Julia Sets

Theory and Algorithms

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MATH/CAAM 435

April 15, 2014

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Remark

The points of S^1 are supersensitive under Q_0. That is, any open ball around $z \in S^1$ has the property that $\bigcup_{n=0}^{\infty} Q_0^{\circ n}(z) = \mathbb{C} \setminus \{p\}$ for at most one point p.

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For the quadratic map $Q_0(z) = z^2$ we saw chaotic behavior only on S^1 by angle doubling ($\theta \to 2\theta$). We also saw that $|Q_0(z)| \to \infty$ for all $|z| > 1$ and $|Q_0(z)| \to 0$ for all $|z| < 1$.

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Remark

Finally, we learned that for $Q_{-2}(z) = z^2 - 2$, we have $J_2 = K_2 = [-2, 2]$, so any $z \in \mathbb{C} \setminus [-2, 2] \to \infty$ as we compose $Q_{-2}(z)$ infinitely many times.

- Goal: discuss and prove parts of:

Theorem

If $|c|$ is sufficiently large, Λ, the set of points whose entire forward orbits lie within the circle $|z| = |c|$, is a Cantor set on which Q_c is topologically conjugate to the shift map on two symbols. All points in $\mathbb{C} - \Lambda$ tend to ∞ under iteration of Q_c. Hence, $J_c = K_c$.

Theorem (The Escape Criterion)

Suppose $2 < |c| \leq |z|$. Then we have that $|Q_c^{\circ n}(z)| \to \infty$ as $n \to \infty$.

Proof.

We use the triangle inequality to get the following estimate:

$$|Q_c(z)| \geq |z|^2 - |c| \geq |z|^2 - |z| = |z|(|z| - 1) \tag{1}$$

Since $|z| > 2$, we know that $|z| - 1 > 1$, so there exists $\lambda > 0$ such that

$$|Q_c^n(z)| \geq (1 + \lambda)^n |z| \tag{2}$$

Since $|z|$ is fixed and $(1 + \lambda)^n$ grows arbitrarily large, $|Q_c^n(z)|$ also grows arbitrarily large, as desired. □

Theorem

Let D be the closed disk (i.e. $\{z : |z| \leq |c|\}$), with $|c| > 2$. Then the filled Julia set of Q_c is given by

$$K_c = \bigcap_{n \geq 0} Q_c^{\circ -n}(D)$$

where $Q_c^{\circ -n}(D)$ denotes the preimage of D under $Q_c^{\circ n}$

Proof.

Consider the case where $z \notin \bigcap_{n \geq 0} Q_c^{\circ -n}(D)$. Then there exists some $k \in \mathbb{N}$ such that $z \notin Q_c^{\circ -k}(D)$, so we have that $Q_c^{\circ k}(z) \notin D$. Thus by the escape criterion, z escapes to infinity under iteration of Q_c, so $z \notin K_c$.

Otherwise, if $z \in \bigcap_{n \geq 0} Q_c^{\circ -n}(D)$, then $Q_c^{\circ n}(z) \in D$ for all $n \in \mathbb{N}$. Thus z is bounded under iteration of Q_c, so $z \in K_c$. $\square$

Algorithm 1: Algorithm to plot K_c

Input: Grid, a list of evenly-spaced complex numbers in a rectangular region of the complex plane

for $j \leftarrow 1$ *to N* **do**
    for *z in grid* **do**
        $z \leftarrow Q_c(z)$

for *z in grid* **do**
    if $|z| \leq max(|c|, 2)$ **then**
        `paintBlack(`z`)`

Algorithm 2: Algorithm to plot J_c

Input: MaxIter, the maximum number of iterations to perform

$z \leftarrow$ `randomComplexNumber()`
for $j \leftarrow 1$ *to MaxIter* **do**
  binaryRand $\leftarrow$ `randomBit()`
  if *binaryRand* $= 0$ **then**
    $z \leftarrow \sqrt{(z - c)}$
  else
    $z \leftarrow -\sqrt{(z - c)}$
  if $j > 100$ **then**
    `paintBlack(`z`)`