



Quick refresher from last time:

Quick refresher from last time:

## Definition

*A set  $\Lambda \subseteq \mathbb{C}$  is a Cantor set if it is a closed, totally disconnected, and perfect subset of  $\mathbb{C}$ . (Adapted from the version for an interval of the real line.)*

Quick refresher from last time:

## Definition

*A set  $\Lambda \subseteq \mathbb{C}$  is a Cantor set if it is a closed, totally disconnected, and perfect subset of  $\mathbb{C}$ . (Adapted from the version for an interval of the real line.)*

## Definition

*An orbit is bounded if there exists a  $K$  such that  $|Q_c^{on}(z)| < K$  for all  $n$ . Otherwise the orbit is unbounded.*

Quick refresher from last time:

## Definition

*A set  $\Lambda \subseteq \mathbb{C}$  is a Cantor set if it is a closed, totally disconnected, and perfect subset of  $\mathbb{C}$ . (Adapted from the version for an interval of the real line.)*

## Definition

*An orbit is bounded if there exists a  $K$  such that  $|Q_c^{on}(z)| < K$  for all  $n$ . Otherwise the orbit is unbounded.*

## Remark

*The points of  $S^1$  are supersensitive under  $Q_0$ . That is, any open ball  $B$  around  $z \in S^1$  has the property that  $\bigcup_{n=0}^{\infty} Q_0^{on}(B) = \mathbb{C} \setminus \{p\}$  for at most one point  $p$ .*

## Definition

*The Julia Set  $J_c$  is the boundary of the filled Julia set  $K_c$ . (The filled Julia set is the set of bounded points of  $Q_c$ .) We could alternatively define  $J_c$  as the closure of the set of repelling points of  $Q_c$ .*

## Definition

*The Julia Set  $J_c$  is the boundary of the filled Julia set  $K_c$ . (The filled Julia set is the set of bounded points of  $Q_c$ .) We could alternatively define  $J_c$  as the closure of the set of repelling points of  $Q_c$ .*

## Remark

*For the quadratic map  $Q_0(z) = z^2$  we saw chaotic behavior only on  $S^1$  by angle doubling ( $\theta \rightarrow 2\theta$ ). We also saw that  $|Q_0(z)| \rightarrow \infty$  for all  $|z| > 1$  and  $|Q_0(z)| \rightarrow 0$  for all  $|z| < 1$ .*

## Definition

*The Julia Set  $J_c$  is the boundary of the filled Julia set  $K_c$ . (The filled Julia set is the set of bounded points of  $Q_c$ .) We could alternatively define  $J_c$  as the closure of the set of repelling points of  $Q_c$ .*

## Remark

*For the quadratic map  $Q_0(z) = z^2$  we saw chaotic behavior only on  $S^1$  by angle doubling ( $\theta \rightarrow 2\theta$ ). We also saw that  $|Q_0(z)| \rightarrow \infty$  for all  $|z| > 1$  and  $|Q_0(z)| \rightarrow 0$  for all  $|z| < 1$ .*

## Remark

*Finally, we learned that for  $Q_{-2}(z) = z^2 - 2$ , we have  $J_2 = K_2 = [-2, 2]$ , so any  $z \in \mathbb{C} \setminus [-2, 2] \rightarrow \infty$  as we compose  $Q_{-2}(z)$  infinitely many times.*



- Goal: discuss and prove parts of:

## Theorem

*If  $|c|$  is sufficiently large,  $\Lambda$ , the set of points whose entire forward orbits lie within the circle  $|z| = |c|$ , is a Cantor set on which  $Q_c$  is topologically conjugate to the shift map on two symbols. All points in  $\mathbb{C} - \Lambda$  tend to  $\infty$  under iteration of  $Q_c$ . Hence,  $J_c = K_c$ .*

How large is 'sufficiently large'? We will discuss the case where  $|c| > 2$ .

