Mixing of States

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1. Introduction

This short note focuses on modeling the mixing of states according to the fashion of the [CDR99] paper. Here I provide two ways of modeling the mixing, which are: (a) using transition matrix and (b) using master equations. Both of the modeling schemes are modified base on Moran's Process.

2. Transition Matrix

a. Notations, Definitions and Notes

I. We assume that we have **N** processors in total with each of the processor indexed by **k**, where $\mathbf{k} \in \{1, 2, ..., N\}$.

II. $\mathbf{E_k}$: the energy at processor indexed k before any mixing of states happen.

III. $\mathbf{E}(\mathbf{k})$: the current energy at processor k. Notice that it keeps changing (along the mixing), but no new energy other than $E_1 \sim E_N$ will be produced during the mixing process.

IV. $(\mathbf{P_k})_{\mathbf{n},\mathbf{n}'}$: the transition probability of the number of occurrence of E_k among the N processors. This can be modeled by Moran Process. $(\mathbf{P_k})$ is the transition matrix.

b. The Transition Matrix

Notice that for each processor $\mathbf{k} \in \{1, 2, ..., N\}$, we have a transition matrix. Note there are two absorbing states. Thus we have

$$\begin{cases} (P_k)_{0,0} = (P_k)_{N,N} = 1 \\ (P_k)_{n,n-1} = \frac{n}{N}(1 - n \cdot \frac{e^{-E_k/T}}{\sum_{j=1}^{N} e^{-E(j)/T}}),\ n \in \{1, 2, ..., N\} \\ (P_k)_{n,n} = \frac{n}{N}(n \cdot \frac{e^{-E_k/T}}{\sum_{j=1}^{N} e^{-E(j)/T}}) + (1 - \frac{n}{N})(1 - n \cdot \frac{e^{-E_k/T}}{\sum_{j=1}^{N} e^{-E(j)/T}}),\ n \in \{1, 2, ..., N\} \\ (P_k)_{n,n+1} = (1 - \frac{n}{N})(n \cdot \frac{e^{-E_k/T}}{\sum_{j=1}^{N} e^{-E(j)/T}}),\ n \in \{1, 2, ..., N-1\}. \end{cases}$$

c. Properties

There are two tricky issues for us: 1. $\sum_{j=1}^{N} e^{-E(j)/T}$ keeps changing along the mixing; 2. our case is different from Moran Process because of the energy states (instead of particle type a or A). If we can have a fair approximation for $\sum_{j=1}^{N} e^{-E(j)/T}$, denote it Z, we can simplify the problem a lot. A potential thing to discuss is how to carry out such an approximation.

Now suppose we have such an Z. According to Greg Lawler's textbook, we can calculate the absorbing probability of state 0 and N for each of the processor according to their corresponding transition matrices. Now we define sub-stochastic matrix Q_k to be an (N-1)*(N-1) matrix that characterize the transition probability from i to j, where i and j are still the number occurrence of E_k among the N processors. The difference is now we have $i, j \in \{1, 2, ..., N-1\}$.

The illustration below is useful.

$$\mathbf{T_k} = \begin{array}{c} \\ 0 \\ N \\ 1 \\ \vdots \\ N-1 \end{array} \begin{array}{c} \begin{array}{cc|cccc} 0 & N & 1 & 2 & \dots & N-1 \end{array} \\ \left(\begin{array}{cc|cccc} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ \hline (P_k)_{1,0} & 0 & (P_k)_{1,1} & (P_k)_{1,2} & \dots & 0 \\ 0 & 0 & \dots & \dots & \dots & \\ 0 & (P_k)_{N-1,N} & & & & \end{array} \right) \end{array}$$

The right-bottom block is Q_k. We also define S_k to be the left-bottom block. Then define $M_k = (I - Q)^{-1}$, we will have $R_k = M_k S_k$ to be the probability matrix of absorbing. Note R_k will have size (N-1)*2. Actually, for our case, we are only interested in the first row of each $M_k S_k$ since the occurrence of each E_k is exactly 1 before mixing. $(R_k)_{1,:}$ will simply be a row vector with two entries. The first entry denotes the probability of the long-time probability that will finally get to 0 occurrence and the second entry denotes the probability of the long-time probability that will finally get to N. I think there might be some way to use it and I propose a possible scheme on how to use it to simulate the long-time mixing in the written part.

3. Master Equation

a. Notations

I. $(p_k)_n$: probability of having n states with E_k.

II. $(r_k)_n$: probability per unit time that being at n, a jump occurs to n-1.

III. $(g_k)_n$: probability per unit time that being at n, a jump occurs to n+1.

b. Master Equation

$\frac{\partial (p_k)_n}{\partial t} = (r_k)_{n+1}(p_k)_{n+1} + (g_k)_{n-1}(p_k)_{n-1} - [(r_k)_n + (g_k)_n](p_k)_n$, where

I. $(r_k)_n = (\frac{n}{N})[1 - n \cdot \frac{e^{-E_k/t}}{\sum_{j=1}^{N} e^{-E(j)/T}}]$,

II. $(g_k)_n = (\frac{N-n}{N})[n \cdot \frac{e^{-E_k/t}}{\sum_{j=1}^{N} e^{-E(j)/T}}]$.

Using this, we may construct

$$\frac{\partial p_k}{\partial t} = \begin{bmatrix} \partial (p_k)_0 \\ \vdots \\ \partial (p_k)_N \end{bmatrix} = A_k \begin{bmatrix} (p_k)_0 \\ \vdots \\ (p_k)_N . \end{bmatrix} \tag{1}$$

Still we will have k of such systems. If we approximate $\sum_{j=1}^{N} e^{-E(j)/T}$ using Z, then we will have a constant matrix A_k, so that we can further analyze the stability related properties.

3. Possible Readings

Maybe I need to read about Multivariate Moran Process (or Multivariate Birth-and-Death Process). Also more stability analysis should be reviews (in Van Kampen textbook).