A Connected Sum Formula for the Seiberg-Witten Invariant of 4-Manifold Families

Joshua Tomlin

School of Computer and Mathematical Sciences
Mathematical Sciences Discpline

June 2025

Contents

Abstract

In this thesis, we derive a general formula for the families Seiberg-Witten invariant of a fibrewise connected sum of 4-manifold families which incorporates both the families blow-up formula of Liu [43] and the gluing formula of Baraglia-Konno [6]. This is accomplished through the families Bauer-Furuta invariant. The ordinary Bauer-Furuta invariant is a cohomotopy refinement of the Seiberg-Witten invariant. This refinement is the stable cohomotopy class of a finite dimensional approximation of the Seiberg-Witten monopole map. We analyse monopole behaviour on a cylinder $S^3 \times [-L, L]$ to prove a connected sum formula for the ordinary Bauer-Furuta invariant. This formula states that the Bauer Furuta invariant of a connected sum is the smash product of the Bauer-Furuta invariants of the summands. We extend this formula to the setting of 4-manifold families. Then, we use a cohomological description of the families Seiberg-Witten invariant to obtain a general connected sum formula for the families Seiberg-Witten invariant.

Declaration

I certify that this work contains no material which has been accepted for the award of any other degree or diploma in my name, in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. In addition, I certify that no part of this work will, in the future, be used in a submission in my name, for any other degree or diploma in any university or other tertiary institution without the prior approval of the University of Adelaide and where applicable, any partner institution responsible for the joint award of this degree.

I give permission for the digital version of my thesis to be made available on the web, via the University's digital research repository, the Library Search and also through web search engines, unless permission has been granted by the University to restrict access for a period of time.

I acknowledge the support I have received for my research through the provision of an Australian Government Research Training Program Scholarship.

Joshua Tomlin

June 30th, 2025

Introduction

Since 1982, gauge theory has been a central tool in the area of 4-manifold topology. One of the first significant applications was by Donaldson and Freedman. Donaldson used $SU(2)$-instanton theory to show that if the intersection form of a compact, oriented, smooth 4-manifold is definite, then it must be diagonalisable [17,18]. On the other hand, Freedman showed that every unimodular symmetric bilinear form can be realised as the intersection form of a closed, oriented, topological 4-manifold [23]. Combining these results shows that any indefinite, non-diagonalisable intersection form determines a topological 4-manifold that has no smooth structure. In particular, they discovered that the E_8 manifold with intersection form the E_8 lattice is a topological manifold with no smooth structure. They both received a Fields medals in 1986 for their individual contributions to the field.

Towards the end of 1994, Nathan Seiberg and Edward Witten pioneered the field of Seiberg-Witten theory. In two papers [57,58], Seiberg and Witten studied $N = 2$ supersymmetric Yang-Mills theory and defined the Seiberg-Witten equations. Later that year, Witten formally defined the integer-valued Seiberg-Witten invariant as a map on spin^c structures [62]. This approach revolutionised the field of 4-manifold topology. It simplified proofs from Donaldson theory, including Donaldson's above theorem, and provided new striking results including a relatively straightforward proof of the Thom conjecture [37].

Seiberg-Witten theory studies closed, oriented, Riemannian 4-manifolds equipped with a spin^c structure. A key advantage of this theory is the fact that the moduli space of solutions to the Seiberg-Witten equations is a smooth, compact, orientable, finite dimensional manifold. In particular, Donaldson's instanton theory often had to deal with compactness issues, forming delicate arguments to circumvent these problems. Further, Seiberg-Witten theory comes with several practically useful properties including the Weitzenböck formula and bounds related to scalar curvature.

Despite these nice properties, it is still difficult to calculate Seiberg-Witten invariants even in the simplest settings. Some basic tools for calculating Seiberg-Witten invariants are the vanishing formula [54] and the blow-up formula [26]. The vanishing formula applies to the connected sum of two 4-manifolds both with positive first Betti number. The blowup formula applies to connected sums with $\overline{\mathbb{CP}^2}$, however the techniques used to prove it can be lifted to obtain to a general connected sum formula [48]. The proofs of these formulas are rooted in analysis with careful arguments and tedious estimates.

In 2003, Stefan Bauer and Mikio Furuta developed a cohomotopy refinement of the integer valued Seiberg-Witten invariant called the Bauer-Furuta invariant [10]. Their approach was to study the stable cohomotopy class of the Seiberg-Witten monopole map, rather than its solutions. This refinement contains strictly more information than the Seiberg-Witten invariant [34, 35]. The monopole map μ is an S^1-equivariant Fredholm map between infinite dimensional Hilbert bundles. The equivariant stable cohomotopy class of μ is defined by approximating μ by restrictions to finite dimensional subspaces. Chapter 1.2 and Chapter 2 are devoted to carefully defining equivariant stable homotopy groups in this setting and two equivalent methods of finite dimensional approximation. Applying tools from algebraic topology to this refinement provided a proof of the $\frac{10}{8}$-theorem [29] and an alternate proof of Donaldson's theorem.

In [9], Bauer followed up on his work by deriving a connected sum formula for the Bauer-Furuta invariant. His approach was to analyse behaviour of monopoles on a manifold with a separating neck of varying length. He showed that given disjoint unions of 4-manifolds with separating necks, ends of the necks can be permuted without changing the homotopy class of μ. While the general strategy of Bauer's proof was sound, some of his arguments were adhoc and incomplete. Chapter 3 provides a detailed reworking of his ideas to obtain an original, complete proof of his connected sum formula. This is accomplished by proving Theorem 3.8 in Chapter 3.4, from which the connected sum formula Theorem 4.13 follows.

In a research report published in 1996 [19], Donaldson suggested studying Seiberg-Witten equations on 4-manifold families. Two years later, Ruberman used Seiberg-Witten equations over 1 parameter 4-manifold families to construct examples of diffeomorphisms that are continuously homotopic to the identity, but not smoothly homotopic to the identity. Since then, several authors including Li-Liu, Nakamura and Ruberman have generalised Seiberg-Witten invariants to 4-manifold families [41, 47, 53]. This body of work involves wall crossing formulas, existence of positive scalar curvature metrics, and a families blow-up formula [43, Theorem 2.2]. However, a general connected sum formula has been a significant gap in the literature.

Since 2019, Baraglia has built on the work of Li-Liu, Ruberman and Nakamura [4–6]. In [6], Baraglia and Konno proved a connected sum formula for the families Seiberg-Witten invariant under some restrictive assumptions. These assumptions simplified the moduli space of one of the summands and avoided the need for chambers. The primary result of this thesis is a new, completely general connected sum formula for families Seiberg-Witten invariants (see Theorem 5.27). In particular, it applies to more complicated monopole maps and can accommodate chambers.

The Bauer-Furuta invariant naturally extends to the families setting and has been developed by Szymik [59]. In 2021, Baraglia and Konno demonstrated how to recover the families Seiberg-Witten invariant from the families Bauer-Furuta invariant via a formulation of the families Seiberg-Witten invariant in S^1-equivariant cohomology [7]. This formulation is discussed in detail in Chapter 5.

Our proof of the Bauer-Furuta connected sum formula in Chapter 3.4 immediately extends to the families setting. Theorem 4.3 and 4.15 in particular are new results which extend Theorems 3.8 and 4.13 respectively. Chapter 5 outlines Baraglia's method of recovering the families Seiberg-Witten invariant from the families Bauer-Furuta invariant in. In Chapter 5.3 and Chapter 5.4, the cohomological formulation is leveraged to obtain a connected sum formula for the families Seiberg-Witten invariant from the families Bauer-Furuta invariant. Theorem 5.26 is a new result which implies Theorem 5.27, a completely general families Seiberg-Witten connected sum formula.

Chapter 1

Background

1.1 Seiberg-Witten theory

Seiberg-Witten theory studies the moduli space of Seiberg-Witten monopoles on a 4-manifold X up to gauge equivalence. A monopole is a pair (ψ, A) of a spinor ψ and spin^c connection A that satisfies the Seiberg-Witten equations. In order to define (ψ, A), it is necessary to equip X with a spin^c structure. In this chapter, we discuss the basics of spin geometry, spin^c structures and the integer valued Seiberg-Witten invariant.

1.1.1 Spin geometry

Let V be an n-dimensional real vector space equipped with a positive definite inner product. The Clifford algebra $Cl(V)$ is the tensor algebra $T(V)$ with the relation

$$v \otimes w + w \otimes v = -2 \langle v, w \rangle$$

for $v, w \in V$. We write $v \cdot w$ to denote Clifford multiplication in this algebra, often abbreviated to just vw. The inner product on V natural extends to a positive definite bilinear form on $Cl(V)$.

Let $\{e_1, ..., e_n\}$ be an orthonormal basis for V. Define $e_{i_1 \cdots i_k} = e_{i_1} e_{i_2} \cdots e_{i_k}$ for $1 \leq i_j \leq n$. Then the elements $e_0 = 1$ and $e_{i_1 \cdots i_k}$ with $i_1 < ... < i_k$ form a linear basis for $Cl(V)$. Consequently, the dimension of $Cl(V)$ is 2^n.

Remark 1.1: More generally, Clifford algebras can be defined for any quadratic form $q : V \to \mathbb{R}$ using the relation $v^2 = q(V)$. Notice that $Cl(V)$ and the exterior algebra ΛV are isomorphic as vector spaces, but are not isomorphic as algebras unless $q = 0$.

Let I denote a multi-index with $1 \leq i_1 < ... < i_k \leq n$. There is a $\mathbb{Z}_2$-grading $Cl(V) = Cl^-(V) \oplus Cl^+(V)$ where $Cl^-(V)$ is the linear span of the elements e_I with an odd number of indices and $Cl^+(V)$ is the linear span of the elements e_I with an even number of indices, including e_0. Write x^+ and x^- to denote the even and odd component of $x \in Cl(V)$ respectively.

There is a natural anti-involution $(\tilde{\cdot})$ of $Cl(V)$ defined on basis vectors defined by

$$\begin{aligned}\tilde{e}_I &= (-1)^k e_{\mathrm{rev}(I)} \\ &= (-1)^{\frac{k(k+1)}{2}} e_I\end{aligned}$$

where $\mathrm{rev}(V)$ is I with reversed order. This is an anti-involution in the sense that $\tilde{\tilde{x}} = x$ and $\widetilde{xy} = \tilde{y}\tilde{x}$ for any $x, y \in Cl(V)$. Clifford algebras have the universal property that any linear map $f : V \to A$ into an associative, unital, $*$-algebra A that satisfies $f(v)f(w) + f(w)f(v) = -2\langle v, w\rangle$ and $f(v)f(v)^* = |v|^2$ extends uniquely to a $*$-algebra homomorphism $g : Cl(V) \to A$.

