MA3H0 Assignment 3

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We consider the problem:

$$\begin{cases} -\partial_x(d(x)\partial_x\tilde{u}(x)) + q(x)\tilde{u}(x) = f(x) & \text{in } \Omega = (0,1), \\ \partial_x\tilde{u}(0) = \partial_x\tilde{u}(1) = 0, \end{cases} \tag{1}$$

with $f \in L^2(\Omega), d, q \in C^0(\Omega)$. Assume that there exist positive constants $\bar{d}, \underline{d}, \underline{q}, \bar{q}$ such that

$$0 < \underline{d} \leq d(x) \leq \bar{d}, \quad 0 < \underline{q} \leq q(x) \leq \bar{q} \quad \forall x \in \overline{\Omega}.$$

1. To convert (1) to a variational problem we use integration by parts, let $\varphi \in H^1(\Omega)$. We then calculate:

$$\int_0^1 f(x)\varphi(x)dx = \int_0^1 -\partial_x(d(x)\partial_x\tilde{u}(x))\varphi(x) + q(x)\tilde{u}(x)\varphi(x)dx$$

$$= \int_0^1 d(x)\partial_x\tilde{u}(x)\partial_x\varphi(x) + q(x)\tilde{u}(x)\varphi(x)dx - [d(x)\varphi(x)\partial_x\tilde{u}(x)]_0^1$$

Note that $[\varphi(x)d(x)\partial_x\tilde{u}(x)]_0^1 = 0$ due to zero Neumann boundary conditions. Therefore we have shown that (1) can be formulated as:

$$\int_0^1 d(x)\partial_x\tilde{u}(x)\partial_x\varphi(x) + q(x)\tilde{u}(x)\varphi(x)dx = \int_0^1 f(x)\varphi(x)dx,$$

for all $\varphi \in H^1(\Omega)$.

2. We define the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ and a linear form $\ell : H^1(\Omega) \to \mathbb{R}$ via :

$$a(u,v) = \int_0^1 d(x)\partial_x v(x)\partial_x u(x) + q(x)u(x)v(x)dx,$$

$$\ell(v) = \int_0^1 f(x)v(x)dx.$$

We first show that a is bounded. We calculate via the triangle inequality:

$$|a(u,v)| = \left|\int_0^1 d(x)\partial_x v(x)\partial_x u(x) + q(x)u(x)v(x)dx\right|$$

$$\leq \max\{\bar{d},\bar{q}\}\left|\int_0^1 \partial_x v(x)\partial_x u(x) + u(x)v(x)dx\right|$$

$$= \max\{\bar{d},\bar{q}\}\left|\langle u,v\rangle_{H^1(\Omega)}\right| \leq \max\{\bar{d},\bar{q}\}\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}.$$

where to get the final line we applied Cauchy Schwartz. Therefore a is bounded.

Similarly, to show a is coercive :

$$a(v,v) = \int_0^1 d(x)(\partial_x v(x))^2 + q(x)(v(x))^2 dx$$

$$\geq \min\{\underline{q},\underline{d}\}\int_0^1 (\partial_x v(x))^2 + (v(x))^2 dx = \min\{\underline{q},\underline{d}\}\|v\|^2_{H^1(\Omega)}.$$

Thus a is also coercive.

Finally we show that ℓ is bounded in the H^1 norm:

$$|\ell(v)| = \left|\int_0^1 f(x)v(x)dx\right| = \left|\langle f,v\rangle_{L^2(\Omega)}\right|$$

$$\leq \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)},$$

where again we used Cauchy-Schwartz in the last line.

3. We recall the definition of the test functions $\psi_j,\ j\in\{0,\dots,N_h\}$: We define $h = 1/N_h$, set $x_j = jh$. Then

$$\psi_j(x) = \begin{cases} (x-x_{j-1})/h & x\in[x_{j-1},x_j], \\ (x_{j+1}-x)/h & x\in[x_j,x_{j+1}], \\ 0 & \text{otherwise.}\end{cases}$$

We then calculate for $j\in\{1,\dots,N_h-1\}$ that

$$a(\psi_j,\psi_j) = \int_0^1 d(x)(\partial\psi_j(x))^2 + q(x)(\psi_j(x))^2 dx$$

$$= \frac{1}{h^2}\int_{x_{j-1}}^{x_j} d(x)+q(x)(x-x_{j-1})^2 dx + \frac{1}{h^2}\int_{x_j}^{x_{j+1}} d(x)+q(x)(x_{j+1}-x)^2 dx.$$

