1. a) $3x - 13z = 1 \quad (1)$

$4x + 5y = 3 \quad (2)$

$(1) \Rightarrow 13z = 3x - 1 \Rightarrow 13z \equiv -1 \pmod 3$

$\Rightarrow z \equiv 2 \pmod 3$

$\therefore z = 2 + 3t$ for $t \in \mathbb{Z} \quad (3)$

Sub (3) into (1):

$3x - 13(2 + 3t) = 1$

$3x - 26 - 39t = 1$

$3x = 27 + 39t$

$x = 9 + 13t \quad (4)$

Sub (4) into (2)

$4(9 + 13t) + 5y = 3 \Rightarrow 36 + 52t \equiv 3 \pmod 5$

$\Rightarrow 52t \equiv -33 \pmod 5$

$2t \equiv 2 \pmod 5$

$t \equiv 1 \pmod 5$

$\therefore t = 1 + 5u$ for $u \in \mathbb{Z} \quad (5)$

Sub (5) into (3), (4) to get z, x

$(3): z = 2 + 3(1 + 5u) = 5 + 15u$

$(4): x = 9 + 13(1 + 5u) = 22 + 65u \quad (6)$

Sub (6) into (2) to get y

$4(22 + 65u) + 5y = 3$

$88 + 260u + 5y = 3$

$5y = -260u - 85$

$y = -52u - 17$

$$\therefore \begin{cases} x = 22 + 65u \\ z = 5 + 15u \quad \text{for } u \in \mathbb{Z} \\ y = -17 - 52u \end{cases}$$

1.b) For $|y| \leq 100 \Rightarrow |47-52u| \leq 100$

$\therefore u = 0, 1, -1, -2$

Sub these values of u into a)

$\therefore$ Solutions are:

$\begin{cases} x = -108 \\ y = 87 \\ z = -25 \end{cases}$, $\begin{cases} x = -43 \\ y = 35 \\ z = -10 \end{cases}$, $\begin{cases} x = 22 \\ y = -17 \\ z = 5 \end{cases}$, $\begin{cases} x = 87 \\ y = -69 \\ z = 20 \end{cases}$

2.a) Find $\phi(55)$

$\phi(55) = \phi(5 \cdot 11) = 4 \cdot 10 = 40$. Also, $\gcd(19, 55) = 1$.

$\therefore 19^{40} \equiv 1 \pmod{55}$. (Euler's Theorem)

Now find $3^{177} \pmod{40}$. Note $\gcd(3, 40) = 1$.

$\phi(40) = \phi(2^3 \cdot 5) = \phi(2^3) \cdot \phi(5) = 2^2(2-1) \cdot 4 = 16$

$3^{16} \equiv 1 \pmod{40}$ (Euler's Theorem)

$3^{177} \equiv (3^{16})^{11} \cdot 3 \equiv 1 \cdot 3 \equiv 3 \pmod{40}$

$\therefore 19^{3^{177}} \equiv 19^3 \equiv 39 \pmod{55}$.

b) Because we cannot compute $2^{177} \pmod{40}$ with Euler's Theorem as $\gcd(2, 40) \neq 1$.

c) $\phi(2p^2) = \phi(2 \cdot p^2) = \phi(2) \cdot \phi(p^2)$

Note that $\gcd(2, p^2) = 1$ as p is prime ≥ 3.

$\phi(2) \cdot \phi(p^2) = 1 \cdot p(p-1) = p^2 - p$

d) Find n s.t. $\phi(n) = p^2 - p$

Since $\phi(2p^2) = \phi(2) \cdot \phi(p^2) = 1 \cdot \phi(p^2)$,

we can use $n = p^2$ seeing as $\phi(p^2) = \phi(2p^2)$

3. a) $\phi(493) = \phi(17 \cdot 29) = (17-1)(29-1) = 448$.

b) Because $\gcd(3, \phi(M)) = \gcd(3, 448) = 1$

$\therefore$ 3 can be used to encrypt $E: m \mapsto c$ where $c = m^e \pmod{M}$

c) Find 3^{-1} in $\mathbb{Z}_{448}$.

$448 = 149 \cdot \underline{3} + \underline{1}$

$\underline{1} = \underline{448} - 149 \cdot \underline{3}$

$\therefore 3^{-1} \equiv -149 \equiv 299 \pmod{448}$.

$\therefore d = 299$.