## Theorem (The Escape Criterion)

*Suppose  $2 < |c| \leq |z|$ . Then we have that  $|Q_c^{on}(z)| \rightarrow \infty$  as  $n \rightarrow \infty$ .*

## Theorem (The Escape Criterion)

*Suppose  $2 < |c| \leq |z|$ . Then we have that  $|Q_c^n(z)| \rightarrow \infty$  as  $n \rightarrow \infty$ .*

### Proof.

We use the triangle inequality to get the following estimate:

$$|Q_c(z)| \geq |z|^2 - |c| \geq |z|^2 - |z| = |z|(|z| - 1) \quad (1)$$

Since  $|z| > 2$ , we know that  $|z| - 1 > 1$ , so there exists  $\lambda > 0$  such that

$$|Q_c^n(z)| \geq (1 + \lambda)^n |z| \quad (2)$$

Since  $|z|$  is fixed and  $(1 + \lambda)^n$  grows arbitrarily large,  $|Q_c^n(z)|$  also grows arbitrarily large, as desired.  $\square$

## Theorem

*Let  $D$  be the closed disk (i.e.  $\{z : |z| \leq |c|\}$ ), with  $|c| > 2$ . Then the filled Julia set of  $Q_c$  is given by*

$$K_c = \bigcap_{n \geq 0} Q_c^{\circ - n}(D)$$

*where  $Q_c^{\circ - n}(D)$  denotes the preimage of  $D$  under  $Q_c^{\circ n}$*

## Theorem

Let  $D$  be the closed disk (i.e.  $\{z : |z| \leq |c|\}$ ), with  $|c| > 2$ . Then the filled Julia set of  $Q_c$  is given by

$$K_c = \bigcap_{n \geq 0} Q_c^{\circ - n}(D)$$

where  $Q_c^{\circ - n}(D)$  denotes the preimage of  $D$  under  $Q_c^{\circ n}$

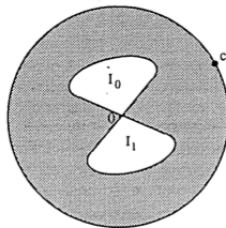
## Proof.

Consider the case where  $z \notin \bigcap_{n \geq 0} Q_c^{\circ - n}(D)$ . Then there exists some  $k \in \mathbb{N}$  such that  $z \notin Q_c^{\circ - k}(D)$ , so we have that  $Q_c^{\circ k}(z) \notin D$ . Thus by the escape criterion,  $z$  escapes to infinity under iteration of  $Q_c$ , so  $z \notin K_c$ .

Otherwise, if  $z \in \bigcap_{n \geq 0} Q_c^{\circ - n}(D)$ , then  $Q_c^{\circ n}(z) \in D$  for all  $n \in \mathbb{N}$ . Thus  $z$  is bounded under iteration of  $Q_c$ , so  $z \in K_c$ . □

Let's return to the theorem we wish to prove.

Take  $C$  to be the circle around the origin of radius  $|c|$ . What does  $Q_c^{-1}(C)$  look like? Recall from last week that  $\pm\sqrt{C-c}$  is the following Lemniscate-like shape (by the Boundary Mapping Principle).

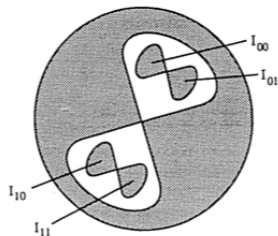


Where each 'lobe' is mapped diffeomorphically (continuously bijected) to  $C$ .

Repeating this process yields nested lobes, each a diffeomorphism to its 'parent' lobe from the previous iteration. Below is a visualization of this process for two backwards-iterations.



Repeating this process yields nested lobes, each a diffeomorphism to its 'parent' lobe from the previous iteration. Below is a visualization of this process for two backwards-iterations.



Each lobe of the  $k^{\text{th}}$  iteration of this process can be identified by a 0 or 1, as each lobe diffeomorphically maps to two inner lobes during each iteration.