For any unit $x \in Cl(V)$, there is an adjoint action $\mathrm{ad}(x) : Cl(V) \to Cl(V)$ defined by

$$\mathrm{ad}(x)y = (x^+ - x^-)y\tilde{x}. \tag{1.1}$$

Define

$$\mathrm{Spin}(V) = \{x \in Cl^+(V) \mid \tilde{x}x = 1, xV\tilde{x} = V\}.$$

Note that $\mathrm{Spin}(V)$ is a subset of the units of $Cl(V)$ and $\mathrm{ad}(x)V \subset V$, hence the adjoint action defines a representation $\mathrm{ad} : \mathrm{Spin}(V) \to SO(V)$. If $\dim V \geq 3$ then there is an exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}(V) \xrightarrow{\mathrm{ad}} SO(V) \longrightarrow 1. \tag{1.2}$$

Moreover when $\dim V \geq 3$, $\mathrm{Spin}(V)$ is compact, connected and simply connected, hence $\mathrm{Spin}(V)$ is the universal cover of $SO(V)$ [55, Lemma 4.25]. The Lie algebra of $\mathrm{Spin}(V)$ is $Cl_2(V)$, the degree 2 elements of $Cl(V)$.

The group $\mathrm{Spin}^c(V)$ is defined in a similar fashion

$$\mathrm{Spin}^c(V) = \{z \in Cl^+(V) \otimes_{\mathbb{R}} \mathbb{C} \mid \tilde{x}x = 1, xV\tilde{x} = V\}.$$

Every element $z \in \mathrm{Spin}^c(V)$ can be written as $z = e^{i\theta}x$ for some $e^{i\theta} \in S^1$ and $x \in \mathrm{Spin}(V)$. This decomposition is not unique since $e^{i\theta}x = (-e^{i\theta})(-x)$, instead we have $\mathrm{Spin}^c(V) = \mathrm{Spin}(V) \times_{\mathbb{Z}_2} S^1$. The adjoint action (1.7) extends to $\mathrm{Spin}^c(V)$ and there is an analogous exact sequence

$$1 \longrightarrow S^1 \longrightarrow \mathrm{Spin}^c(V) \xrightarrow{\mathrm{ad}} SO(V) \longrightarrow 1. \tag{1.3}$$

Define the squaring map $\delta : \mathrm{Spin}^c(V) \to S^1$ by

$$\delta(e^{i\theta}x) = e^{2i\theta}. \tag{1.4}$$

Note the necessity of squaring to ensure the map is well defined. There is another exact sequence

$$1 \longrightarrow \mathrm{Spin}(V) \longrightarrow \mathrm{Spin}^c(V) \xrightarrow{\delta} S^1 \longrightarrow 1. \tag{1.5}$$

When $\dim V \geq 3$, the map δ induces an isomorphism of fundamental groups and $\pi_1(\mathrm{Spin}^c(V)) \cong \mathbb{Z}$ [55, Lemma 4.30].

The most relevant case for our purposes is when $\dim V = 4$ and $\mathrm{Spin}(V) = \mathrm{Spin}(4) = SU(2) \times SU(2)$. As a fundamental example, let $V = \mathbb{H}$ denote the space of quaternions. That is, $\mathbb{H}$ is the 4-dimensional real algebra generated by $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k} \in \mathbb{H}$ satisfying the multiplication relations

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -\mathbf{1}. \tag{1.6}$$

Write $q \in \mathbb{H}$ as $q = t\mathbf{1} + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ with $t, x, y, z \in \mathbb{R}$. The conjugate of q is $\overline{q} = t\mathbf{1} - x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$. The imaginary quaternions is the space $\mathrm{Im}\,\mathbb{H}$ spanned by $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$. The standard inner product on $\mathbb{H}$ makes $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ an orthonormal basis.

Let $W^+ = \mathbb{C}^2$ equipped with the standard Hermitian metric. Also let $W^- = \mathbb{C}^2$, identified as the dual of W^+, and set $W = W^+ \oplus W^- = \mathbb{C}^4$. There is a faithful representation $\gamma : \mathbb{H} \to \mathrm{Hom}(W^+, W^-)$ defined by

$$\gamma(q) = \begin{pmatrix} t + ix & y + iz \\ -y + iz & t - ix \end{pmatrix}$$

In particular, letting $\mathbb{1} = \gamma(\mathbf{1}), I = \gamma(\mathbf{i}), J = \gamma(\mathbf{j})$ and $K = \gamma(\mathbf{k})$ we have

$$\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Of course, these matrices satisfy the relations (1.6), hence γ is an algebra homomorphism. Moreover $\gamma(\overline{q}) = \gamma(q)^*$, the conjugate transpose of $\gamma(q)$. The matrices I, J and K generate $SU(2)$, hence γ identifies the space of unit quaternions with $SU(2) = \mathrm{Spin}(3)$. To distinguish between these two interpretations, we write upper case letters to denote 2×2 unitary matrices and lower case letters to denote the corresponding unit quaternions. For $(U, V) \in \mathrm{Spin}(4) = SU(2) \times SU(2)$ with $U, V \in SU(2)$, the adjoint action $\mathrm{ad} : \mathrm{Spin}(4) \to SO(4)$ is

$$\mathrm{ad}(U, V)q = uq\overline{v}. \tag{1.7}$$

Note that this is an orthogonal transformation since u and v have norm 1. Similarly, the central extension $\mathrm{Spin}^c(4) = \mathrm{Spin}(4) \times_{\mathbb{Z}_2} S^1$ is the set of pairs pairs $(\lambda U, \lambda V)$ with $U, V \in SU(2)$ and $\lambda \in S^1$, under the $\mathbb{Z}_2$ relation described above. The adjoint action $\mathrm{ad} : \mathrm{Spin}(4) \to SO(4)$ is independent of λ and is also described by (1.7). The map $\gamma : \mathbb{H} \to \mathrm{Hom}(W^+, W^-)$ is adjoint equivariant in the sense that

$$\gamma(\mathrm{ad}(U, V)q) = U\gamma(q)V^*$$

Define a representation $\Gamma : \mathbb{H} \to \mathrm{End}(W)$ by

$$\Gamma(q) = \begin{pmatrix} 0 & -\gamma(q)^* \\ \gamma(q) & 0 \end{pmatrix}$$

By [55, Proposition 4.33], the map Γ extends uniquely to an algebra isomorphism $\Gamma : Cl^c(\mathbb{H}) \to \mathrm{End}(W)$. This is the standard model that we will use to define spin^c structures on vector bundles in the next section. Note that $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{Hom}_{\mathbb{C}}(W^+, W^-)$.

Label $e_0 = \mathbf{1}, e_1 = \mathbf{i}, e_2 = \mathbf{j}, e_3 = \mathbf{k}$ and let $\{e_0^*, e_1^*, e_2^*, e_3^*\}$ be the corresponding dual basis of $\mathbb{H}^*$. Let $\Lambda^2\mathbb{H}^*$ be the space of degree two forms on $\mathbb{H}^*$. The forms $e_i^* \wedge e_j^*$ with $i < j$ form a basis for $\Lambda^2\mathbb{H}^*$. Let $Cl_2(\mathbb{H})$ be the subspace of degree 2 elements of $Cl(\mathbb{H})$, which is the Lie algebra of $\mathrm{Spin}(4)$. This subspace is spanned by products e_ie_j with $i < j$. There is an obvious linear isomorphism $Cl_2(\mathbb{H}) \cong \Lambda^2\mathbb{H}^*$ which maps e_ie_j to $e_i \wedge e_j$. Define a linear map $\rho : \Lambda^2\mathbb{H}^* \to \mathrm{End}(W)$ on basis vectors by

$$\rho(e_i \wedge e_j) = \Gamma(e_ie_j).$$

The map ρ defines a representation of the Lie algebra of $\mathrm{Spin}(4)$.

Let $* : \Lambda^2\mathbb{H}^* \to \Lambda^2\mathbb{H}^*$ be the Hodge star operator which satisfies $*^2 = 1$ and

$$*(e_0 \wedge e_1) = e_2 \wedge e_3, \quad *(e_0 \wedge e_2) = e_3 \wedge e_1, \quad *(e_0 \wedge e_3) = e_1 \wedge e_2.$$

Let $\Lambda^2_{\pm}\mathbb{H}^*$ denote the positive and negative eigenspace of $*^2$ respectively. For $\eta \in \Lambda^2\mathbb{H}^*$, write the eigenspace decomposition of η as $\eta = \eta_+ + \eta_-$. Denote by $\rho^{\pm}(\eta) \in$

$\mathrm{End}(W^\pm)$ the restriction of $\rho(\eta)$ to $W^\pm$. Recall that $\Gamma(e_i)W^\pm \subset W^\mp$, hence the image of $\rho^\pm(\eta)$ is contained in $W^\pm$. [55, Lemma 4.55] highlights the observation that $\rho^\pm(\eta) = 0$ if and only if $\eta^\pm = 0$, so in fact

$$\rho^\pm : \Lambda^2_\pm \mathbb{H}^* \to \mathrm{End}(W^\pm).$$

Further, the complexified map $\rho^\pm : \Lambda^2_\pm \mathbb{H}^* \otimes_\mathbb{R} \mathbb{C} \to \mathrm{End}_0(W^\pm)$ is an isomorphism from $\Lambda^2_\pm \mathbb{H}^* \otimes_\mathbb{R} \mathbb{C}$ to the space of trace-free, complex linear endomorphisms of $W^\pm$. This isomorphism maps real valued forms to skew-Hermitian endomorphisms and imaginary valued forms to Hermitian endomorphisms. The inverse isomorphism

$$(\rho^+)^{-1} : \mathrm{End}_0(W^+) \to \Lambda^2_+ \mathbb{H}^* \otimes_\mathbb{R} \mathbb{C} \tag{1.8}$$

identifies traceless complex 2×2 matrices with complex valued self-dual two forms on $\mathbb{H}$.

1.1.2 Spin$^\mathrm{c}$ structures on 4-manifolds

Let X be a closed, connected, oriented, 4-dimensional Riemannian manifold. The given orientation and metric on X means that the structure group of the bundle TX can be reduced to $SO(4)$. That is, the space of oriented, orthogonal frames of TX is a principle $SO(4)$-bundle $\pi_\mathcal{F} : \mathcal{F} \to X$. A spin structure on X is a double covering of $\mathcal{F}$ by a Spin(4)-principle bundle in the following sense.

