Julia Sets

Theory and Algorithms

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MATH/CAAM 435

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Quick refresher from last time:

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A set $\Lambda \subseteq \mathbb{C}$ is a Cantor set if it is a closed, totally disconnected, and perfect subset of $\mathbb{C}$. (Adapted from the version for an interval of the real line.)

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Remark

The points of S^1 are supersensitive under Q_0. That is, any open ball B around $z \in S^1$ has the property that $\bigcup_{n=0}^{\infty} Q_0^{\circ n}(B) = \mathbb{C} \setminus \{p\}$ for at most one point p.

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The Julia Set J_c is the boundary of the filled Julia set K_c. (The filled Julia set is the set of bounded points of Q_c.) We could alternatively define J_c as the closure of the set of repelling points of Q_c.

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For the quadratic map $Q_0(z) = z^2$ we saw chaotic behavior only on S^1 by angle doubling ($\theta \to 2\theta$). We also saw that $|Q_0(z)| \to \infty$ for all $|z| > 1$ and $|Q_0(z)| \to 0$ for all $|z| < 1$.

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Remark

Finally, we learned that for $Q_{-2}(z) = z^2 - 2$, we have $J_2 = K_2 = [-2, 2]$, so any $z \in \mathbb{C} \setminus [-2, 2] \to \infty$ as we compose $Q_{-2}(z)$ infinitely many times.

- Goal: discuss and prove parts of:

Theorem

If $|c|$ is sufficiently large, Λ, the set of points whose entire forward orbits lie within the circle $|z| = |c|$, is a Cantor set on which Q_c is topologically conjugate to the shift map on two symbols. All points in $\mathbb{C} - \Lambda$ tend to ∞ under iteration of Q_c. Hence, $J_c = K_c$.

How large is 'sufficiently large'? We will discuss the case where $|c| > 2$.

Theorem (The Escape Criterion)

Suppose $2 < |c| \leq |z|$. *Then we have that* $|Q_c^{\circ n}(z)| \to \infty$ *as* $n \to \infty$.

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Proof.

We use the triangle inequality to get the following estimate:

$$|Q_c(z)| \geq |z|^2 - |c| \geq |z|^2 - |z| = |z|(|z| - 1) \tag{1}$$

Since $|z| > 2$, we know that $|z| - 1 > 1$, so there exists $\lambda > 0$ such that

$$|Q_c^n(z)| \geq (1 + \lambda)^n |z| \tag{2}$$

Since $|z|$ is fixed and $(1 + \lambda)^n$ grows arbitrarily large, $|Q_c^n(z)|$ also grows arbitrarily large, as desired. $\square$

Theorem

Let D be the closed disk (i.e. $\{z : |z| \leq |c|\}$), with $|c| > 2$. Then the filled Julia set of Q_c is given by

$$K_c = \bigcap_{n \geq 0} Q_c^{\circ -n}(D)$$

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Proof.

Consider the case where $z \notin \bigcap_{n \ge 0} Q_c^{\circ -n}(D)$. Then there exists some $k \in \mathbb{N}$ such that $z \notin Q_c^{\circ -k}(D)$, so we have that $Q_c^{\circ k}(z) \notin D$. Thus by the escape criterion, z escapes to infinity under iteration of Q_c, so $z \notin K_c$.
Otherwise, if $z \in \bigcap_{n \ge 0} Q_c^{\circ -n}(D)$, then $Q_c^{\circ n}(z) \in D$ for all $n \in \mathbb{N}$. Thus z is bounded under iteration of Q_c, so $z \in K_c$. □

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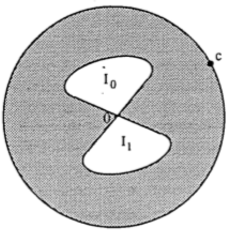
Take C to be the circle around the origin of radius $|c|$. What does $Q_c^{\circ -1}(C)$ look like? Recall from last week that $\pm\sqrt{C-c}$ is the following Lemniscate-like shape (by the Boundary Mapping Principle).



Where each 'lobe' is mapped diffeomorphically (continuously bijected) to C.

Repeating this process yields nested lobes, each a diffeomorphism to its 'parent' lobe from the previous iteration. Below is a visualization of this process for two backwards-iterations.

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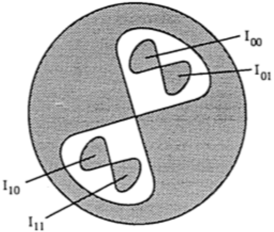

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Define

$$I_{s_0 s_1 \ldots s_n} = \{z \in D \mid z \in I_{s_0}, Q_c(z) \in I_{s_1}, \ldots, Q_c^{\circ n}(z) \in I_{s_n}\}$$

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As you may have noticed, this is identical to our previous use of symbolic dynamics (in relation to the Cantor Middle-Thirds set). By similar logic to our discussion before, the following intersection is a nonempty set.

$$\bigcap_{n \geq 0} I_{s_0 s_1 \ldots s_n}$$

This gives us that, for any $z \in \bigcap_{n \geq 0} I_{s_0 s_1 \ldots s_n}$, $Q_c^{\circ k}(z) \in D$ for all k. Hence, $z \in K_c$.

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Conversely, any $z \in K_c$ must lie in exactly one of these components, since any infinite string of 1s and 0 maps to any z by $S(z) = s_0 s_1 \ldots s_n$ provided $z \in \bigcap_{n\geq 0} I_{s_0 s_1 \ldots s_n}$.

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Similar arguments to those we used for the Cantor set yield that [1] this correspondence is continuous and [2] that each of these infinite intersections is exactly a single point in the complex plane. The mechanics behind this exceed the scope of this lecture (and class), so we will not include them here.

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When $|c| > 2$, the orbit of 0 tends to ∞. This will be important on Thursday.

This website gives an amazing visual intuition for the process we just described:

- Acko - How to Fold a Julia Fractal

Algorithm 1: Algorithm to plot K_c

Input: grid, a list of evenly-spaced complex numbers in a rectangular region of the complex plane. MaxIter, the number of iterations to perform

for $j \leftarrow 1$ *to MaxIter* do
    for *z in grid* do
        $z \leftarrow Q_c(z)$

`// Shade all points which didn't escape`

for *z in grid* do
    if $|z| \leq max(|c|, 2)$ then
        `paintBlack(`z`)`

Algorithm 2: Algorithm to plot J_c

Input: MaxIter, the maximum number of iterations to perform

```
z ← randomComplexNumber()
for j ← 1 to MaxIter do
    binaryRand ← randomBit()
                    // Pick a random backwards iteration
    if binaryRand = 0 then
        z ← √(z − c)
    else
        z ← −√(z − c)
                    // Don't plot stray points
    if j > 100 then
        paintBlack(z)
```

Explanation for an alternative plotting algorithm

The above algorithm works well for K_c but as well for J_c. To remedy this, recall that Q_c is supersensitive on J_c. This means that any neighborhood B of some point $z \in J_c$ has the property that $\bigcup_{n=0}^{\infty} Q_0^{\circ n}(B) = \mathbb{C} \setminus \{p\}$ for at most one point p.

So supersensitive points serve as ``attractors'' for the backward iteration of Q_c, in the sense that for each supersensitive point, all of $\mathbb{C}$ except at most one point must come arbitrarily close to it on reverse iteration.

- A demo of Julia-Mandelbrot correspondence
- Some pictures of generalized Julia sets