Each lobe of the  $k^{\text{th}}$  iteration of this process can be identified by a 0 or 1, as each lobe diffeomorphically maps to two inner lobes during each iteration.

We therefore have  $2^k$  unique lobes after  $k$  iterations.

Each lobe of the  $k^{\text{th}}$  iteration of this process can be identified by a 0 or 1, as each lobe diffeomorphically maps to two inner lobes during each iteration.

We therefore have  $2^k$  unique lobes after  $k$  iterations.

Define

$$I_{s_0 s_1 \dots s_n} = \{z \in D \mid z \in I_{s_0}, Q_c(z) \in I_{s_1}, \dots, Q_c^{s_n}(z) \in I_{s_n}\}$$

Each lobe of the  $k^{\text{th}}$  iteration of this process can be identified by a 0 or 1, as each lobe diffeomorphically maps to two inner lobes during each iteration.

We therefore have  $2^k$  unique lobes after  $k$  iterations.

Define

$$I_{s_0 s_1 \dots s_n} = \{z \in D \mid z \in I_{s_0}, Q_c(z) \in I_{s_1}, \dots, Q_c^{\circ n}(z) \in I_{s_n}\}$$

As you may have noticed, this is identical to our previous use of symbolic dynamics (in relation to the Cantor Middle-Thirds set). By similar logic to our discussion before, the following intersection is a nonempty set.

$$\bigcap_{n \geq 0} I_{s_0 s_1 \dots s_n}$$

This gives us that, for any  $z \in \cap_{n \geq 0} I_{s_0 s_1 \dots s_n}$ ,  $Q_c^{ok}(z) \in D$  for all  $k$ .  
Hence,  $z \in K_c$ .

This gives us that, for any  $z \in \cap_{n \geq 0} I_{s_0 s_1 \dots s_n}$ ,  $Q_c^{ok}(z) \in D$  for all  $k$ .  
Hence,  $z \in K_c$ .

A *component* of  $K_c$  is defined as an infinite intersection of figure-eights and their lobes. Two components are necessarily disjoint.

This gives us that, for any  $z \in \cap_{n \geq 0} I_{s_0 s_1 \dots s_n}$ ,  $Q_c^{ok}(z) \in D$  for all  $k$ . Hence,  $z \in K_c$ .

A *component* of  $K_c$  is defined as an infinite intersection of figure-eights and their lobes. Two components are necessarily disjoint.

Conversely, any  $z \in K_c$  must lie in exactly one of these components, since any infinite string of 1s and 0 maps to any  $z$  by  $S(z) = s_0 s_1 \dots s_n$  provided  $z \in \cap_{n \geq 0} I_{s_0 s_1 \dots s_n}$ .



This gives us that, for any  $z \in \cap_{n \geq 0} I_{s_0 s_1 \dots s_n}$ ,  $Q_c^{ok}(z) \in D$  for all  $k$ .  
Hence,  $z \in K_c$ .

A *component* of  $K_c$  is defined as an infinite intersection of figure-eights and their lobes. Two components are necessarily disjoint.

Conversely, any  $z \in K_c$  must lie in exactly one of these components, since any infinite string of 1s and 0 maps to any  $z$  by  $S(z) = s_0 s_1 \dots s_n$  provided  $z \in \cap_{n \geq 0} I_{s_0 s_1 \dots s_n}$ .

Similar arguments to those we used for the Cantor set yield that [1] this correspondence is continuous and [2] that each of these infinite intersections is exactly a single point in the complex plane. The mechanics behind this exceed the scope of this lecture (and class), so we will not include them here.

To wrap this up, here are a few remarks:

To wrap this up, here are a few remarks:

## Remark

$Q_c$  is supersensitive on  $J_c$ . Roughly, (as handwavy as Devaney), this is because we can, for any point  $z \in I_{s_0 s_1 \dots s_k}$ , find a small enough disk that contains  $z$  such that for an arbitrarily large  $k$  we have  $Q_c^{\circ k} = D$ . Further iterations of  $Q_c$  will eventually diffeomorphically map that disk to all of  $\mathbb{C}$  (except for at most 1 point).