Definition 1.2. *A spin structure $\mathfrak{s}$ on X is a principle Spin(4)-bundle $\pi_\mathcal{P} : \mathcal{P} \to X$ along with an adjoint-equivariant double covering $\pi : \mathcal{P} \to \mathcal{F}$ such that $\pi_\mathcal{F} \circ \pi = \pi_\mathcal{P}$.*

The equivariance condition means that for every $g \in \mathrm{Spin}(4)$ and $x \in \mathcal{P}$,

$$\pi(x \cdot g) = \pi(x) \cdot \mathrm{ad}(g).$$

For a trivialising open cover $\{U_\alpha\}$ of X, the transition functions $\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Spin}(4)$ of $\mathcal{P}$ have the property that $g_{\alpha\beta} = \mathrm{ad} \circ \tilde{g}_{\alpha\beta}$ defines transition functions for $\mathcal{F}$. Moreover, the adjoint representation determines an identification $TX = \mathcal{P} \times_{\mathrm{ad}} \mathbb{R}^4$ as Riemannian vector bundles. To define a spin^c structure, replace the group Spin(4) with Spin^c(4) in Definition 1.2. Let $\mathcal{S}(X)$ denote the set of spin^c structures on X up to Spin^c(4)-bundle isomorphism.

An obstruction to the existence of a spin structure on X is the second Stiefel-Whitney class $w_2(TX) \in H^2(X; \mathbb{Z}_2)$. That is, X is spin structure if and only if

$w_2(TX) = 0$ [40, Theorem 2.1]. However, X admits a spin^c structure if and only if $w_2(TX) \in H^2(X;\mathbb{Z}_2)$ admits a lift to $H^2(X;\mathbb{Z})$. This is a more lenient condition and a fundamental fact about 4-manifold topology that every compact, oriented 4-manifold admits a spin^c structure [45, Theorem 2.2].

Fix a spin^c structure $\mathfrak{s}$ on X with spin^c frame bundle $\mathcal{P} \to X$. The associated spinor bundle is the complex 4-dimensional bundle W defined by

$$W = \mathcal{P} \times_{\text{ad}} \mathbb{C}^4.$$

This bundle comes with a spinor multiplication map $\Gamma : TX \to \text{End}(W)$ which extends uniquely to an algebra isomorphism $\Gamma : Cl^c(TX) \to \text{End}(W)$ [55, Proposition 4.33]. Here $Cl^c(TX) \to X$ is the complexified Clifford bundle of X, with fibre $Cl^c(T_xX)$ at $x \in X$. Since TX is four dimensional and oriented, there is a natural splitting

$$W = W^+ \oplus W^-.$$

This splitting has the property that $\Gamma(v) \cdot W^{\pm} \subset W^{\mp}$. Further, there are isomorphisms $\rho^{\pm} : \Lambda^2_{\pm}(T^*X) \otimes \mathbb{C} \to \text{End}_0(W^{\pm})$ as constructed in (1.8).

Recall from (1.4) that $\delta : \text{Spin}^c(4) \to S^1$ denotes the squaring representation. The characteristic line bundle L of $\mathfrak{s}$ is

$$L = \mathcal{P} \times_{\delta} \mathbb{C}.$$

The first chern class $c_1(L) \in H^2(X;\mathbb{Z})$ is an integral lift of $w_2(TX)$ and classifies $\mathfrak{s}$ up to isomorphism.

Let $\nabla : C^\infty(X,TX) \times C^\infty(X,TX) \to C^\infty(X,TX)$ be the Levi-Civita connection on TX. For any vector bundle $E \to X$, let $\Omega^1(X,E) = C^\infty(X, T^*X \otimes E)$ denote the space of E-valued one-forms. A spin^c connection $\nabla_A : C^\infty(X,W) \to \Omega^1(X,W)$ is a hermitian connection on X that is compatible with the Levi-Civita connection ∇ in the sense that

$$\nabla_{A,v}(\Gamma(u)\psi) = \Gamma(\nabla_v u)\psi + \Gamma(u)\nabla_{A,v}\psi$$

for any $u, v \in C^\infty(X,TX)$ and $\psi \in C^\infty(X,W)$. This compatibility condition implies that ∇_A is completely determined by the imaginary valued one-form $A = \frac{1}{4}\,\text{Trace}\,\nabla_A$. Let $\mathcal{A}_{\mathfrak{s}}$ denote the space of spin^c connections on X, which is affine and parallel to $\Omega^1(X,i\mathbb{R})$. The form $2A$ induces a $U(1)$ connection on the determinant line bundle $\det(W^+)$. Write $F_A \in \Omega^2(X,i\mathbb{R})$ to denote the curvature of A. For any other connection $A' = A + a$ with $a \in \Omega^1(X,i\mathbb{R})$ we have

$$F_{A'} = F_A + da.$$

Given that X is oriented, let $* : \Omega^2(X) \to \Omega^2(X)$ denote the Hodge star operator. This operator squares to the identity and determines an eigenspace decomposition $\Omega^2(X) = \Omega^2_+(X) \oplus \Omega^2_-(X)$. Write F_A^+ to denote the self-dual component of F_A and note that $F_{A'}^+ = F_A^+ + d^+a$ where $d^+ : \Omega^2(X) \to \Omega^2_+(X)$ is the projection of d to $\Omega^2_+(X)$. Define a map $\sigma : C^\infty(X, W^+) \to \Omega^2_+(X)$ by

$$\sigma(\psi) = (\rho^+)^{-1}\left(\psi \otimes \psi^* - \frac{1}{2}|\psi|^2 \mathrm{Id}\right). \tag{1.9}$$

Here $\psi \otimes \psi^* - \frac{1}{2}|\psi|^2\mathrm{Id}$ is a traceless endomorphism of W^+ and $(\rho^+)^{-1}$ is defined in (1.8).

The metric on X determines a canonical isomorphism $TX \cong T^*X$. This isomorphism defines an action $\Gamma : T^*X \to \mathrm{End}(W)$, which defines an interior multiplication $c : \Omega^1(X, W) \to C^\infty(X, W)$ that maps a pure tensor $\eta \otimes \psi$ to $\Gamma(\eta)\psi$. The Dirac operator $\mathcal{D}_A : C^\infty(X, W) \to C^\infty(X, W)$ is the map $\mathcal{D}_A = c \circ \nabla_A$. This operator is formally self-adjoint with respect to the L^2-inner product and is elliptic with index zero. Let $D_A : C^\infty(X, W^+) \to C^\infty(X, W^-)$ denote the restriction of $\mathcal{D}$ to W^+. This operator is also elliptic and hence Fredholm with index

$$\mathrm{ind}\, D_A = \frac{c_1(L)^2 - \sigma(X)}{4}.$$

The index of D_A is a topological invariant of X calculated by the Atiyah-Singer index theorem. Here $\sigma(X)$ is the signature of X. To define $\sigma(X)$, let b_i denote the ith Betti number of X, that is, the real dimension of $H^i(X;\mathbb{R})$. Let b^+ be the dimension of a maximal subspace of $H_2(X)$ on which the intersection paring is positive definite and set $b^- = b_2 - b^+$. Then the signature of X is the difference $\sigma(X) = b^+ - b^-$. Note that $c_1(L)^2 \in H^4(X;\mathbb{Z})$ is interpreted as an integer by evaluation on the fundamental class $[X] \in H_4(X)$.

1.1.3 The Seiberg-Witten invariant

Let $\mathcal{G} = \mathrm{Map}(X, S^1)$ denote the group of gauge transformations on X. A gauge transformation $u \in \mathcal{G}$ acts on a connection A by pullback, $u^*A = A + u^{-1}du$, and on spinors by pointwise multiplication with u^{-1}. Direct calculations show that, for $A \in \mathcal{A}_\mathfrak{s}$ and $\psi \in C^\infty(X, W^+)$,

$$\begin{aligned} D_{u^*A}(u^{-1}\psi) &= u^{-1}D_A\psi. \\ F_{u^*A} &= F_A \\ \sigma(u^{-1}\psi) &= \sigma(\psi). \end{aligned} \tag{1.10}$$

For $(A, \psi) \in \mathcal{A}_{\mathfrak{s}} \times C^\infty(X, W^+)$, the Seiberg-Witten equations are

$$\begin{aligned} D_A \psi &= 0 \\ F_A^+ &= \sigma(\psi). \end{aligned} \tag{1.11}$$

Let $\mathcal{M} \subset (\mathcal{A}_{\mathfrak{s}} \times C^\infty(X, W^+))/\mathcal{G}$ denote the set of solutions to (1.10) modulo gauge equivalence. The striking result of Seiberg-Witten theory is that after a generic perturbation, $\mathcal{M}$ is a smooth, compact, orientable, finite dimensional manifold. A small caveat is that for a suitable perturbation to exist, it is necessary that $b^+ \geq 1$. Proving that $\mathcal{M}$ has these properties takes a significant amount of work, but the main ideas are as follows.

First, we solve the Seiberg-Witten equations over appropriately chosen Sobolev spaces. Then regularity results from elliptic operator theory can be used to show that these solutions are gauge equivalent to smooth solutions. Next, a generic perturbation is chosen so that the gauge group acts transversely on solutions. This implies that the quotient of the solution space by the gauge action is smooth. Proving compactness involves introducing the gauge fixing condition $d^*A = 0$, restricting attention to harmonic gauge transformations and applying Uhlenbeck's theorem [55, Theorem 7.14]. Calculating the Fredholm index of the linearised Seiberg-Witten equations gives

$$\begin{aligned} \dim \mathcal{M} &= \frac{c_1(L)^2 - 2\chi(X) - 3\sigma(X)}{4} \\ &= \frac{c_1(L)^2 - \sigma(X)}{4} + b_1 - b^+ - 1. \end{aligned} \tag{1.12}$$

Finally, an orientation of $H^1(X; \mathbb{R}) \oplus H^2_+(X; \mathbb{R})$ determines an orientation of $\mathcal{M}$. In the case that $\dim \mathcal{M} = 0$, the moduli space $\mathcal{M}$ is the disjoint union of a finite number of oriented points and $SW_X(\mathfrak{s}) \in \mathbb{Z}$ is defined to be the number of points counted with sign. If $b^+ > 1$, then $SW_X(\mathfrak{s})$ is a diffeomorphism invariant that depends only on the isomorphism class of the spin^c structure $\mathfrak{s}$ [48, Theorem 2.3.5]. That is, this construction defines a map

$$SW_X : \mathcal{S}(X) \to \mathbb{Z} \tag{1.13}$$

Here $\mathcal{S}(X)$ is the set of isomorphism classes of spin^c structures on X. When $b^+ = 1$, $SW_X(\mathfrak{s})$ also depends on a choice of chamber and when $b^+ = 0$, the moduli space $\mathcal{M}$ always contains singular points corresponding to reducible solutions.

