MA3H0 Assignment 1

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1. We define the differential operator $\mathcal{L}(v)(x) := -\partial_x^2 v(x) + q(x)v(x)$, then we can reformulate the given problem as

$$\begin{cases} \mathcal{L}(\tilde{u})(x) = f(x), \quad x \in (0,1) \\ \partial_x \tilde{u}(0) = \alpha, \\ \tilde{u}(1) = \beta \end{cases} \tag{1}$$

In order to apply the 'ghost' point technique, we consider (1) but where the interval on which the problem is defined is slightly extended to $(-h, 1)$, i.e.

$$\begin{cases} \mathcal{L}(\tilde{u})(x) = f(x), \quad x \in (-h,1) \\ \partial_x \tilde{u}(0) = \alpha, \\ \tilde{u}(1) = \beta \end{cases} \tag{2}$$

Let $N \in \mathbb{N}$ and we define $h = \frac{1}{N}$, we then create a mesh for $(-h, 1)$, denoted Ω_h, where $\Omega_h = \{x_j\}$, where $x_j := jh,\ j = -1, 0, \ldots, N$. Let $u_h \in S_h$ and define $u_i = u_h(x_i)$.

We then calculate the second difference of u_h at 0 to be

$$\begin{aligned} D_x^2 u_h(0) &= \frac{u_h(-h) - 2u_h(0) + u_h(h)}{h^2} \\ &= \frac{1}{h^2}(u_{-1} - 2u_0 + u_1) \end{aligned} \tag{3}$$

We then rearrange $\alpha = \partial_x \tilde{u}(0) \approx \frac{1}{2h}(u_1 - u_{-1})$ to get $u_{-1} \approx u_1 - 2h\alpha$. Inserting this into (3) gives us

$$D_x^2 u_h(0) = \frac{2}{h^2}(u_1 - u_0 - h\alpha).$$

We can define the discrete operator $\mathcal{L}_h : S_h \to S_h$ given by

$$\begin{cases} \mathcal{L}_h(u_h)_0 = -\frac{2}{h^2}(u_1 - u_0) + q(x_0)u_0 \\ \mathcal{L}_h(u_h)_i = -\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} + q(x_i)u_i, \quad i = 1, \ldots, N-1 \\ \mathcal{L}_h(u_h)_N = u_N \end{cases} \tag{4}$$

Therefore we can discretise (2) using $\mathcal{L}_h$ by

$$\begin{cases} \mathcal{L}_h(u_h)_0 = f(x_0) - 2\alpha/h \\ \mathcal{L}_h(u_h)_i = f(x_i), & i = 1, \dots, N-1 \\ \mathcal{L}_h(u_h)_N = \beta \end{cases} \tag{5}$$

2. Using (4) and (5) we define the matrix

$$A = \begin{pmatrix} \frac{2}{h^2} + q(x_0) & \frac{-2}{h^2} & 0 & \dots & 0 \\ \frac{-1}{h^2} & \frac{2}{h^2} + q(x_1) & \frac{-1}{h^2} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \frac{-1}{h^2} & \frac{2}{h^2} + q(x_{N-1}) & \frac{-1}{h^2} \\ 0 & \dots & 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{(N+1)\times(N+1)},$$

and the vectors

$$U = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \\ u_N \end{pmatrix} \in \mathbb{R}^{N+1}, \; B = \begin{pmatrix} f(x_0) - \frac{2\alpha}{h} \\ f(x_1) \\ \vdots \\ f(x_{N-1}) \\ \beta \end{pmatrix} \in \mathbb{R}^{N+1}.$$

So (4) and (5) can be rewritten as $AU = B$

3. Let $z_h := \mathcal{L}_h(v_h)$, recall that $q(x_i) \geq Q^* > 0$, then we calculate that

$$\left| \left(\frac{2}{h^2} + Q^* \right) v_i \right| \leq \left(\frac{2}{h^2} + q(x_i) \right) |v_i|. \tag{6}$$

