Julia Sets

Theory and Algorithms

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MATH/CAAM 435

April 15, 2014

Quick refresher from last time:

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Remark

The points of S^1 are supersensitive under Q_0. That is, any open ball B around $z \in S^1$ has the property that $\bigcup_{n=0}^{\infty} Q_0^{\circ n}(B) = \mathbb{C} \setminus \{p\}$ for at most one point p.

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The Julia Set J_c is the boundary of the filled Julia set K_c. (The filled Julia set is the set of bounded points of Q_c.) We could alternatively define J_c as the closure of the set of repelling points of Q_c.

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Remark

For the quadratic map $Q_0(z) = z^2$ we saw chaotic behavior only on S^1 by angle doubling ($\theta \to 2\theta$). We also saw that $|Q_0(z)| \to \infty$ for all $|z| > 1$ and $|Q_0(z)| \to 0$ for all $|z| < 1$.

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Remark

Finally, we learned that for $Q_{-2}(z) = z^2 - 2$, we have $J_2 = K_2 = [-2, 2]$, so any $z \in \mathbb{C} \setminus [-2, 2] \to \infty$ as we compose $Q_{-2}(z)$ infinitely many times.

- Goal: discuss and prove parts of:

Theorem

If $|c|$ is sufficiently large, Λ, the set of points whose entire forward orbits lie within the circle $|z| = |c|$, is a Cantor set on which Q_c is topologically conjugate to the shift map on two symbols. All points in $\mathbb{C} - \Lambda$ tend to ∞ under iteration of Q_c. Hence, $J_c = K_c$.

How large is 'sufficiently large'? We will discuss the case where $|c| > 2$.

Theorem (The Escape Criterion)

Suppose $2 < |c| \leq |z|$. Then we have that $|Q_c^{\circ n}(z)| \to \infty$ as $n \to \infty$.

Proof.

We use the triangle inequality to get the following estimate:

$$|Q_c(z)| \geq |z|^2 - |c| \geq |z|^2 - |z| = |z|(|z| - 1) \tag{1}$$

Since $|z| > 2$, we know that $|z| - 1 > 1$, so there exists $\lambda > 0$ such that

$$|Q_c^n(z)| \geq (1 + \lambda)^n |z| \tag{2}$$

Since $|z|$ is fixed and $(1 + \lambda)^n$ grows arbitrarily large, $|Q_c^n(z)|$ also grows arbitrarily large, as desired. □

Theorem

Let D be the closed disk (i.e. $\{z : |z| \leq |c|\}$), with $|c| > 2$. Then the filled Julia set of Q_c is given by

$$K_c = \bigcap_{n \geq 0} Q_c^{\circ -n}(D)$$

where $Q_c^{\circ -n}(D)$ denotes the preimage of D under $Q_c^{\circ n}$

Proof.

Consider the case where $z \notin \bigcap_{n \geq 0} Q_c^{\circ -n}(D)$. Then there exists some $k \in \mathbb{N}$ such that $z \notin Q_c^{\circ -k}(D)$, so we have that $Q_c^{\circ k}(z) \notin D$. Thus by the escape criterion, z escapes to infinity under iteration of Q_c, so $z \notin K_c$.
Otherwise, if $z \in \bigcap_{n \geq 0} Q_c^{\circ -n}(D)$, then $Q_c^{\circ n}(z) \in D$ for all $n \in \mathbb{N}$. Thus z is bounded under iteration of Q_c, so $z \in K_c$. $\square$

Let's return to the theorem we wish to prove.

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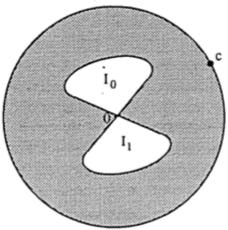
Take C to be the circle around the origin of radius $|c|$. What does $Q_c^{\circ -1}(C)$ look like? Recall from last week that $\pm\sqrt{C - c}$ is the following Lemniscate-like shape (by the Boundary Mapping Principle).



Where each 'lobe' is mapped diffeomorphically (continuously bijected) to C.

Repeating this process yields nested lobes, each a diffeomorphism to its 'parent' lobe from the previous iteration. Below is a visualization of this process for two backwards-iterations.

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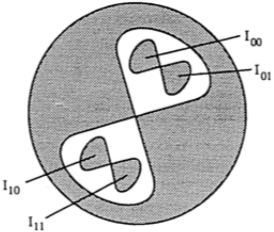

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Define

$$I_{s_0 s_1 \ldots s_n} = \{z \in D \mid z \in I_{s_0}, Q_c(z) \in I_{s_1}, \ldots, Q_c^{\circ n}(z) \in I_{s_n}\}$$

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As you may have noticed, this is identical to our previous use of symbolic dynamics (in relation to the Cantor Middle-Thirds set). By similar logic to our discussion before, the following intersection is a nonempty set.

$$\bigcap_{n \geq 0} I_{s_0 s_1 \ldots s_n}$$

This gives us that, for any $z \in \cap_{n \geq 0} I_{s_0 s_1 \ldots s_n}$, $Q_c^{\circ k}(z) \in D$ for all k. Hence, $z \in K_c$.

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Conversely, any $z \in K_c$ must lie in exactly one of these components, since any infinite string of 1s and 0 maps to any z by $S(z) = s_0 s_1 \ldots s_n$ provided $z \in \cap_{n\geq 0} I_{s_0 s_1 \ldots s_n}$.

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Similar arguments to those we used for the Cantor set yield that [1] this correspondence is continuous and [2] that each of these infinite intersections is exactly a single point in the complex plane. The mechanics behind this exceed the scope of this lecture (and class), so we will not include them here.

To wrap this up, here are a few remarks:

Remark

Q_c is supersensitive on J_c. (As handwavy as Devaney)

Remark

Since K_c is a Cantor set (each component of K_c is a point) when $|c| > 2$, the Julia and filled Julia sets are identical ($K_c = J_c$).

Remark

When $|c| > 2$, the orbit of 0 tents to ∞. This will be important on Thursday.

Algorithm 1: Algorithm to plot K_c

Input: grid, a list of evenly-spaced complex numbers in a rectangular region of the complex plane

for $j \leftarrow 1$ *to* N **do**
    for z *in grid* **do**
        $z \leftarrow Q_c(z)$

`// Shade all points which didn't escape`

for z *in grid* **do**
    if $|z| \leq max(|c|, 2)$ **then**
        `paintBlack(`z`)`

Explanation for an alternative plotting algorithm

The above algorithm works well for K_c but as well for J_c. To remedy this, recall that Q_c is supersensitive on J_c. This means that any neighborhood B of some point $z \in J_c$ has the property that $\bigcup_{n=0}^{\infty} Q_0^{\circ n}(B) = \mathbb{C} \setminus \{p\}$ for at most one point p.

So supersensitive points serve as ``attractors'' for the backward iteration of Q_c, in the sense that for each supersensitive point, all of $\mathbb{C}$ except at most one point must come arbitrarily close to it on reverse iteration.

Algorithm 2: Algorithm to plot J_c

Input: MaxIter, the maximum number of iterations to perform

```
z ← randomComplexNumber()
for j ← 1 to MaxIter do
    binaryRand ← randomBit()
                        // Pick a random backwards iteration
    if binaryRand = 0 then
        z ← √(z − c)
    else
        z ← −√(z − c)
                        // Don't plot stray points
    if j > 100 then
        paintBlack(z)
```