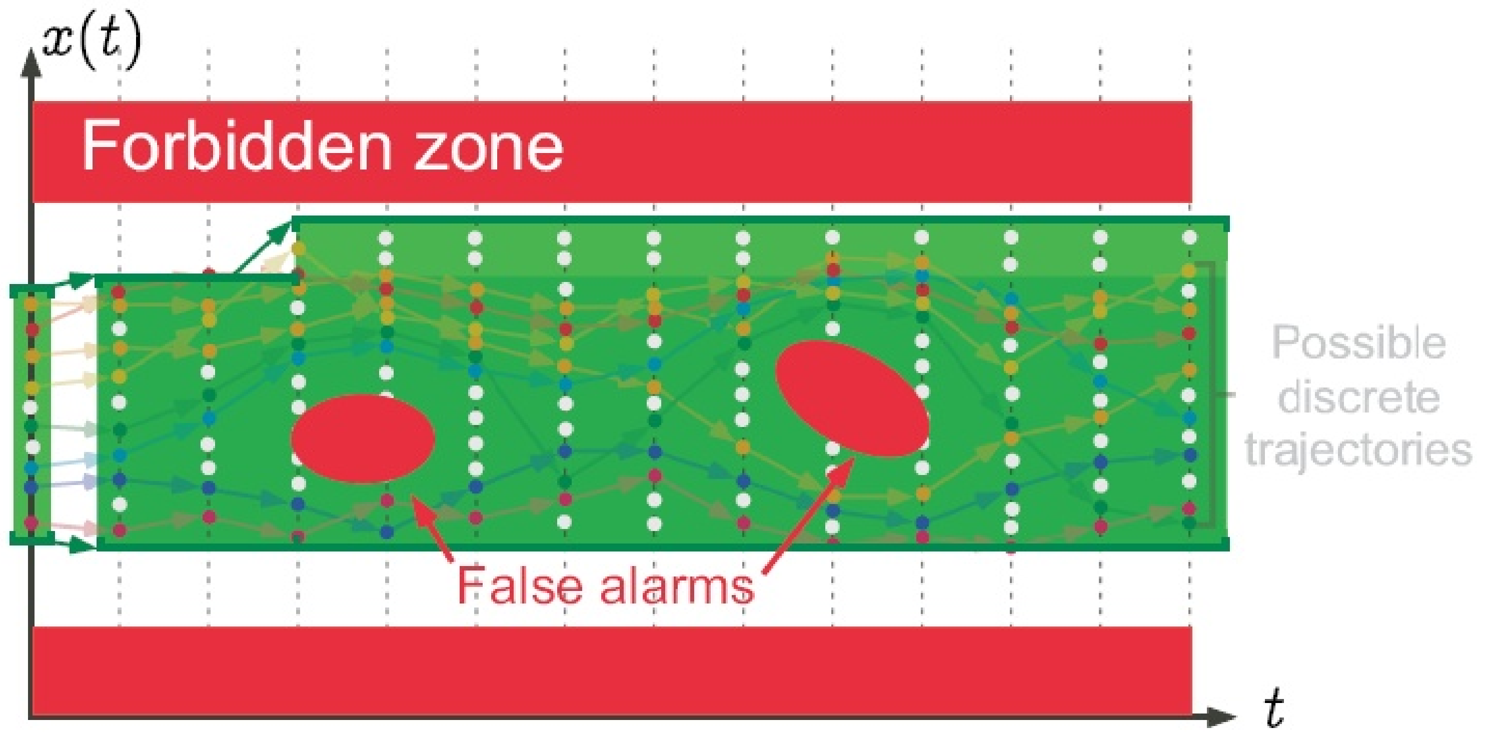

Figure by P.Cousot

Refining the Abstract Semantics: Partitioning

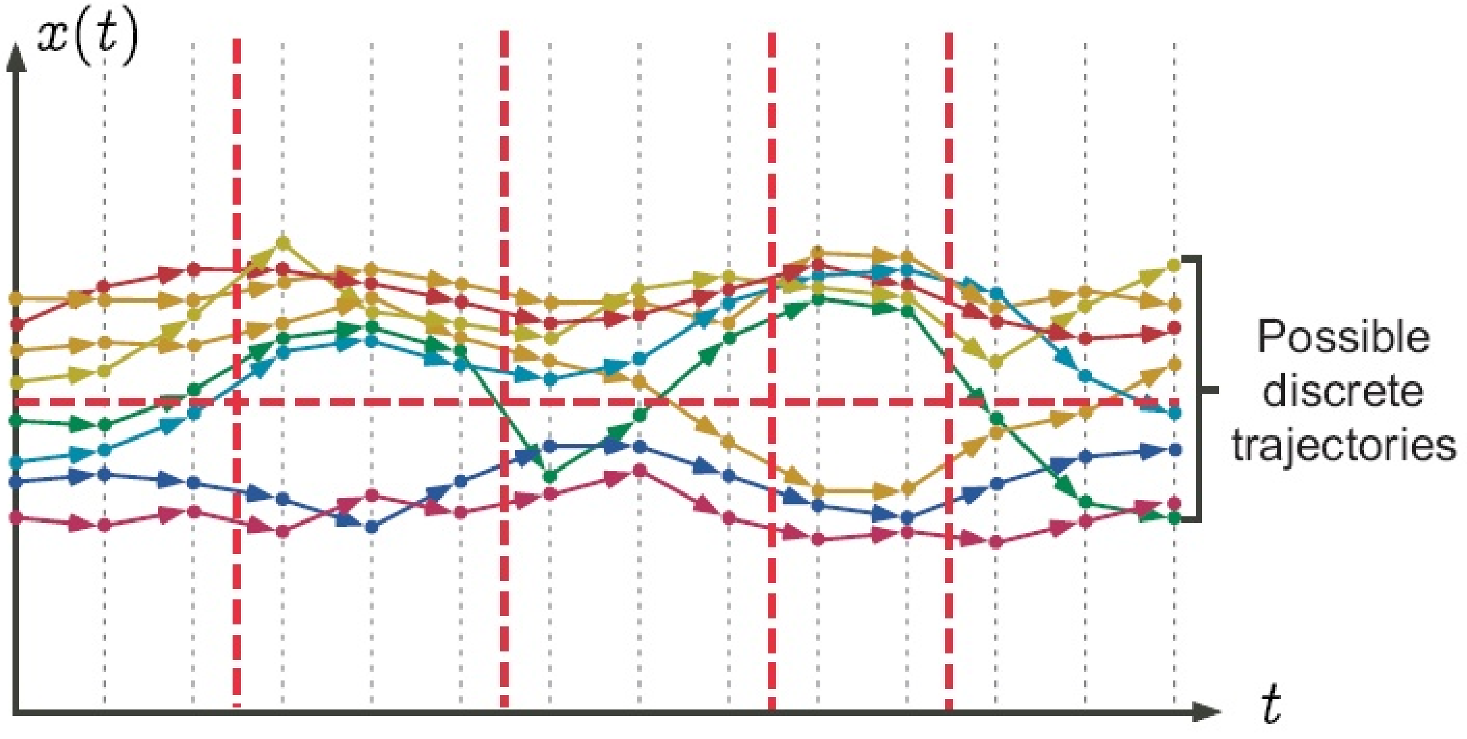

Figure by P.Cousot