Furthermore, we calculate

$$a(\psi_0,\psi_0) = \frac{1}{h^2}\int_0^h d(x) + q(x)(h-x)^2 dx,$$

$$a(\psi_{N_h}, \psi_{N_h}) = \frac{1}{h^2} \int_{1-h}^{1} d(x) + q(x)(x-1+h)^2 dx.$$

We again calculate that for $j \in \{1, \dots, N_h\}$

$$a(\psi_{j-1}, \psi_j) = a(\psi_j, \psi_{j-1}) = \frac{1}{h^2} \int_{x_{j-1}}^{x_j} -d(x) + q(x)(x_j - x)(x - x_{j-1}) dx.$$

We also have that for $i \neq j-1, j, j+1$:

$$a(\psi_i, \psi_j) = 0.$$

We also note:

$$\ell(\psi_j) = \frac{1}{h} \int_{x_{j-1}}^{x_j} f(x)(x - x_{j-1}) dx + \frac{1}{h} \int_{x_j}^{x_{j+1}} f(x)(x_{j+1} - x) dx.$$

We set

$$A = \begin{pmatrix} a(\psi_0, \psi_0) & \dots & a(\psi_0, \psi_{N_h}) \\ \vdots & \ddots & \vdots \\ a(\psi_{N_h}, \psi_0) & \dots & a(\psi_{N_h}, \psi_{N_h}) \end{pmatrix}, \; B = \begin{pmatrix} \ell(\psi_0) \\ \vdots \\ \ell(\psi_{N_h}) \end{pmatrix}$$

$$U = \begin{pmatrix} u_0 \\ \vdots \\ u_{N_h} \end{pmatrix}$$

Then the discrete problem can be solved by solving the tridiagonal matrix system :

$$AU = B$$

4. By applying Cea (lemma 4.13) in the notes, we calculate

$$\|\tilde{u} - u_h\|_{H^1(\Omega)} \leq \frac{\max\{\bar{d}, \bar{q}\}}{\min\{\underline{q}, \underline{d}\}} \|\tilde{u} - P_h(\tilde{u})\|_{H^1(\Omega)},$$

where $P_h(\tilde{u}) = \sum_{j=0}^{N_h} \tilde{u}(x_j)\psi_j \in V_h$. We then apply the interpolation estimate (lemma 4.17) to get that there is some $\tilde{C}$ such that:

$$\frac{\max\{\bar{d}, \bar{q}\}}{\min\{\underline{q}, \underline{d}\}} \|\tilde{u} - P_h(\tilde{u})\|_{H^1(\Omega)} \leq \tilde{C}h \frac{\max\{\bar{d}, \bar{q}\}}{\min\{\underline{q}, \underline{d}\}} \|\tilde{u}\|_{H^2(\Omega)}.$$

The result follows from taking $C = \tilde{C}\frac{\max\{\bar{d}, \bar{q}\}}{\min\{\underline{q}, \underline{d}\}}$.

5. By plugging in the integrals from the solution to question three into Wolfram Alpha, we calculate that for the given $q(x), d(x), f(x)$:

$$a(\psi_0, \psi_0) = \frac{7h}{20},$$

$$\begin{aligned} a(\psi_{N_h}, \psi_{N_h}) &= \frac{h}{20}(N_h^2 - N_h + 7), \\ a(\psi_j, \psi_j) &= \frac{hj^2 + 7h}{10}, \quad j \in \{1, \ldots, N_h - 1\}, \\ a(\psi_j, \psi_{j-1}) = a(\psi_{j-1}, \psi_j) &= \frac{h}{20}(-j^2 + j + 3). \end{aligned}$$

For the vector B, we calculate that :

$$\ell(\psi_0) = \frac{24h^4 - 35h^3}{600},$$

$$\begin{aligned} \ell(\psi_{N_h}) &= \frac{h^4}{25}(10N_h^3 - 10N_h^2 + 5N_h - 1) - \frac{7h^3}{120}(6N_h^2 - 4N_h + 1)), \\ \ell(\psi_j) &= \frac{2h^4}{5}(2j^3 + j) - \frac{7h^3}{60}(6j^2 + 1) \quad j \in \{1, \ldots, N_h - 1\}. \end{aligned}$$

In the jupyter notebook we calculate the EOC for the $\|\cdot\|_\infty$ norm and see that it is ≈ 2.