In the case that the dimension of the moduli space is odd, then we define the Seiberg-Witten invariant to vanish. However, in the case that $\dim \mathcal{M} = 2d > 0$, the Seiberg-Witten invariant of X can still be defined in the following manner. Let $\mathcal{G}_0 \subset \mathcal{G}$ be the subgroup of based gauge transformations. These are the gauge transformations $u \in \mathcal{G}$ that satisfy $u(x_0) = 1$ for some chosen basepoint $x_0 \in X$. Consequently $\mathcal{G}/\mathcal{G}_0 \cong S^1$. Let $\widetilde{\mathcal{M}} \subset (\mathcal{A}_{\mathfrak{s}} \times C^\infty(X, W^+))/\mathcal{G}_0$ denote the set of solutions to (1.10) modulo $\mathcal{G}_0$ so that $\widetilde{\mathcal{M}} \to \mathcal{M}$ is a principle S^1-bundle. Let $\mathcal{L} \to \mathcal{M}$ denote the complex line bundle associated to $\widetilde{\mathcal{M}}$ and define $\tau = c_1(\mathcal{L}) \in H^2(\mathcal{M}; \mathbb{Z})$. The Seiberg-Witten invariant of X is

$$SW_X(\mathfrak{s}) = \int_{\mathcal{M}} \tau^d. \tag{1.14}$$

Once again, the map $SW_X : \mathcal{S}(X) \to \mathbb{Z}$ is a diffeomorphism invariant if $b^+ > 1$ and depends only on a choice of chamber when $b^+ = 1$. Note that if $d = 0$, then this integral is the oriented point count of $\mathcal{M}$.

Remark 1.3 (Simple Type Conjecture)**:** Let $\operatorname{Supp}(SW_X) \subset \mathcal{S}(X)$ denote the support of SW_X. We say that X is of simple type if for each $\mathfrak{s} \in \operatorname{Supp}(SW_X)$, we have

$$\frac{c_1(L)^2 - \sigma(X)}{4} = b^+ - b_1 + 1.$$

That is, the expected dimension of the moduli space associated to $\mathfrak{s}$ is zero. In particular, this implies that if $\dim \mathcal{M}_{\mathfrak{s}} > 0$, then $SW_X(\mathfrak{s}) = 0$. Currently, the large number of known examples of smooth 4-manifolds with $b^+ > 1$ are of simple type. It is conjectured [38, Conjecture 1.6.2] that every simply connected 4-manifold with $b^+ > 1$ is of simple type, which is known to at least be true for symplectic manifolds. In 2023, it was shown that the mod 2 simple type conjecture holds given a small assumption on the cohomology ring of M [36].

The Seiberg-Witten invariant has many useful properties which make it practically simpler to work with than other gauge theories. The most notable properties are the listed below.

- **Finiteness:** The map $SW_X : \mathcal{S}(X) \to \mathbb{Z}$ is supported on a finite set. That is, $\mathrm{SW}_X(\mathfrak{s})$ is only non-zero for finitely many $\mathfrak{s} \in \mathcal{S}(X)$.

- **Weitzenböck forumla:** Let $\psi \in C^\infty(X, W^+)$ be a spinor, $A \in \mathcal{A}_{\mathfrak{s}}$ a connection and let $s : X \to \mathbb{R}$ denote the scalar curvature of X. Write ∇_A^* to denote the formal L^2-adjoint of ∇_A. The Weitzenböck formula [55, Theorem 6.19] is

$$\mathcal{D}_A \mathcal{D}_A \psi = \nabla_A^* \nabla_A \psi + \frac{1}{4} s\psi + \rho(F_A)\psi. \tag{1.15}$$

The operator $\nabla_A^*\nabla_A$ is called the Bochner-Laplacian. If (ψ, A) is a monopole then

$$\Delta_g|\psi|^2 + \frac{1}{4}s|\psi|^2 + \frac{1}{2}|\psi|^4 \leq 0.$$

Here $\Delta_g = d^*d$ is the positive definite Laplace-Beltrami operator, which is positive at a maximum of $|\psi|^2$. This implies that $|\psi| \leq S$ where $S = \max_X\{-s, 0\}$. These formulas will be used extensively in Chapter 3.

- **Scalar curvature:** The above estimate $|\psi| \leq S$ implies that if X admits a metric with non-negative scalar curvature, then the map SW_X is identically zero.

- **Gluing formulas:** Let $X = X_1\#X_2$ be a connected sum of compact, oriented 4-manifolds. Salamon [54] showed that if $b^+(X_1) \geq 1$ and $b^+(X_2) \geq 1$, then the Seiberg-Witten invariant of X vanishes on all spin^c structures. A striking consequence of this formula is that X cannot be symplectic [48, Corollary 4.6.2]. For $X = X_1\#\overline{\mathbb{CP}^2}$, there is the blowup formula [48, Theorem 4.6.7]. Notice that in this case $b^+(\overline{\mathbb{CP}^2}) = 0$. The techniques used in the proof of this formula can be used to obtain a general connected sum formula SW_X.

1.2 Stable homotopy theory

In [10], Bauer-Furuta defined a refinement of the Seiberg-Witten invariant which is known as the Bauer-Furuta invariant. This invariant is derived from the S^1-equivariant stable cohomotopy class of the Seiberg-Witten monopole map. Proving results about the Seiberg-Witten invariant often requires arguments draped in difficult analysis, however techniques from algebraic topology can be used on the Bauer-Furuta invariant to circumvent these needs. In this section, we review the fundamentals of stable homotopy theory and Bauer's description [8] of equivariant stable cohomotopy groups.

1.2.1 The classical setting

Let X and Y denote pointed topological spaces. That is, X and Y have distinguished basepoints $x_0 \in X$ and $y_0 \in Y$. We will assume that all maps $f : X \to Y$ are continuous and basepoint preserving. Denote by $[X, Y]$ the set of based homotopy classes of maps between X and Y. Let S^n denote the unit sphere in $\mathbb{R} \oplus \mathbb{R}^n$ and choose $(1, 0) \in S^n$ as the basepoint. The n-th homotopy group of X is

$$\pi_n(X) = [S^n, X].$$

Recall that the smash product of pointed spaces is defined as

$$X \wedge Y = X \times Y/(X \times \{y_0\} \cup \{x_0\} \times Y).$$

The suspension ΣX of X is given by the formula $\Sigma X = S^1 \wedge X$. For any map $f : X \to Y$, let $\Sigma f : \Sigma X \to \Sigma Y$ be defined by $\Sigma f = \mathrm{id} \wedge f$. Thus suspension defines an endofunctor on the category of pointed topological spaces. Notice that $S^n \wedge S^m = S^{n+m}$ and so n-fold suspension is given by $\Sigma^n X = S^n \wedge X$.

Given a homotopy class $[f] \in \pi_n(X)$, the suspension $\Sigma f : S^{n+1} \to \Sigma X$ defines a class $[\Sigma f] \in \pi_{n+1}(\Sigma X)$. It is clear that the class $[\Sigma f]$ is independent of the choice of representative f. Thus, suspension defines a map of homotopy groups

$$\Sigma : \pi_n(\Sigma^n X) \to \pi_{n+1}(\Sigma^{n+1} X).$$

A fundamental result in homotopy theory is the Freudenthal suspension theorem [24], which states that this map is an isomorphism for large enough n. This theorem is the motivation behind stable homotopy theory, the objective being to capture behaviour that persists under arbitrary suspension. With this principle in mind, we make the following definition.

Definition 1.4. *For any $n \geq 0$, the n-th stable homotopy group $\pi_n^s(X)$ of X is*

$$\begin{aligned}\pi_n^s(X) &= \underset{\longrightarrow k}{\mathrm{Colim}}\, \pi_{n+k}(\Sigma^k X) \\ &= \underset{\longrightarrow k}{\mathrm{Colim}} [S^{n+k}, \Sigma^k X].\end{aligned}$$

Note that this sequence is constant for large enough k, hence the limit is guarenteed to exist. In the stable range, the homotopy group $\pi_{n+k}(\Sigma^k X) = [S^{n+k}, \Sigma^k X]$ does not depend on the dimension of the domain and target, but only on the difference in dimensions.

In the opposite fashion, the n-th cohomotopy set of X is $\pi^n(X) = [X, S^n]$. The functor π^n is now contravariant, but suspension still defines a map $\Sigma : \pi^n(X) \to \pi^{n+1}(\Sigma X)$. Denote the n-th stable cohomotopy group of X as $\pi_s^n(X)$, which is given by the formula

$$\begin{aligned}\pi_s^n(X) &= \underset{\longrightarrow k}{\mathrm{Colim}}\, \pi^{n+k}(\Sigma^k X) \\ &= \underset{\longrightarrow k}{\mathrm{Colim}} [\Sigma^k X, S^{n+k}].\end{aligned}$$

1.2.2 Spanier-Whitehead spectra

The modern approach to studying stable homotopy theory is to work in the homotopy category of spectra. In a loose sense, a spectrum is a sequence of topological spaces indexed by the natural numbers with certain suspension compatibility conditions. The stable cohomotopy groups above define a generalised cohomology theory, and Brown's representability theorem [15] guarantees this cohomology theory is representable. Spectra are the natural objects for representing stable cohomotopy groups. In particular, the sphere spectrum $\mathbb{S}^n$ represents ordinary stable cohomotopy groups.

In order to define the Bauer-Furuta invariant, it will be more practical for our purposes to work with spaces that are indexed by finite dimensional subspaces of an infinite dimensional Hilbert space. The objects that fulfil this purpose are Spanier-Whitehead spectra and this approach was outlined by Bauer in [8]. In this chapter, we unpack Bauer's description of Spanier-Whitehead spectra and demonstrate how these objects represent equivariant stable cohomotopy groups

To begin with, fix a universe $\mathcal{U}$. That is, $\mathcal{U}$ is an infinite dimensional separable Hilbert space that will keep track of suspension coordinates. For any subspace $U \subset \mathcal{U}$, let S_U denote the unit sphere in $\mathbb{R} \oplus U$. The space S_U has a natural basepoint $(1, 0) \in \mathbb{R} \oplus U$. If U is finite dimensional, then S_U is the one-point compactification of U and the basepoint is the point at infinity. For any direct sum $V \oplus U$, we have $S_{V \oplus U} = S_V \wedge S_U$. Now instead of suspending spaces by taking a smash product with a sphere S^n, we take the smash product with S_U for some n-dimensional subspace $U \subset \mathcal{U}$.