Refining the Abstract Semantics: Partitioning

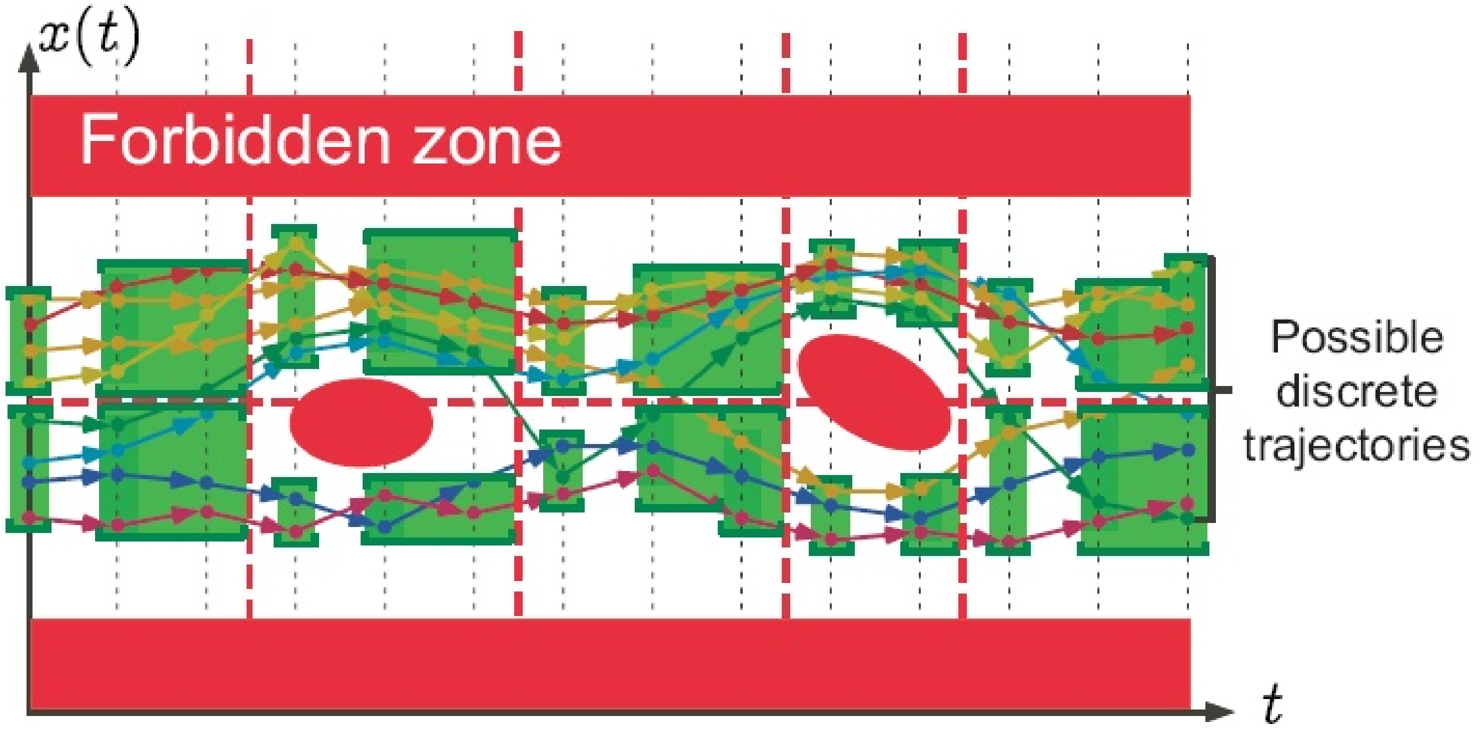

Figure by P.Cousot

Overall Architecture

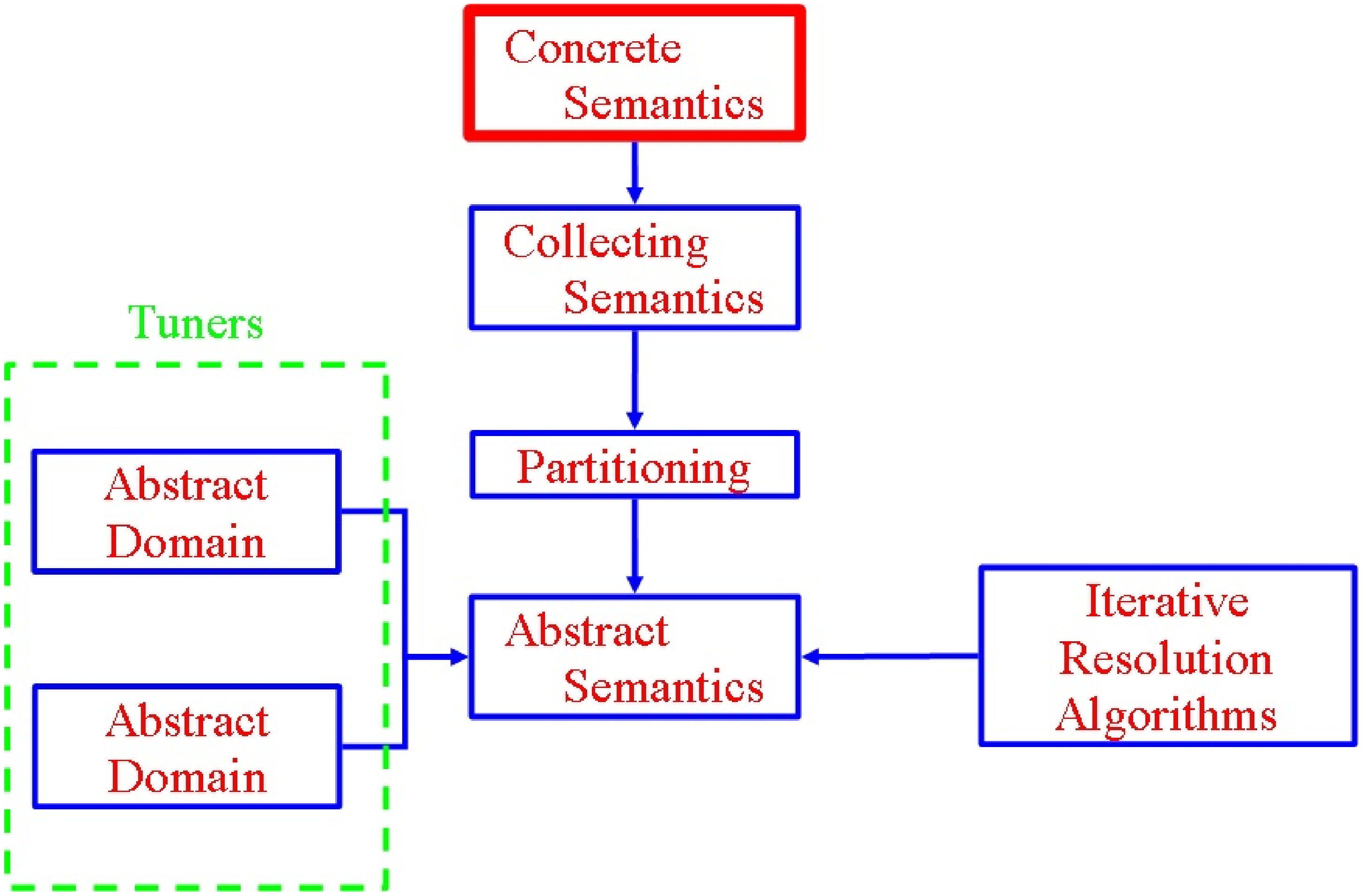

Figure by A.Venet

References

- Home page of Patrick Cousot:

  `http://www.di.ens.fr/˜cousot/`

- My papers:

  `http://www.dsi.unive.it/˜cortesi/publ.htm`