Lecture Notes – The Julia Set

Josh Lipschultz & Ricky LeVan

Preliminaries

As a quick recap of what we saw last week, recall the following facts. First, for the quadratic map $Q_0(z) = z^2$ we saw chaotic behavior only on S^1 by angle doubling ($\theta \to 2\theta$). We also saw that $|Q_0(z)| \to \infty$ for all $|z| > 1$ and $|Q_0(z)| \to 0$ for all $|z| < 1$.

We also defined the Julia Set J_c as the boundary of the filled Julia set K_c. (The filled Julia set is the set of bounded points of Q_c.) We could alternatively define J_c as the closure of the set of repelling points of Q_c (in fact, this definition isn't limited to the quadratic map; any polynomial will do, and is denoted $J(P)$ for some $P : \mathbb{C} \to \mathbb{C}$).

The *stable set* of a a complex polynomial P, denoted $S(P)$, is the complement of $J(C)$.

Another useful definition was that of a *bounded orbit*. An orbit is bounded if there exists a K such that $|Q_c^{\circ n}(z)| < K$ for all n. Otherwise the orbit is *unbounded*.

The previous group also discussed how the points of S^1 were *supersensitive*. That is, any open ball around $z \in S^1$ has the property that $\bigcup_{n=0}^{\infty} Q_0^{\circ n}(z) = \mathbb{C} \setminus \{p\}$ for at most one point p.

Finally, as a reminder, the *filled Julia set* is the set of bounded points of Q_c, and the Julia set

16.3 – The Julia Set as a Cantor Set

For this section of the talk, we will consider the family of quadratic functions $Q_c(z) = z^2 + c$ when $|c| > 2$.

The first big Theorem we will discuss, not in complete detail, is the following.

Theorem 1. *Suppose $|c| > 2$. Then $J_c = K_c$ is a Cantor set. Moreover, the quadratic map Q_c, when restricted to J_c, is conjugate to the shift map on two symbols.*

–?– So we're not proving this theorem, but Devaney seems ambiguous here. What is the definition of a Cantor set in $\mathbb{C}$ anyway? Is he referring to $J_c = K_c$ being a totally disconnected set?

Theorem 2 (The Escape Criterion)**.** *Suppose $|z| \geq |c| > 2$. Then we have that $|Q_c^n(z)| \to \infty$ as $n \to \infty$.*

Proof. We use the triangle inequality to get the following estimate:

$$|Q_c^n(z)| \geq |z|^2 - |c| \geq |z|^2 - |z| = |z|(|z| - 1) \tag{1}$$

Since $|z| > 2$, we know that $|z| - 1 > 1$, so there exists $\lambda > 0$ such that

$$|Q_c^n(z)| \geq (1+\lambda)^n |z| \tag{2}$$

Since $|z|$ is fixed and $(1+\lambda)^n$ grows arbitrarily large, $|Q_c^n(z)|$ also grows arbitrarily large, as desired.

□

Corollary 1.

Corollary 2.

Corollary 3.

16.4 – Computing the Filled Julia Set

16.5 – Computing the Julia Set Another Way