Definition 1.5. *A Spanier-Whitehead spectrum $\mathcal{A} = \{\mathcal{A}_U\}$ indexed by $\mathcal{U}$ is the following data:*

1. *For each finite dimensional subspace $U \subset \mathcal{U}$, a pointed topological space $\mathcal{A}_U$.*
2. *For any finite dimensional subspace $W \supset U$ with orthogonal decomposition $W = V \oplus U$, a structure homeomorphism*

$$\sigma_{U,W} : S_V \wedge A_U \to A_W.$$

These structure maps have the property that for any finite dimensional subspace $W' \supset W$ with $W' = V' \oplus W$ orthogonally, the following diagram com-

mutes up to homotopy.

$$\begin{tikzcd} S_{V'\oplus V} \wedge A_U \arrow[rr, "\sigma_{U,W'}"] \arrow[d, "="'] & & A_{W'} \\ S_{V'} \wedge S_V \wedge A_U \arrow[rr, "id\wedge\sigma_{U,W}"] & & S_{V'} \wedge A_W \arrow[u, "\sigma_{W,W'}"'] \end{tikzcd} \tag{1.16}$$

We will refer to Spanier-Whitehead spectra as simply as spectra. Morphisms between spectra are defined stably as in ordinary stable homotopy theory.

Definition 1.6. *The set of morphisms $Hom_{\mathcal{U}}(\mathcal{A}, \mathcal{B})$ between two spectra $\mathcal{A}$ and $\mathcal{B}$, both indexed by $\mathcal{U}$, is*

$$Hom_{\mathcal{U}}(\mathcal{A}, \mathcal{B}) = \underset{U\subset\mathcal{U}}{\mathrm{Colim}}[\mathcal{A}_U, \mathcal{B}_U].$$

This colimit is taken over morphisms of the form

$$[\mathcal{A}_U, \mathcal{B}_U] \overset{id_{S_V}\wedge -}{\longrightarrow} [S_V \wedge \mathcal{A}_U, S_V \wedge \mathcal{B}_U] = [\mathcal{A}_W, \mathcal{B}_W]$$

for $W = V \oplus U$ orthogonally. The identification of $[S_V \wedge \mathcal{A}_U, S_V \wedge \mathcal{B}_U]$ with $[\mathcal{A}_W, \mathcal{B}_W]$ is given by the structure maps $\sigma^{\mathcal{A}}_{U,W}$ and $\sigma^{\mathcal{B}}_{U,W}$.

Note that morphisms between spectra are only defined up to homotopy and are only defined stably.

Example 1.7 (Suspension Spectrum)**:** For any space A, define the suspension spectrum ΣA by

$$(\Sigma A)_U = S_U \wedge A.$$

For $W = V \oplus U$ orthogonally, the structure map $\sigma_{U,W} : S_V \wedge (S_U \wedge A) \to S_W \wedge A$ is just the identity. Further, a map $f : A \to B$ induces a map $\Sigma f : \Sigma A \to \Sigma B$ of spectra by taking smash products with the identity. Thus Σ embeds pointed topological spaces as a full subcategory inside the category of spectra. We write $\mathbb{S}^n$ to denote the suspension spectrum of S^n.

More generally, for any fixed finite dimensional subspace $V \subset \mathcal{U}$ define the suspension $\Sigma^V \mathcal{A}$ of a spectrum $\mathcal{A}$ by

$$(\Sigma^V \mathcal{A})_U = S_V \wedge \mathcal{A}_U.$$

The associated structure maps are the obvious ones induced by smash products with the identity.

Example 1.8 (Desuspension)**:** Fix a finite dimensional subspace $V \subset \mathcal{U}$. For any finite dimensional subspace W containing V, write $W = V \oplus U$ orthogonally and define the desuspension $\Sigma^{-V}\mathcal{A}$ by

$$(\Sigma^{-V}\mathcal{A})_W = \mathcal{A}_U.$$

This defines $\Sigma^{-V}A$ up to isomorphism since the directed system of finite dimensional subspaces containing V is cofinal. For $W' = V' \oplus W$ orthogonally, the structure map $\sigma_{W,W'} : S_{V'} \wedge (\Sigma^{-V}\mathcal{A})_W \to (\Sigma^{-V}\mathcal{A})_{W'}$ is simply equal to $\sigma^{\mathcal{A}}_{U,V'\oplus U} : S_{V'} \wedge \mathcal{A}_U \to \mathcal{A}_{V'\oplus U}$. The set of morphisms between $\Sigma^{-V}\mathcal{A}$ and another spectrum $\mathcal{B}$ is given by

$$\mathrm{Hom}_{\mathcal{U}}(\Sigma^{-V}\mathcal{A}, \mathcal{B}) = \mathrm{Hom}_{\mathcal{U}}(\mathcal{A}, \Sigma^{V}\mathcal{B}).$$

That is, Σ^{-V} is the left adjoint of Σ^{V}.

Example 1.9 (Smash product of spectra)**:** Let $\mathcal{A}$ be a spectrum indexed by $\mathcal{U}$ and $\mathcal{B}$ be a spectrum indexed by $\mathcal{V}$. The smash product $\mathcal{A} \wedge \mathcal{B}$ is a spectrum indexed by the universe $\mathcal{U} \oplus \mathcal{V}$, defined for subspaces $U \subset \mathcal{U}$ and $V \subset \mathcal{V}$ by

$$(\mathcal{A} \wedge \mathcal{B})_{U\oplus V} = \mathcal{A}_U \wedge \mathcal{B}_V.$$

Let $W_U = U' \oplus U$ and $W_V = V' \oplus V$ orthogonally. The structure map $\sigma_{U\oplus V, W_U \oplus W_V}$ is defined by the following diagram.

$$\begin{tikzcd}
S_{U'\oplus V'} \wedge (\mathcal{A} \wedge \mathcal{B})_{U\oplus V} \arrow[r, "\sigma_{U\oplus V, W_U\oplus W_V}"] \arrow[d, "="] & (\mathcal{A} \wedge \mathcal{B})_{W_U \oplus W_V} \arrow[d, "="] \\
(S_{U'} \wedge \mathcal{A}_U) \wedge (S_{V'} \wedge \mathcal{B}_V) \arrow[r, "\sigma_{U,W_U} \wedge \sigma_{V,W_V}"] & \mathcal{A}_{W_U} \wedge \mathcal{B}_{W_V}
\end{tikzcd} \tag{1.17}$$

The motivating principle behind defining these objects is that spectra represent stable cohomology theories. In this case, let B be a compact topological space and fix a universe $\mathcal{U}$. Let λ be a real K-theory element $\lambda \in KO^0(B)$. Write $\lambda = E - F$ where E and F are honest finite dimensional vector bundles over B. Assume without loss generality that $F = B \times V$ is trivial with $V \subset \mathcal{U}$. Let TE be the Thom space of E with $T\lambda = \Sigma^{-V}TE$ the Thom spectrum of λ.

Definition 1.10. *The n-th stable cohomotopy group of B with coefficients in $\lambda \in KO^0(B)$ is*

$$\begin{aligned} \pi^n_{\mathcal{U}}(B;\lambda) &= Hom_{\mathcal{U}}(T\lambda, \mathbb{S}^n) \\ &= \operatorname*{Colim}_{U\perp V}[S_U \wedge TE, S_U \wedge S_V \wedge S^n]. \end{aligned} \tag{1.18}$$

1.2.3 Equivariant stable cohomotopy groups

The definitions in the previous section naturally extend to the equivariant setting of G-spaces for G a compact lie group. A G-space is a pointed topological space X with a continuous left action $G \times X \to X$ that fixes the basepoint. For two G-spaces X and Y, let $[X, Y]^G$ denote the set of homotopy classes through equivariant pointed maps. The diagonal subgroup of $G \times G$ naturally defines a G-action on the smash product $X \wedge Y$. In this setting, the definition of a G-universe requires a little more care.

Definition 1.11. *A G-universe $\mathcal{U}$ is an infinite dimensional separable Hilbert space which G acts on by isometric bijections. It is required that $\mathcal{U}$ contains the trivial representation, and for any irreducible G-module M, $Hom_G(M,\mathcal{U})$ is either zero or infinite dimensional.*

The condition that $\mathrm{Hom}_G(M,\mathcal{U})$ is either zero or infinite dimensional guarantees that if we ever suspend by an irreducible representation M, then we can suspend by M an arbitrary number of times. A G-universe is called complete if it contains a copy of every irreducible representation of G [32].

For a subrepresentation $U \subset \mathcal{U}$, extend the G-action on U to $S_U \subset \mathbb{R} \oplus U$ by acting trivially on the $\mathbb{R}$ component. Since G acts by isometries on U, the basepoint $\infty = (1, 0) \in S_U$ is fixed.

Definition 1.12. *A G-spectrum $\mathcal{A} = \{\mathcal{A}_U\}$ (indexed by a G-universe $\mathcal{U}$) is a collection of G-spaces indexed by finite dimensional subrepresentations $U \subset \mathcal{U}$. Additionally, for any finite dimensional subrepresentation $W \supset U$ with orthogonal decomposition $W = V \oplus U$, there is an equivariant structure homeomorphism*

$$\sigma_{U,W} : S_V \wedge A_U \to A_W.$$

The structure maps must satisfy the same homotopy commutativity requirement of Definition 1.5.

The set of morphisms $Hom_{G,\mathcal{U}}(\mathcal{A}, \mathcal{B})$ between two G-spectra is

$$Hom_{G,\mathcal{U}}(\mathcal{A}, \mathcal{B}) = \operatorname*{Colim}_{U \subset \mathcal{U}} [\mathcal{A}_U, \mathcal{B}_U]^G.$$

This colimit is identical to the one in Definition 1.6.

For any G-space X, the equivariant suspension spectrum ΣX can be defined as in

Example 1.7. Once again let $\mathbb{S}^n = \Sigma S^n$ denote the sphere spectrum where S^n is given the trivial G-action.

Finally, to define equivariant stable cohomotopy groups, let B be a compact space and let $\lambda \in RO(B)$ be an equivariant K-theory class. This class λ can be represented as a difference of finite dimensional vector bundles $\lambda = E - F$ with $F \cong B \times V$ for some finite dimensional subrepresentation $V \subset \mathcal{U}$. As in 1.13, the equivariant stable cohomotopy groups of B are defined as maps into equivariant sphere spectra.

Definition 1.13. *The n-th equivariant stable cohomotopy group of B with coefficients in λ is*

$$\begin{aligned} \pi^n_{G,\mathcal{U}}(B;\lambda) &= Hom_{G,\mathcal{U}}(T\lambda, \mathbb{S}^n) \\ &= \operatorname*{Colim}_{U \perp V} [S_U \wedge TE, S_U \wedge S_V \wedge S^n]^G. \end{aligned} \tag{1.19}$$

Chapter 2

Finite dimensional approximation

In [10], Bauer-Furuta described two methods of finite dimensional approximation which they used to define the stable cohomotopy class of a Fredholm map $f : H' \to H$ between Hilbert spaces. One method is a construction due to Schwarz [60] and the other is their own construction. Bauer clarified their construction in [8] using Spanier-Whitehead categories. In this chapter, we carefully define the two different methods of finite dimensional approximation and give a detailed proof of their equivalence. We begin with the unparameterised case, then work up to the parameterised and equivariant setting.

