Lecture Notes – The Julia Set

Josh Lipschultz & Ricky LeVan

Preliminaries

As a quick recap of what we saw last week, recall the following facts.

The *stable set* of a a complex polynomial $P : \mathbb{C} \to \mathbb{C}$, denoted $S(P)$, is the complement of $J(C)$.

Another useful definition was that of a *bounded orbit*. An orbit is bounded if there exists a K such that $|Q_c^{\circ n}(z)| < K$ for all n. Otherwise the orbit is *unbounded*.

The previous group also discussed how the points of S^1 were *supersensitive*. That is, any open ball around $z \in S^1$ has the property that $\bigcup_{n=0}^{\infty} Q_0^{\circ n}(z) = \mathbb{C} \setminus \{p\}$ for at most one point p.

We also defined the Julia Set J_c as the boundary of the filled Julia set K_c. (The filled Julia set is the set of bounded points of Q_c.) We could alternatively define J_c as the closure of the set of repelling points of Q_c (in fact, this definition isn't limited to the quadratic map; any polynomial will do, and is denoted $J(P)$).

For the quadratic map $Q_0(z) = z^2$ we saw chaotic behavior only on S^1 by angle doubling ($\theta \to 2\theta$). We also saw that $|Q_0(z)| \to \infty$ for all $|z| > 1$ and $|Q_0(z)| \to 0$ for all $|z| < 1$.

Finally, we learned that for $Q_{-2}(z) = z^2 - 2$, we have $J_2 = K_2 = [-2, 2]$, so any $z \in \mathbb{C}\ [-2, 2] \to \infty$ as we compose $Q_{-2}(z)$ infinitely many times.

16.3 – The Julia Set as a Cantor Set

Our goal for this section of the talk is to discuss and prove parts of the following theorem.

Theorem 1. *If $|c|$ is sufficiently large, Λ, the set of points whose entire forward orbits lie within the circle $|z| = |c|$, is a Cantor set on which Q_c is topologically conjugate to the shift map on two symbols. All points in $\mathbb{C} - \Lambda$ tend to ∞ under iteration of Q_c. Hence, $J_c = K_c$.*

Points Which Escape

Theorem 2. *[The Escape Criterion] Suppose $2 < |c| \leq |z|$. Then we have that $|Q_c^n(z)| \to \infty$ as $n \to \infty$.*

Proof. We use the triangle inequality to get the following estimate:

$$|Q_c^n(z)| \geq |z|^2 - |c| \geq |z|^2 - |z| = |z|(|z| - 1) \tag{1}$$

Since $|z| > 2$, we know that $|z| - 1 > 1$, so there exists $\lambda > 0$ such that

$$|Q_c^n(z)| \geq (1+\lambda)^n |z| \tag{2}$$

Since $|z|$ is fixed and $(1+\lambda)^n$ grows arbitrarily large, $|Q_c^n(z)|$ also grows arbitrarily large, as desired.

□

Corollary 1.

The Filled Julia Set

Theorem 3. *Let D be the closed disk (i.e. $\{z : |z| \leq |c|\}$), with $|c| > 2$. Then the filled Julia set of Q_c is given by*

$$K_c = \bigcap_{n \geq 0} Q_c^{\circ -n}(D)$$

where $Q_c^{\circ -n}(D)$ denotes the preimage of D under $Q_c^{\circ n}$

Proof. Consider the case where $z \notin \bigcap_{n \geq 0} Q_c^{\circ -n}(D)$. Then there exists some $k \in \mathbb{N}$ such that $z \notin Q_c^{\circ -k}(D)$, so we have that $Q_c^{\circ k}(z) \notin D$. Thus by Theorem 2, z escapes to infinity under iteration of Q_c, so $z \notin K_c$.

Otherwise, if $z \in \bigcap_{n \geq 0} Q_c^{\circ -n}(D)$, then $Q_c^{\circ n}(z) \in D$ for all $n \in \mathbb{N}$. Thus z is bounded under iteration of Q_c, so $z \in K_c$. □

The above characterization of Julia sets provides a method of construction based on a given function Q_c. The key idea is that inverting the quadratic function and applying the Boundary Mapping Principle gives us a system of nested "figure-eight" lobes within lobes.

16.4 – Computing the Filled Julia Set

16.6 – Computing the Julia Set Another Way