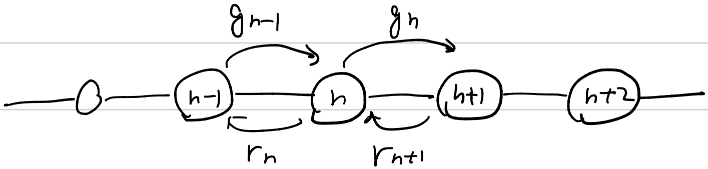
P134 (van Kampen): One step process



$$\frac{\partial P_n}{\partial t} = r_{n+1} \cdot P_{n+1} + g_{n-1} \cdot P_{n-1} - (r_n + g_n) \cdot P_n$$

$n=0$ boundary: $\frac{\partial P_0}{\partial t} = r_1 P_1 - g_0 P_0$ $\rightarrow$ Two absorbing states:

$n=N$ boundary: $\frac{\partial P_N}{\partial t} = g_{N-1} P_{N-1} - r_N P_N$ P_0 and P_N

r_{n+1}, g_{n-1} : $\begin{cases} \text{constant / linear} \\ \text{nonlinear.} \end{cases}$

Two ways of thinking: ①. transition matrix

②. master equation

① Transition matrix.

Notations and Definitions (also notes):

I. N processors with index k, where $k \in \{1, \cdots, N\}$.

II. $E(k)$: $\boxed{\text{Current}}$ energy at processor indexed k (so it keeps changing)

III. E_k: before any mixing starts, the energy at processor indexed k.

Note: during the mixing, no new energy other than $E_1 \sim E_N$ will be produced

IV. $(P_k)_{n,n'}$: transition probability of the number of occurences of E_k among the N processors. This can be modeled by Moran Process.

(P_k) is the transition matrix.

Important note: Instead of focusing on each processor, we track the number of processors that is currently having E_k.

$\rightarrow \quad (P_k)_{0,0} = (P_k)_{N,N} = 1$, absorbing.

$$\begin{cases} (P_k)_{n,n-1} = \frac{n}{N} \cdot \left(1 - n \cdot \frac{e^{-E_k/T}}{\sum_{j=1}^{N} e^{-E(j)/T}}\right) \\ (P_k)_{n,n} = \frac{n}{N} \cdot \left(n \cdot \frac{e^{-E_k/T}}{\sum_{j=1}^{N} e^{-E(j)/T}}\right) + \left(\frac{N-n}{N}\right) \cdot \left(1 - n \cdot \frac{e^{-E_k/T}}{\sum_{j=1}^{N} e^{-E(j)/T}}\right) \\ (P_k)_{n,n+1} = \left(\frac{N-n}{N}\right) \cdot \left(n \cdot \frac{e^{-E_k/T}}{\sum_{j=1}^{N} e^{-E(j)/T}}\right) \end{cases}$$

A tricky issue: $\sum_{j=1}^{N} e^{-E(j)/T}$ keeps changing all the way. If that's a constant we can do more things. Suppose it is constant M. According to Greg Lawler textbook page 27:

$$P = \begin{array}{c|c} & \begin{array}{cccccc} 0 & N & 1 & 2 & 3 \cdots & N-1 \end{array} \\ \begin{array}{c} 0 \\ N \\ 1 \\ 2 \\ \vdots \\ N-1 \end{array} & \left[\begin{array}{cc|ccc} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \hline \boxed{\begin{array}{c} 0 \\ 0 \\ \vdots \\ 0 \end{array}} & \begin{array}{c} 0 \\ 0 \\ \vdots \\ \end{array} & & Q & \end{array}\right] \end{array}$$

$\rightarrow$ Q: substochastic matrix

$\rightarrow S$

$M = (I - Q)^{-1}$, MS = probability matrix of absorbing.

$$Q_{n,n-1} = \frac{n}{N}\left(1 - n \cdot \frac{e^{-E_k/T}}{M}\right)$$

$$Q_{n,n} = \frac{n^2}{N} \cdot \frac{e^{-E_k/T}}{M} + \left(1 - \frac{n}{N}\right) \cdot \left(1 - n \cdot \frac{e^{-E_k/T}}{M}\right) = \frac{2n^2}{N} \cdot \frac{e^{-E_k/T}}{M} + 1 - \frac{n}{N} - n \cdot \frac{e^{-E_k/T}}{M}$$

$$Q_{n,n+1} = \left(1 - \frac{n}{N}\right) \cdot n \cdot \frac{e^{-E_k/T}}{M}$$

$I - Q = M^{-1}$

$$(M^{-1})_{n,n-1} = \frac{n}{N}\left(n \cdot \frac{e^{-E_k/T}}{M} - 1\right)$$

$$(M^{-1})_{n,n} = \frac{n}{N} + n \cdot \frac{e^{-E_k/T}}{M} - \frac{2n^2}{N} \cdot \frac{e^{-E_k/T}}{M}$$

$$(M^{-1})_{n,n+1} = n \cdot \left(\frac{n}{N} - 1\right) \cdot \frac{e^{-E_k/T}}{M}$$

MS = prob. of absorbing

this entry is what we want.

