Lecture Notes – The Julia Set

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Preliminaries

As a quick recap of what we saw last week, recall the following facts.

The *stable set* of a a complex polynomial $P : \mathbb{C} \to \mathbb{C}$, denoted $S(P)$, is the complement of $J(C)$.

Another useful definition was that of a *bounded orbit*. An orbit is bounded if there exists a K such that $|Q_c^{\circ n}(z)| < K$ for all n. Otherwise the orbit is *unbounded*.

The previous group also discussed how the points of S^1 were *supersensitive*. That is, any open ball B around $z \in S^1$ has the property that $\bigcup_{n=0}^{\infty} Q_0^{\circ n}(B) = \mathbb{C} \setminus \{p\}$ for at most one point p.

We also defined the Julia Set J_c as the boundary of the filled Julia set K_c. (The filled Julia set is the set of bounded points of Q_c.) We could alternatively define J_c as the closure of the set of repelling points of Q_c (in fact, this definition isn't limited to the quadratic map; any polynomial will do, and is denoted $J(P)$).

For the quadratic map $Q_0(z) = z^2$ we saw chaotic behavior only on S^1 by angle doubling ($\theta \to 2\theta$). We also saw that $|Q_0(z)| \to \infty$ for all $|z| > 1$ and $|Q_0(z)| \to 0$ for all $|z| < 1$.

Finally, we learned that for $Q_{-2}(z) = z^2 - 2$, we have $J_2 = K_2 = [-2, 2]$, so any $z \in \mathbb{C}\ [-2, 2] \to \infty$ as we compose $Q_{-2}(z)$ infinitely many times.

16.3 – The Julia Set as a Cantor Set

Our goal for this section of the talk is to discuss and prove parts of the following theorem.

Theorem 1. *If $|c|$ is sufficiently large, Λ, the set of points whose entire forward orbits lie within the circle $|z| = |c|$, is a Cantor set on which Q_c is topologically conjugate to the shift map on two symbols. All points in $\mathbb{C} - \Lambda$ tend to ∞ under iteration of Q_c. Hence, $J_c = K_c$.*

Points Which Escape

Theorem 2. *[The Escape Criterion] Suppose $2 < |c| \leq |z|$. Then we have that $|Q_c^n(z)| \to \infty$ as $n \to \infty$.*

Proof. We use the triangle inequality to get the following estimate:

$$|Q_c^n(z)| \geq |z|^2 - |c| \geq |z|^2 - |z| = |z|(|z| - 1) \tag{1}$$

Since $|z| > 2$, we know that $|z| - 1 > 1$, so there exists $\lambda > 0$ such that

$$|Q_c^n(z)| \geq (1+\lambda)^n |z| \tag{2}$$

Since $|z|$ is fixed and $(1+\lambda)^n$ grows arbitrarily large, $|Q_c^n(z)|$ also grows arbitrarily large, as desired.

□

Points Which Escape

The Filled Julia Set

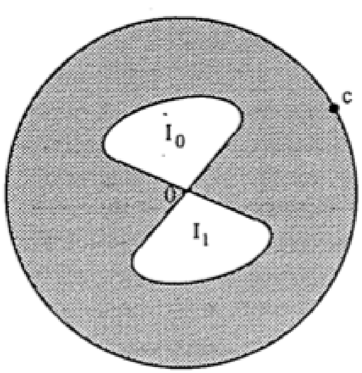
Theorem 3. *Let D be the closed disk (i.e. $\{z : |z| \leq |c|\}$), with $|c| > 2$. Then the filled Julia set of Q_c is given by*

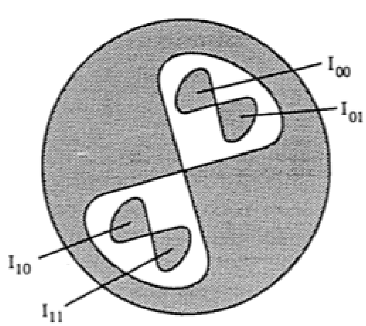
$$K_c = \bigcap_{n \geq 0} Q_c^{\circ -n}(D)$$

where $Q_c^{\circ -n}(D)$ denotes the preimage of D under $Q_c^{\circ n}$

Proof. Consider the case where $z \notin \bigcap_{n\geq 0} Q_c^{\circ -n}(D)$. Then there exists some $k \in \mathbb{N}$ such that $z \notin Q_c^{\circ -k}(D)$, so we have that $Q_c^{\circ k}(z) \notin D$. Thus by Theorem 2, z escapes to infinity under iteration of Q_c, so $z \notin K_c$.

Otherwise, if $z \in \bigcap_{n\geq 0} Q_c^{\circ -n}(D)$, then $Q_c^{\circ n}(z) \in D$ for all $n \in \mathbb{N}$. Thus z is bounded under iteration of Q_c, so $z \in K_c$. □





THE ABOVE CHARACTERIZATION of Julia sets provides a method of construction based on a given function Q_c. The key idea is that inverting the quadratic function and applying the Boundary Mapping Principle gives us a system of nested "figure-eight" lobes within lobes.