2.1 Bounded Fredholm maps

Let H and H' be separable Hilbert spaces and fix a linear Fredholm map $l : H' \to H$.

Definition 2.1. *A map $f : H' \to H$ is Fredholm (relative to l) if $c = f - l : H' \to H$ is continuous and compact.*

Note that f is not assumed to be linear, so we will be careful to distinguish between Fredholm maps and linear Fredholm maps. Recall that a compact map is one that sends bounded sets to pre-compact sets. In other words, the map f decomposes as $f = l + c$ for some continuous, compact map c. Our interest is in bounded Fredholm maps in the following sense.

Definition 2.2. *A Fredholm map $f : H' \to H$ is bounded if for any bounded set $D \subset H$, the preimage $f^{-1}(D)$ is bounded.*

An alternative definition is that if $f^{-1}(D)$ is unbounded, then D must be unbounded. Consequently f maps unbounded sets to unbounded sets. This definition is quite different to the usual notion of boundedness for linear maps. In the finite dimensional case, a bounded Fredholm map is exactly a continuous and proper map.

For any normed space V, write $S(V) \subset V$ to denote the unit sphere of V. Recall that $S_H = S(\mathbb{R} \oplus H)$ denotes the unit sphere in $\mathbb{R} \oplus H$. There is a natural identification $H \subset S_H$ determined by the map

$$\begin{aligned} \iota_H &: H \to S(\mathbb{R} \oplus H) \\ \iota_H(h) &= \frac{1}{|h|^2 + 1}(|h|^2 - 1, 2h). \end{aligned} \tag{2.1}$$

A simple calculation shows that ι_H is injective and surjects onto $S(\mathbb{R}\oplus H) \setminus \{(1,0)\}$. The inverse map $\iota_H^{-1} : S(\mathbb{R} \oplus H) \setminus \{(1,0)\} \to H$ is given by stereographic projection

$$\iota_H^{-1}(t,h) = \frac{1}{1-t}h. \tag{2.2}$$

The point $(1,0) \in S_H$ is called the point at infinity and gives S_H a natural pointed structure with basepoint $\infty \in S_H$. If H is finite dimensional, then S_H is the one-point compactification of H. In the infinite dimensional case, S_H is notably not compact.

Lemma 2.3. *A Fredholm map $f : H' \to H$ extends continuously to a map $f : S_{H'} \to S_H$ if and only if f is bounded.*

Proof. Extend f to a pointed map $f : S_{H'} \to S_H$ by defining $f(\infty_{H'}) = \infty_H$. It remains to show that this extension is continuous at infinity if and only if f is bounded. First suppose that f is bounded and let $h_n \in H'$ be a sequence such that $|f(h_n)|$ does not converge to infinity. This means that $f(h_n)$ has a bounded subsequence and therefore, by the boundedness of f, h_n has a bounded subsequence. Thus h_n does not converge to infinity and by contrapositive, f is continuous at infinity.

Now assume that $f : S_{H'} \to S_H$ is continuous at infinity. Let $D \subset H$ be a subset such that $f^{-1}(D)$ is unbounded. Then there is a sequence $h_n \in f^{-1}(D)$ with $|h_n| \to \infty$. But continuity of f implies $|f(h_n)| \to \infty$, thus D is unbounded and consequently f is bounded. $\square$

Let $\mathcal{P}_l(H', H)$ denote the set of bounded Fredholm maps $f : H' \to H$, relative to l. Equip $\mathcal{P}_l(H', H)$ with the topology induced by the metric

$$d(f, g) = \sup_{h \in H} |i_H(f(h)) - i_H(g(h))|. \tag{2.3}$$

Later, we will analyse $\pi_0(\mathcal{P}_l(H', H))$ using finite dimensional approximation.

Let $W \subset H$ be a closed subspace with $p_W : H \to W$ the orthogonal projection. The orthogonal decomposition $H = W \oplus W^\perp$ gives a decomposition $S_H = S(\mathbb{R} \oplus W \oplus W^\perp)$. The inclusion $W \to H$ identifies $S_W \subset S_H$ as $S(\mathbb{R} \oplus W \oplus 0)$ and similarly the inclusion $W^\perp \to H$ identifies $S_{W^\perp} \subset S_H$ as $S(\mathbb{R} \oplus 0 \oplus W^\perp)$.

Consider the unit sphere $S(W^\perp)$ in $W^\perp$ and notice that $S(W^\perp)$ and S_W are disjoint subsets of S_H. Define a deformation retraction $\rho_W : S_H \setminus S(W^\perp) \to S_W$ by

$$\rho_W(t, w, w') = \frac{1}{\sqrt{t^2 + |w|^2}}(t, w, 0). \tag{2.4}$$

Note that for $(t, w, w') \in S_H \setminus S(W^\perp)$, $w' \neq 1$ and therefore $t^2 + |w|^2 \neq 0$.

Proposition 2.4. *The map $\rho_W : S_H \setminus S(W^\perp) \to S_W$ is a deformation retraction. If $h \in H \setminus W^\perp$, then $\rho_W(h) = \lambda(h)p(h)$ for some positive and continuous function $\lambda : H \setminus W^\perp \to \mathbb{R}$.*

Proof. First notice that $S_W \subset \mathbb{R} \oplus W \oplus W^\perp$ is the set of points $(t, w, 0)$ with $t^2 + |w|^2 = 1$, thus $\rho_W|_{S_W} = \mathrm{id}$ and $\rho_W(S_H \setminus S(W^\perp)) = S_W$. Next, define a homotopy

$$\rho_{W,s}(t, w, w') = \sqrt{\frac{1 - (1-s)^2|w'|^2}{1 - |w'|^2}}(t, w, 0) + (0, 0, (1-s)w').$$

Note that $1 - |w'|^2 = t^2 + |w|^2$. This is a homotopy from $\rho_{W,0} = \mathrm{id}$ to $\rho_{W,1} = \rho_W$, hence ρ_W is a deformation retraction.

For $h = w + w' \in H \setminus W^\perp$ with $|h|^2 = |w|^2 + |w'|^2$ we have

$$\iota_H(h) = \frac{1}{1 + |h|^2}(|h|^2 - 1, 2w, 2w')$$

$$\rho_W(\iota_H(h)) = \frac{1}{\sqrt{(|h|^2 - 1)^2 + 4|w|^2}}(|h|^2 - 1, 2w, 0).$$

Since $h \notin W^\perp$, $w \neq 0$ and $\rho_W(\iota_H(h)) \neq \infty$. Apply stereographic projection to obtain

$$\iota_H^{-1}(\rho_W(\iota_H(h))) = \frac{2w}{\sqrt{(|h|^2-1)^2+4|w|^2}-(|h|^2-1)}.$$

Define

$$\lambda(h) = \frac{2}{\sqrt{(|h|^2-1)^2+4|w|^2}-(|h|^2-1)}.$$

This is a continuous and positive function so long as $|w| > 0$. $\square$

2.2 Bauer-Furuta approximation

Let $f : H' \to H$ be a fixed bounded Fredholm map with $f = l + c$ for l linear Fredholm and c compact. For any finite dimensional subspace $V \subset H$, set $V' = l^{-1}(V)$ and let $p_V, p_{V'}$ be orthogonal projections onto V and V' respectively. We say that V surjects onto $\operatorname{coker} l$ if the restriction $\pi|_V$ of the projection map $\pi : H \to H/\operatorname{im} l$ is surjective. In this case, $V + (\operatorname{im} l)^\perp$ spans H and $\dim V' = \dim V + \operatorname{ind}(l)$. Moreover, suppose that $f|_{S_{V'}}$ misses the sphere $S(V^\perp) \subset S_H$. Now $\rho_V \circ f|_{S_{V'}} : S_{V'} \to S_V$ is a map between spheres with dimensions that differ by $\operatorname{ind}(l)$.

Definition 2.5. *The map $\varphi_f = \rho_V \circ f|_{S_{V'}} : S_{V'} \to S_V$ is called the (Bauer-Furuta) finite dimensional approximation of f.*

Notice that this definition of finite dimensional approximation depends on the choice of subspace V and choice of decomposition $f = l + c$. We will show that the stable homotopy class of φ_f is independent of V, l and c.

For any finite dimensional subspace $W \supset V$, write W as an orthogonal sum $W = U \oplus V$ with U the orthogonal complement of V inside W. Let $W' = l^{-1}(W)$ with $W' = \widetilde{U} \oplus V'$ where $\widetilde{U}$ is the orthogonal complement of V' in W'. Notice that $l|_{\widetilde{U}} : \widetilde{U} \to U$ is an isomorphism.

Definition 2.6. *A finite dimensional subspace $V \subset H$ is admissible (with respect to f) if it satisfies the following three conditions:*

1. *V surjects onto* $\operatorname{coker} l$.

2. *For any finite dimensional subspace $W \supset V$, the image of $f|_{S_{W'}} : S_{W'} \to S_H$ is disjoint from the unit sphere $S(W^\perp)$ in $W^\perp$. Consequently the deformation retract $\rho_W : S_H \setminus S(W^\perp) \to S_W$ determines a map*

$$\rho_W f|_{S_{W'}} : S_{W'} \to S_W.$$

3. *The maps $\rho_W f|_{S_{W'}}$ and $l|_{S_{\widetilde{U}}} \wedge \rho_V f|_{S_{V'}}$ are homotopic as pointed maps under the identification $S_{W'} = S_{\widetilde{U}} \wedge S_{V'}$.*

$$\begin{tikzcd} S_{W'} \arrow[r, "\rho_W f|_{S_{W'}}"] \arrow[d, "="'] & S_W \arrow[d, "="] \\ S_{\widetilde{U}} \wedge S_{V'} \arrow[r, "l|_{S_{\widetilde{U}}} \wedge \rho_V f|_{S_{V'}}"] & S_U \wedge S_V \end{tikzcd} \tag{2.5}$$

Proposition 2.7 ([10] Lemma 2.3)**.** *For any bounded Fredholm map $f = l + c : H' \to H$, there exists an admissible subspace $V \subset H$.*

Proof. To construct one such V, let $D \subset H$ be the closed unit disk in H. By the boundedness condition, $f^{-1}(D)$ is bounded. Let D'_R be a closed disk of radius R containing $f^{-1}(D)$. Consequently, if $|h'| > R$, then $|f(h)| > 1$. Let C be the closure of $c(D'_R)$, which is compact since c is a compact map. Let $0 < \varepsilon \leq \frac{1}{4}$ and choose a finite covering of C by balls of radius ε with centers v_i for $i = 1, ..., N$. Now define $V = \mathrm{span}\{v_1, ..., v_N\} + \mathrm{im}(l)^\perp$. Note that V is finite dimensional since $\dim \mathrm{im}(l)^\perp = \dim \mathrm{coker}(l) < \infty$. The subspace V satisfies (1) by construction, it remains to show that it satisfies (2) and (3).