→ After computing MS:

$$\begin{array}{cc} & 2 \\ N-1 & \left[\begin{array}{c} \circ \\ \\ \\ \end{array}\right] \end{array}$$

↳ Final mixing scheme: $\pi = [\pi_{(1)} \cdots \pi_{(n)}]$

where $\pi_{(k)} = \dfrac{(M_k S_k)_{1,2}}{\sum_{j=1}^{N} (M_j S_j)_{1,2}}$. ▨

↓

Why not use Boltzmann again?

Since those are already probabilities.

② CME

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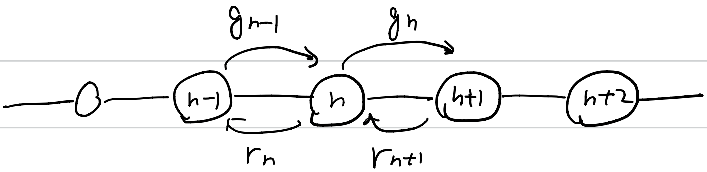

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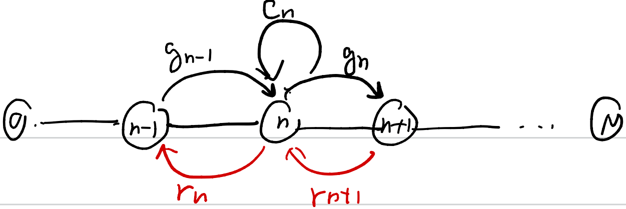
Our case:



notations: ① $(P_k)_n$: probability of having n states with E_k.

② r_n: probability per unit time that being at n, a jump occurs to n-1

g_n: probability per unit time that being at n, a jump occurs to n+1

c_n: probability per unit time that stays at n.

$\hookrightarrow$ $(r_k)_n$, $(g_k)_n$, $(c_k)_n$

$$\frac{d(P_k)_n}{dt} = (r_k)_{n+1} \cdot (P_k)_{n+1} + (g_k)_{n-1}(P_k)_{n-1} - \left[(r_k)_n + (g_k)_n\right](P_k)_n$$

$$(r_k)_{n+1} = \left(\frac{n+1}{N}\right) \cdot \left[1 - (n+1) \cdot \frac{e^{-E_k/T}}{\sum_{j=1}^{N} e^{-E(j)/T}}\right]$$

$$(g_k)_{n-1} = \left(\frac{n-1}{N}\right) \cdot \left[(n-1) \cdot \frac{e^{-E_k/T}}{\sum_{j=1}^{N} e^{-E(j)/T}}\right]$$

$$(r_k)_n = \left(\frac{n}{N}\right) \cdot \left[1 - n \cdot \frac{e^{-E_k/T}}{\sum_{j=1}^{N} e^{-E(j)/T}}\right]$$

$$(g_k)_n = \left(\frac{N-n}{N}\right)\left[n \cdot \frac{e^{-E_k/T}}{\sum_{j=1}^{N} e^{-E(j)/T}}\right]$$

$$\frac{d(p_k)_0}{dt} = r_1 p_1 ,$$

$$\frac{d(p_k)_N}{dt} = g_{N-1} p_{N-1}$$

$\rightarrow$

$(r_k)_1 + (g_k)_1$ $\quad$ $(r_k)_2$

$$\begin{bmatrix} d(p_k)_0 \\ \vdots \\ d(p_k)_N \end{bmatrix} = \begin{array}{cc} & N+1 \\ N+1 & \begin{bmatrix} 0 & r_1 & \cdots & 0 \\ (g_k)_0 & & \ddots & \\ & \ddots & & \ddots \\ 0 & \cdots & & \end{bmatrix} \end{array} \begin{bmatrix} (p_k)_0 \\ (p_k)_1 \\ \vdots \\ (p_k)_N \end{bmatrix}$$