Professor Tiago Januario
Teaching Fellow Islam Faisal

Homework 1 – Due Tuesday, May 28th, 2024 09:00 pm

- Provide step-by-step explanations, not just answers. Answers without explanations will earn a small fraction of the points.
- Submit your solutions on Gradescope. Don't forget to include information about your collaborators (or say "Collaborators: none").

Problems

0. (**0 points**) The following steps are required to get you started in the course. Please complete them **today**.

   (a) Make sure you are signed up on Piazza at `https://piazza.com/bu/summer2024/cascs330s` using your BU email address.

   (b) Sign up on Gradescope using your BU email address and the code **PWDGPG**.

   (c) Read and sign the Collaboration and Honesty Policy and submit it on Gradescope. We will be able to grade your homework only after you complete this step.

   (d) (**Nameplate**) Please print out (or make by hand) a nameplate with your name and bring it to every lecture and lab. A template is available at the bottom of the course web page.

   (e) Check out the course webpage: `https://cs-people.bu.edu/januario/teaching/cs330/su24/index.html`;

   (f) Familiarize yourself with the homework template files at the bottom of the course webpage. Each problem must include a note about collaborators (even if you did the problem by yourself).

1. (**Tracing Algorithms, 10 points**)

   Consider the following algorithm:

```
// A is an array of integers, indexed from 0 to n-1.
mystery( int A[] ){
    int i = 1
    while(i < A.length){
        int j = i
        while(j > 0 && A[j-1] > A[j]) {
            swap(A[j],A[j - 1])
            j = j - 1
        }
       i = i + 1
    }
}
```

   (a) Trace the run of the algorithm on the following inputs. You can draw a table showing the progress of the nested loop (the progress of the counters i, j) and the contents of the array A as the algorithm progresses. If the contents of A do not change, you can leave the corresponding cell blank.

   - $A = [1, 2, 3, 4, 5]$
   - $A = [4, 1, 5, 3, 2]$
   - $A = [5, 4, 3, 2, 1]$

   Solution:

   $A = [1, 2, 3, 4, 5]$:

i, j	4	3	2	1
1				
2				
3				
4				

   $A = [4, 1, 5, 3, 2]$:

i, j	4	3	2	1
1				[1,4,5,3,2]
2				
3		[1,4,3,5,2]	[1,3,4,5,2]	
4	[1,3,4,2,5]	[1,3,2,4,5]	[1,2,3,4,5]	

   $A = [5, 4, 3, 2, 1]$:

i, j	4	3	2	1
1				[4,5,3,2,1]
2			[4,3,5,2,1]	[3,4,5,2,1]
3		[3,4,2,5,1]	[3,2,4,5,1]	[2,3,4,5,1]
4	[2,3,4,1,5]	[2,3,1,4,5]	[2,1,3,4,5]	[1,2,3,4,5]

(b) Briefly describe what the algorithm does in general. One word can be enough!

Sort

(c) For an input array A of length n, how many steps does the algorithm need to finish (in the worst case)? For simplicity, count the number of times the variable j is updated.

$$\frac{n(n-1)}{2} \cdot 2 = \boxed{n(n+1)}$$

2. (**Proof Techniques: Contradiction and Contraposition, 15 points**) In this problem, let n be a positive integer. You will prove the following claim using two different proof techniques.

Claim 1. *If $3n + 2$ is odd, then n is odd.*

(a) Prove the above claim by **contradiction**.

Assume $3n+2$ is odd and n is even. So $n = 2k$ for some integer k. $3n+2 = 3(2k)+2 = 6k+2$. $6k+2$ must be even as it can be written as $2(3k+1)$. Therefore $3n+2$ is even, condratiction. □

(b) Rewrite the contrapositive equivalent of the claim.

if n is even, then $3n+2$ is even

(c) Prove the claim you wrote in part (b). Please do not use a proof by contradiction in this question item because we want you to train on both techniques.

assume n is even. Therefore $n = 2k$ for some k. if you multipy n by 3 and add 2 you get $3n+2 = 6k+2$. $6k+2$ can be written as $2(3k+1)$ and therefore is even. □

3. (**Proofs by Induction, 15 points**) A *binary tree* is a rooted tree in which each node has at most two children. Let T be a binary tree with n nodes and let L denote the set of leaves and I the set of nodes with exactly two children. Prove by induction that it's always true that $|I| = |L| - 1$.

Hint: Start by writing down the statement you want to prove formally. Start by proving the base case. Then, prove the inductive case by first identifying the inductive hypothesis.

Proof. **Base case:**

Let T be a binary tree with only one node. Then $|I| = 0$, $|L| = 1$.

So, $|I| = |L| - 1$ is $0 = 1 - 1$ which holds.

Inductive Step: Assume that $|I| = |L| - 1$ holds for all binary trees with n nodes where $n \leq K$ for some integer K.

Inductive Hypothesis:
We must prove this holds for any tree w/ $K+1$ nodes. When adding a new node, there are two resulting cases.

Case 1: The new node is an only child.
In this case, $|I_{K+1}| = |I_K| + 0$ because no nodes w/ two children were created.
$|L_{K+1}| = (|L_K| - 1) + 1$ because the new leaf node is added as a child of an existing leaf node.
Therefore $|I_{K+1}| = |L_{K+1}| - 1$ equals $|I_K| = |L_K| - 1$ and by IH holds.

Case 2: The new node is a sibling.
In this case, $|I_{K+1}| = |I_K| + 1$ because the added node results in the parent having two children.
$|L_{K+1}| = |L_K| + 1$ because the added node is also a leaf node.
Therefore $|I_{K+1}| = |L_{K+1}| - 1$ equals $|I_K| + 1 = |L_K| + 1 - 1$ which equals $|I_K| = |L_K| + 1$. By IH this holds.

Conclusion:

For any binary tree with n nodes,
$|I| = |L| - 1$ holds.

□

Hint 1: In the inductive step, consider deleting a leaf node.

Hint 2: What happens if it has a sibling?
When you delete the node, what happens if the deleted node is the only child?

4. (**Programming Assignment, 10 points**) Login to Vjudge and submit your solution to the programming assignment. You can choose either Pypy 3 or Python 3 as your language. No other programming language will be accepted.