Recall for $i = 1, \dots, N-1$, we have

$$z_i = -\frac{1}{h^2} v_{i-1} + \left(\frac{2}{h^2} + q(x_i) \right) v_i - \frac{1}{h^2} v_{i+1}$$

which implies, using the triangle rule,

$$\begin{aligned} \left(\frac{2}{h^2} + q(x_i) \right) |v_i| &\leq |z_i| + \frac{1}{h^2} |v_{i-1}| + \frac{1}{h^2} |v_{i+1}| \\ &\leq \frac{2}{h^2} \|v_h\|_{\infty,h} + \|z_h\|_{\infty,h}. \end{aligned} \tag{7}$$

Similarly, for $i = 0$, recall

$$z_0 = \left(\frac{2}{h^2} + q(x_0) \right) v_0 - \frac{2}{h^2} v_1$$

Rearranging and applying the triangle inequality again, gives

$$\left(\frac{2}{h^2} + q(x_0)\right)|v_0| \leq |z_0| + \frac{2}{h^2}|v_1| \leq \frac{2}{h^2}\|v_h\|_{\infty,h} + \|z_h\|_{\infty,h}. \tag{8}$$

Applying (8) and (7) to (6) gives

$$\left(\frac{2}{h^2} + Q^*\right)\|v_h\|_{\infty,h} \leq \frac{2}{h^2}\|v_h\|_{\infty,h} + \|z_h\|_{\infty,h},$$

rearranging the above shows we can take $C = 1/Q^*$ when bounding for $i = 0, \dots, N-1$. For $i = N$, we have $z_N = v_N$ so clearly

$$|v_N| = |z_N| \leq \|z_h\|_{\infty,h}$$

Therefore the inequality holds for $C = \max\{1, 1/Q^*\}$

4. Since this method depends on the second difference, the order of consistency will be 2 when using the supremum norm. However we require our solution, $\tilde{u}$ to be in $C^4([-h,1])$

5. Let $b_h = \mathcal{L}_h(u_h)$. We know by applying the proof of Theorem 1.7, that for $i = 1, \dots, N-1$

$$|\mathcal{L}_h(\tilde{u})_i - b_i| \leq C_c h^2 |\tilde{u}|_{C^4([-h,1])}$$

for some constant C_c. For $i = N$ we have $\mathcal{L}_h(\tilde{u}) = \beta$ and therefore $|\mathcal{L}(\tilde{u})_N - b_N| = 0$. For $i = 0$ we have

$$|\mathcal{L}(\tilde{u})_0 - b_0| = \left|-\frac{2}{h^2}(\tilde{u}(x_1) - \tilde{u}(x_0)) + q(x_0)\tilde{u}(x_0) - f(x_0) + \frac{2\alpha}{h}\right|$$

$$= \left|-D_x^2\tilde{u}(0) + q(x_0)\tilde{u}(x_0) - f(x_0)\right|$$

$$= | - D_x^2\tilde{u}(0) \underbrace{-\partial_x^2\tilde{u}(0) + q(x_0)\tilde{u}(x_0) - f(x_0)}_{=0} +\partial_x^2\tilde{u}(0)|$$

$$= \left|\partial_x^2\tilde{u}(0) - D_x^2\tilde{u}(0)\right| \leq C_c h^2 |\tilde{u}|_{C^4([-h,1])},$$

where in the last step we applied lemma 1.3 in the notes. Thus by applying question three, we have

$$\|\tilde{u} - u_h\|_{\infty,h} \leq C_c\|\mathcal{L}_h(\tilde{u} - u_h)\|_{\infty,h}$$

$$= C_c\|\mathcal{L}_h(\tilde{u}) - \mathcal{L}_h(u_h)\|_{\infty,h} \leq C_s C_c h^2 |\tilde{u}|_{C^4([-h,1])}$$

Therefore proving the order of convergence to be two two when using the supremum norm to calculate error. As $h \to 0$, $C_s C_c h^2 |\tilde{u}|_{C^4([0,1])} \to 0$, therefore $u_h \to \tilde{u}$ as $h \to 0$ in the discrete supremum norm and u_h converges to $\tilde{u}$.

6. Using this new scheme would mean that the approximation around x_0 would not be done using the second difference. It uses a forward difference scheme, which means the new order of convergence for this scheme would be one. However, we would only need the solution to be in $C^3([0,1])$ for this new scheme to work.