To wrap this up, here are a few remarks:

### Remark

$Q_c$  is supersensitive on  $J_c$ . Roughly, (as handwavy as Devaney), this is because we can, for any point  $z \in I_{s_0 s_1 \dots s_k}$ , find a small enough disk that contains  $z$  such that for an arbitrarily large  $k$  we have  $Q_c^{\circ k} = D$ . Further iterations of  $Q_c$  will eventually diffeomorphically map that disk to all of  $\mathbb{C}$  (except for at most 1 point).

### Remark

Since  $K_c$  is a Cantor set (each component of  $K_c$  is a point) when  $|c| > 2$ , the Julia and filled Julia sets are identical ( $K_c = J_c$ ).

To wrap this up, here are a few remarks:

### Remark

$Q_c$  is supersensitive on  $J_c$ . Roughly, (as handwavy as Devaney), this is because we can, for any point  $z \in I_{s_0 s_1 \dots s_k}$ , find a small enough disk that contains  $z$  such that for an arbitrarily large  $k$  we have  $Q_c^{\circ k} = D$ . Further iterations of  $Q_c$  will eventually diffeomorphically map that disk to all of  $\mathbb{C}$  (except for at most 1 point).

### Remark

Since  $K_c$  is a Cantor set (each component of  $K_c$  is a point) when  $|c| > 2$ , the Julia and filled Julia sets are identical ( $K_c = J_c$ ).

### Remark

When  $|c| > 2$ , the orbit of 0 tends to  $\infty$ . This will be important on Thursday.

This website gives an amazing visual intuition for the process we just described:

- ▶ [Acko - How to Fold a Julia Fractal](#)

---

## Algorithm 1: Algorithm to plot $K_c$

---

Input: *grid*, a list of evenly-spaced complex numbers in a rectangular region of the complex plane

```

1 for  $j \leftarrow 1$  to  $N$  do
2   for  $z$  in grid do
3      $z \leftarrow Q_c(z)$ 

                                     // Shade all points which didn't escape
4 for  $z$  in grid do
5   if  $|z| \leq \max(|c|, 2)$  then
6     paintBlack( $z$ )

```

---

---

## Algorithm 2: Algorithm to plot $J_c$

---

Input: *MaxIter*, the maximum number of iterations to perform

---

```

1  $z \leftarrow \text{randomComplexNumber}()$ 
2 for  $j \leftarrow 1$  to  $\text{MaxIter}$  do
3    $\text{binaryRand} \leftarrow \text{randomBit}()$ 
                                     // Pick a random backwards iteration
4   if  $\text{binaryRand} = 0$  then
5      $z \leftarrow \sqrt{z - c}$ 
6   else
7      $z \leftarrow -\sqrt{z - c}$ 
                                     // Don't plot stray points
8   if  $j > 100$  then
9      $\text{paintBlack}(z)$ 

```

---



## Explanation for an alternative plotting algorithm

The above algorithm works well for  $K_c$  but as well for  $J_c$ . To remedy this, recall that  $Q_c$  is supersensitive on  $J_c$ . This means that any neighborhood  $B$  of some point  $z \in J_c$  has the property that  $\bigcup_{n=0}^{\infty} Q_0^{\circ n}(B) = \mathbb{C} \setminus \{p\}$  for at most one point  $p$ .

So supersensitive points serve as "attractors" for the backward iteration of  $Q_c$ , in the sense that for each supersensitive point, all of  $\mathbb{C}$  except at most one point must come arbitrarily close to it on reverse iteration.

- ▶ A demo of Julia-Mandelbrot correspondence
- ▶ Some pictures of generalized Julia sets