Let $W \supset V$ be a subspace of H and let p_W be orthogonal projection onto W. For any $h \in D'_R$, $c(h)$ is within a distance of ε from V and therefore also from W. That is,

$$|(1 - p_W)c(h)| < \varepsilon.$$

Now for $w \in W'$, suppose for a contradiction that $f(w) \in S(W^\perp)$. In particular, $|f(w)| = 1$ and $w \in D'_R$.

$$\begin{aligned} f(w) &= (l + c)(w) \\ &= l(w) + p_W c(w) + (1 - p_W)c(w) \end{aligned}$$

Since $l(w) \in W$ and $p_W c(w) \in W$, the assumption that $f(w) \in S(W^\perp)$ implies $f(w) = (1-p_W)c(w)$. This is impossible since $|f(w)| = 1$ and $|(1-p_W)c(w)| < \varepsilon < 1$. Hence (2) holds.

For (3), a homotopy from $\rho_W f|_{S_{W'}}$ to $l|_{S_{\widetilde{U}}} \wedge \rho_V f|_{S_{V'}}$ is required. Since ρ_W is a deformation retraction it suffices to find a homotopy between $f|_{S_{W'}}$ and $l|_{S_{\widetilde{U}}} \wedge \rho_V f|_{S_{V'}}$. First a homotopy from $f|_{S_{W'}}$ to $l|_{\widetilde{U}} \wedge \rho_V f|_{S_{V'}}$ on the restricted domain $D'_R \cap W' \subset S_{W'}$ is constructed, then we argue that this homotopy can be extended to $S_{W'}$. Define $h : D'_R \times [0,3] \to H \setminus S(W^\perp)$ by

$$h_t = \begin{cases} l + (1-t)c + tp_V c & 0 \le t \le 1 \\ l + p_V c[(2-t)\mathrm{id}_{W'} + (t-1)p_{V'}] & 1 \le t \le 2 \\ p_U l + [(3-t)p_V + (t-2)\rho_V](l+c)p_{V'} & 2 \le t \le 3. \end{cases}$$

The first stage of h_t is a homotopy from $h_0 = f$ to $h_1 = l + p_V c$. The second stage is a homotopy from $l + p_V c$ to $h_2 = l + p_V c p_{V'}$ and the third stage is from $h_2 = p_U l + p_V(l+c)p_{V'}$ to $h_3 = p_U l + \rho_V f p_{V'}$. Note that $p_U l + p_V(l+c)p_{V'} = l + p_V c p_{V'}$ since $p_V l p_{V'} = p_V l$, so h is continuous. Moreover, $p_U l = p_U l p_{\widetilde{U}} = l|_{\widetilde{U}}$, thus $h_3 = (l|_{\widetilde{U}} + \rho_V f|_{V'})$.

Recall that $c(D'_R)$ is contained in an ε-neighbourhood of V. Further, the image of $h_t|_{D'_R \cap W'}$ is contained in an ε-neighbourhood of W for all t. In particular, h_t is valued in $H \setminus S(W^\perp)$. Let $S' = \partial D'_R \cap W'$. To extend h_t from $D'_R \cap W'$ to $S_{W'}$, we show that $h_t|_{S'}$ misses $W^\perp$ for all t.

For $t = 0$, consider $h_0(s) = f(s)$ with $s \in S'$. Since $s \notin f^{-1}(D)$, $|f(s)| \ge 1$. Further, $f(s)$ is in an ε-neighbourhood of W. Therefore $h_0(s) \notin W^\perp$ since $f(s)$ is a distance of at least $1 - \varepsilon = \frac{3}{4}$ from $W^\perp$. For $t \in [0,1]$, $f_t(s)$ only moves a distance of at most ε since

$$\begin{aligned} |h_t(s) - h_0(s)| &= |l(s) + (1-t)c(s) + t \cdot \mathrm{pr}_V(c(s)) - (l+c)(s)| \\ &= t|(1-p_V)c(s)| \\ &< \varepsilon \end{aligned}$$

Thus $h_t(s) \notin W^\perp$ for $t \le 1$.

For $t \ge 1$, $p_U(h_t(s)) = p_U(l(s))$ and since $p_U(W^\perp) = 0$ we can assume $p_U(l(s)) = 0$. This means $l(s) \in V$ and $s \in S' \cap V'$, hence $h_t(s)$ is fixed for $t \in [1,2]$. For $t \in [2,3]$, $h_t(s)$ moves in a straight line from $p_V f(s)$ to $\rho_V f(s)$. These are both non-zero vectors and it has already been shown that $f(s) \in H \setminus W^\perp$. Therefore $\rho_V f(s)$ is a positive scaling of $p_V f(s)$ by Proposition 2.4. Since $p_V f(s) \notin W^\perp$, no vectors on this straight line can be in $W^\perp$. Thus $h_t|_{S'}$ is valued in $S_H \setminus (D \cap W^\perp)$ for all $t \in [0,3]$. Since $S_H \setminus (D \cap W^\perp)$ is contractible, h can be extended to a homotopy over the complement of $D'_R \cap W'$ in $S_{W'}$. This gives the desired homotopy from $\rho_W f|_{S_{W'}}$ to $l|_{\widetilde{U}} \wedge \rho_V f|_{S_{V'}}$. □

Denote by $\mathcal{V}$ the set of admissible subspaces $V \subset H$. It is apparent from the definition that the properties of an admissible subspace translate to a subspace containing it.

Lemma 2.8. *Let $V \in \mathcal{V}$ and suppose $W \supset V$ is a finite dimensional subspace. Then $W \in \mathcal{V}$. Consequently, $\mathcal{V}$ is cofinal in the directed system of finite dimensional subspaces of H.*

Proof. Properties (1) and (2) clearly extend to W since $W \supset V$. To see property (3), let $X \supset W$ be a finite dimensional subspace and write $X = \widetilde{U}_X \oplus \widetilde{U}_W \oplus V$ orthogonally with $W = \widetilde{U}_W \oplus V$. Applying property (3) to $X \supset V$ gives

$$\begin{aligned} \rho_X f|_{S_{X'}} &\simeq l_{S_{\widetilde{U}_X \oplus \widetilde{U}_W}} \wedge \rho_V f|_{S_{V'}} \\ &\simeq l_{S_{\widetilde{U}_X}} \wedge l_{S_{\widetilde{U}_W}} \wedge \rho_V f|_{S_{V'}} \\ &\simeq l_{S_{\widetilde{U}_X}} \wedge \rho_W f|_{S_{W'}}. \end{aligned}$$

In the last line, property (3) was applied to $W \supset V$.

Cofinal in this context means that for any finite dimensional subspace $U \subset H$, there exists $W \in \mathcal{V}$ with $W \supset U$. Given such U, choose some $V \in \mathcal{V}$ and set $W = V + U$. Then $W \in \mathcal{V}$ since $W \supset V$. □

As stated earlier, $\varphi_f : S_{V'} \to S_V$ is a map between finite dimensional spheres with dimensions that satisfy $\dim V' - \dim V = \operatorname{ind} l$. Hence the stable homotopy class of φ_f is an element of $\pi_s^{\dim V}(TV') = \pi^s_{\operatorname{ind} l}(S^0)$.

Corollary 2.9. *Let $f = l + c : H' \to H$ be a bounded Fredholm map and $V \subset H$ an admissible subspace. The stable homotopy class of $\varphi_f = \rho_V f|_{S_{V'}}$ is independent of of the choice of admissible subspace V and Fredholm decomposition $f = l + c$. That is, Bauer-Furuta approximation defines a map*

$$\begin{aligned} \Phi : \mathcal{P}_l(H', H) &\to \pi^s_{\operatorname{ind} l}(S^0) \\ \Phi f &= [\varphi_f]. \end{aligned} \tag{2.6}$$

Proof. Let $U \subset H$ be another choice of admissible finite dimensional subspace with $\varphi_{f,U} = \rho_U f|_{S_{\widetilde{U}}}$. From Lemma 2.8, there is an admissible finite dimensional subspace W containing U and V. Then $\varphi_{f,W}$ is a suspension of both $\varphi_{f,V}$ and $\varphi_{f,U}$ by property (3), hence their stable homotopy classes are equal. The proof that this class is independent of the choice of Fredholm decomposition $f = l + c$ is deferred to the discussion of the parameterised case in Section 2.5. $\square$

2.3 Schwarz approximation

In [60], Schwarz defined a method of finite dimensional approximation with a different flavour to the above Bauer-Furuta method. The Schwarz approximation has practical advantages which will be used in Chapter 3, hence we unpack his construction now. In Chapter 2.6, we prove that it is equivalent to the Bauer-Furuta construction.

Fix a closed disk $D' \subset H'$ with boundary sphere S'. Let $\mathcal{C}_l(D', H)$ denote the set of continuous maps $f : D' \to H$ such that $f - l|_{D'}$ is compact and $f|_{S'}$ is non-vanishing. In particular, we are no longer assuming that f is a bounded Fredholm map. Give $\mathcal{C}_l(D', H)$ the uniform convergence topology so that $\pi_0(\mathcal{C}_l(D', H))$ is the set of compact homotopy classes, which are defined below.

Definition 2.10. *Two Fredholm maps $f_0, f_1 : D' \to H$ are compactly homotopic (relative to l) if there is a homotopy $f_t = l|_{D'} + c_t$ with c_t compact and $(f_t)|_{S'}$ non-vanishing for all $t \in [0, 1]$.*

The homotopy class of a Fredholm map f is not interesting since there is a linear homotopy $l + (1 - t)c$ from f to l. Thus the homotopy class of f is the same as the homotopy class of l, which is classified by $\operatorname{ind} l$ [16]. However, restricting to homotopies through $\mathcal{C}_l(D', H)$ gives more interesting behaviour.