More specifically, begin with a disk of radius $|c|$ centered at the origin. We can take reverse-iterations of Q_c by subtracting c from all points, then taking their square root. The shape this process yields is one similar to the picture on the left. The lemniscate-like shape has two lobes, each of which has a diffeomorphic mapping to its entire 'parent' lobe or disk.

It is plain to see that we have 2^k unique disjoint lobes at the k^{th} iteration. We can precisely identify a unique lobe via the following. Define

$$I_{s_0 s_1 \ldots s_n} = \{z \in D \mid z \in I_{s_0}, Q_c(z) \in I_{s_1}, \ldots, Q_c^{\circ n}(z) \in I_{s_n}\}$$

Just as we did with with the quadratic map in the real interval $[0,1]$, we can identify a unique string on two symbols with each node. When those strings are of infinite length we have that, for any $z \in \bigcap_{n\geq 0} I_{s_0 s_1 \ldots s_n}$, $Q_c^{\circ k}(z) \in D$ for all k. Hence, $z \in K_c$.

This brings us to a more in-depth discussion of K_c. Define a *component* of K_c is defined as an infinite intersection of lemniscates and their lobes. Two components are necessarily disjoint.

Conversely, any $z \in K_c$ must lie in exactly one of these components, since any infinite string of 1s and 0 maps to any z by $S(z) = s_0 s_1 \dots s_n$ provided $z \in \bigcap_{n \geq 0} I_{s_0 s_1 \dots s_n}$.

Similar arguments to those we used for the Cantor set yield that [1] this correspondence is continuous and [2] that each of these infinite intersections is exactly a single point in the complex plane. The mechanics behind this exceed the scope of this lecture (and class), so we will not include them here.

We additionally give the following remarks without proof, since they would go beyond the scope of our talk. They are interesting facts that are good to know, however.

Remark 1. *Q_c is supersensitive on J_c. Roughly, (as handwavy as Devaney), this is because we can, for any point $z \in I_{s_0 s_1 \dots s_k}$, find a small enough disk D_ϵ that contains z such that for an arbitrarily large k we have $Q_c^{\circ k}(D_\epsilon) = D$ ($D = B(|c|, 0)$). Further iterations of Q_c will eventually diffeomorphically map that disk to all of $\mathbb{C}$ (except for at most 1 point).*

Remark 2. *Since K_c is a Cantor set (each component of K_c is a point) when $|c| > 2$, the Julia and filled Julia sets are identical ($K_c = J_c$).*

Remark 3. *When $|c| > 2$, the orbit of 0 tends to ∞. This was important in the talk given by the group that covered the Mandelbrot set.*

In our talk, we also showed some good images of various Julia sets. Here are some of the links we used.

This website gives an amazing visual intuition for the process we just described

- Acko - How to Fold a Julia Fractal

- A demo of Julia-Mandelbrot correspondence

- Some pictures of generalized Julia sets

Finally, in the following two sections, we discuss two algorithms for generating some of the pictures you can find on these websites.

16.4 – Computing the Filled Julia Set

Algorithm 1: Algorithm to plot K_c

Input: grid, a list of evenly-spaced complex numbers in a rectangular region of the complex plane. MaxIter, the number of iterations to perform

for $j \leftarrow 1$ *to* $MaxIter$ **do**
  for z *in grid* **do**
    $z \leftarrow Q_c(z)$

`// Shade all points which didn't escape`

for z *in grid* **do**
  if $|z| \leq max(|c|, 2)$ **then**
    `paintBlack(`z`)`

16.6 – Computing the Julia Set Another Way

If we are only interested in the Julia set itself, and not the filled Julia set, there is an even faster algorithm we can use. Recall that Q_c is supersensitive on J_c. This means that any neighborhood B of some point $z \in J_c$ has the property that $\bigcup_{n=0}^{\infty} Q_0^{\circ n}(B) = \mathbb{C} \setminus \{p\}$ for at most one point p.

Hence, upersensitive points serve as 'attractors' for the backward iteration of Q_c, in the sense that for each supersensitive point, all of $\mathbb{C}$ except at most one point must come arbitrarily close to it on reverse iteration.

Algorithm 2: Algorithm to plot J_c

Input: MaxIter, the maximum number of iterations to perform

$z \leftarrow$ `randomComplexNumber()`
for $j \leftarrow 1$ *to MaxIter* **do**
  $binaryRand \leftarrow$ `randomBit()`
  `// Pick a random backwards iteration`
  if $binaryRand = 0$ **then**
    $z \leftarrow \sqrt{(z-c)}$
  else
    $z \leftarrow -\sqrt{(z-c)}$
  `// Don't plot stray points`
  if $j > 100$ **then**
    `paintBlack(`z`)`