Let $f = l + c \in \mathcal{C}_l(D', H)$. Suppose for now that $c(D')$ is contained in a finite dimensional subspace $V \subset H$ of dimension n. Without loss of generality, we can assume that $(\operatorname{im} l)^\perp \subset V$ since the cokernel of l is finite dimensional. Let $V' = l^{-1}(V)$, which has dimension $\dim V' = n + \operatorname{ind} l$. Denote the restriction $f|_{D' \cap V'}$ by

$$\psi_{f,V} = f|_{D' \cap V'} : (D' \cap V', S' \cap V') \to (V, V \setminus \{0\})$$

Let $W \supset V$ be a finite dimensional subspace containing V with $W = U \oplus V$ orthogonally. Let $W' = l^{-1}(W)$ so that $W' = \widetilde{U} \oplus V$ orthogonally with $l|_{\widetilde{U}} : \widetilde{U} \to U$ an isomorphism. For any map $g : (D' \cap V', S' \cap V') \to (V, V \setminus \{0\})$, define

$$\begin{aligned} &\Sigma^{\widetilde{U}} g : (D' \cap W', S' \cap W') \to (W, W \setminus \{0\}) \\ &\Sigma^{\widetilde{U}} g(u + v) = l(u) + g(v). \end{aligned}$$

Note that for $w = u + v$, if $\Sigma^{\widetilde{U}} g(w) = 0$ then $g(v) = 0$ and $u = 0$, which implies that $w \notin S' \cap W'$. Let $[(A, B); (C, D)]$ denote the set of homotopy classes of maps from (A, B) to (C, D) where the homotopies are through maps of pairs. Then $\Sigma^{\widetilde{U}}$ descends to a map of homotopy classes

$$\Sigma^{\widetilde{U}} : [(D' \cap V', S' \cap V'); (V, V \setminus \{0\})] \to [(D' \cap W', S' \cap W'); (W, W \setminus \{0\})]. \tag{2.7}$$

This map $\Sigma^{\widetilde{U}}$ is a particular instance of suspension. Define

$$\Pi_l(D', H) = \underset{V \subset H}{\operatorname{Colim}} [(D' \cap V', S' \cap V'); (V, V \setminus \{0\})] \tag{2.8}$$

where the colimit is taken over the maps given by (2.7). Any map $g : (D' \cap V', S' \cap V') \to (V, V \setminus \{0\})$ defines a class $[g] \in \Pi_l(D', H)$ by suspension. There is a natural identification $\Pi_l(D', H) = \pi^s_{\operatorname{ind} l}(S^0)$ given by

$$\gamma_{D'} : \Pi_l(D', H) \xrightarrow{\sim} \pi^s_{\operatorname{ind} l}(S^0)$$
$$\gamma_{D'}([f]) = \left[\frac{f|_{S'}}{|f|_{S'}|} \right].$$

The map $\psi_{f,V}$ depends on the choice of subspace V, however the class of $[\psi_{f,V}] \in \Pi_l(D', H)$ does not.

Lemma 2.11. *For $f = l + c \in \mathcal{C}_l(D', H)$, suppose that V and W are finite dimensional subspaces both containing $c(D') \cup (im\, l)^{\perp}$. Then $[\psi_{f,V}]$ and $[\psi_{f,W}]$ are equal in $\Pi_l(D', H)$.*

Proof. Assume without loss of generality that $V \subset W$. As before write $W = U \oplus V$ and $W' = \widetilde{U} \oplus V'$ orthogonally with $l|_{\widetilde{U}} : \widetilde{U} \to U$ an isomorphism. For any element $u + v \in W'$ with $u \in \widetilde{U}$ and $v \in V'$, we have

$$f|_{S' \cap W'}(u + v) = l(u) + l(v) + c(u + v).$$

Define a homotopy

$$F_t(u + v) = l(u) + l(v) + (1 - t)c(v) + tc(v + u).$$

This is a homotopy from $F_0 = \Sigma^{\widetilde{U}} \psi_{f,V}$ to $F_1 = \psi_{f,W}$. Additionally, F_t is non-zero on $S' \cap W'$ for all $t \in [0, 1]$. To see this, recall that $c(D') \subset V$, hence $F_t(v + u) = 0$ implies that $l(u) = 0$. It follows that $u = 0$ and $|v| = 1$. But $f|_{S' \cap W'}(v) = f|_{S' \cap V'}(v)$, which does not vanish. Thus the classes $[\psi_{f,V}]$ and $[\psi_{f,W}]$ are equal in $\Pi_l(D', H)$. $\square$

Lemma 2.12. *Suppose $f_t = l + c_t : [0,1] \to \mathcal{C}_l(D', H)$ is a compact homotopy with $c_0(D') \cup c_1(D') \cup (im\, l)^\perp \subset V$ for some finite dimensional subspace $V \subset H$. Then $\psi_{f_0,V}$ and $\psi_{f_1,V}$ are homotopic as maps of pairs.*

Proof. By Definition 2.10, the restriction

$$(f_t)|_{D' \cap V'} : (D' \cap V', S' \cap V') \to (V, V \setminus \{0\})$$

is a map of pairs for all t. Hence $(f_t)|_{D' \cap V'}$ is a homotopy through maps of pairs from $F_0 = \psi_{f_0,V}$ to $F_1 = \psi_{f_1,V}$. □

Not all elements $f = l + c \in \mathcal{C}_l(D', H)$ are nice enough to have $c(D')$ contained in a finite dimensional subspace, however it is true that every compact homotopy class has such a representative.

Lemma 2.13. *For any $f \in \mathcal{C}_l(D', H)$, there exists $\delta > 0$ such that $|f(h)| > \delta$ for all $h \in S'$.*

Proof. Suppose no such δ exists and let $h_n \in S'$ be a sequence with $|f(h_n)| \to 0$. By the weak compactness of S', after passing to a subsequence it can be assumed that $h_n \to h$ weakly for some $h \in H'$. By the compactness of c, after passing to a further subsequence it can be assumed that $c(h_n) \to a$ strongly for some $a \in H$. Now $l(h_n) = f(h_n) - c(h_n) \to -a$ strongly. Since l is Fredholm, its image is closed and $a = l(v)$ for some $v \in (\ker l)^\perp$. Write $h_n = x_n + y_n$ for $x_n \in \ker l$ and $y_n \in (\ker l)^\perp$. Now $l(h_n) = l(y_n) \to -l(v)$. Since l is an isomorphism from $(\ker l)^\perp$ onto its image, it follows that $y_n \to -v$. Further $x_n = h_n - y_n \to h + v$ weakly, but $\ker l$ is finite dimensional so $x_n \to h + v$ strongly as well. Thus $h_n \to h$ strongly and $h \in S'$ since S' is closed. However $f(h_n) \to 0$ implies that $f(h) = 0$, contradicting the assumption that $f|_{S'} \neq 0$. □

Remark 2.14: In fact, suppose that $f : H' \to H$ is a bounded Fredholm map with $f^{-1}(0) \subset \operatorname{int} D'$. The above argument can be extended to show that there is a $\delta > 0$ with $|f(h)| > \delta$ for every $h \in H' \setminus \operatorname{int}(D')$. First choose a closed disk E' such that $|f(h)| \geq 1$ for all $h \notin E'$, which we can assume contains D'. Now the argument in the lemma easily extends to the closed, bounded set $E' \setminus \operatorname{int}(D')$.

Corollary 2.15. *Every element $f = l + c_0 \in \mathcal{C}_l(D', H)$ is compactly homotopic to a map $g = l + c_1 \in \mathcal{C}_l(D', H)$ with $c_1(D')$ contained in a finite dimensional subspace.*

Proof. From the above lemma, choose $\delta > 0$ such that $|f(h)| > \delta$ for all $h \in S'$. Let $\varepsilon = \frac{\delta}{2}$. Since D' is bounded and c is compact, the closure of $c(D')$ can be covered by finitely many balls of radius ε with centers $v_1, ..., v_n$. Define $V = \operatorname{span}\{v_i\}$, set $V' = l^{-1}(V)$ and let $g = l + p_V c$. By construction, $|(1 - p_V)c(h)| < \varepsilon$ for all $h \in D'$. Define a homotopy $F_t = l + (1-t)c + tp_V c$, $t \in [0, 1]$. Notice that for $h \in S'$,

$$\begin{aligned} |F_t(h)| &= |l(h) + c(h) - t(1 - p_V)c(h)| \\ &\geq |f(h)| - t|(1 - p_V)c(h)| \\ &> \frac{\delta}{2}. \end{aligned}$$

Thus F_t is a compact homotopy from $F_0 = f$ to $F_1 = g$. Note that this also shows that $g \in \mathcal{C}_l(D', H)$. □

For any $f \in \mathcal{C}_l(D', H)$, define $\psi_f = \psi_{g,V}$ for some choice of g compactly homotopic to f with finite dimensional subspace V satisfying $c(D') \cup (\operatorname{im} l)^{\perp} \subset V$. The map $f \mapsto [\psi_f]$ identifies $\pi_0(\mathcal{C}_l(D', H))$ with a subset of $\Pi_l(D', H)$, which is a result originally due to Schwarz [60].

Theorem 2.16 ([11] Theorem 5.3.20)**.** *Let $l : H' \to H$ be a linear Fredholm operator and fix a closed disk $D' \subset H'$ with $S' = \partial D'$. The map*

$$\begin{aligned} \Psi_{D'} : \pi_0(\mathcal{C}_l(D', H)) &\to \Pi_l(D', H) \\ [f] &\mapsto [\psi_f] \end{aligned} \tag{2.9}$$

is well-defined and injective.

Proof. Lemma 2.11 and 2.12 show that $\Psi_{D'} : \pi_0(\mathcal{C}_l(D', H)) \to \Pi_l(D', H)$ is well defined. To prove injectivity, suppose $f = l + c_0$ and $g = l + c_1$ are elements of $\mathcal{C}_l(D', H)$ with $[\psi_f] = [\psi_g]$. After applying Σ if necessary, we can assume that there is a finite dimensional subspace $V \subset H$ containing $c_0(D') \cup c_1(D')$ and a compact homotopy $F : (D' \cap V') \times [0, 1] \to V$ with $F_0 = f$ and $F_1 = g$. To show that f and g are compactly homotopy, we must extend F to $D' \times [0, 1]$.

Let $v_1, ..., v_n$ be an orthonormal basis for V and write $F_t(v) = l(v) + \sum_{i=1}^{n} c_t^i(v)v_i$ with $c_t^i(v) = \langle c_t(v), v_i\rangle$. Each component c^i is a map $c^i : (D' \cap V') \times [0, 1] \to \mathbb{R}$. Since $(D' \cap V') \times [0, 1]$ is a closed subset of $D' \times [0, 1]$, the Tietze extension theorem guarantees the existence of a continuous extension $c^i : D' \times [0, 1] \to \mathbb{R}$. Define

$$H_t = l + \sum_{i=1}^{n} c_t^i v_i : D' \times [0, 1] \to H.$$

It remains to show that H_t is non-vanishing on S' for all t. If $H_t(h) = 0$ for $h \in S'$, then $l(h) \in V$. Thus $h \in l^{-1}(V) = V'$ and $h \in S' \cap V'$. Therefore $H_t(h) = F_t(h) \neq 0$. Thus H_t is a compact homotopy from f to g. □

The map $\Psi_{D'}$ extends to a map $\Psi : \mathcal{P}(H', H) \to \pi^s_{\mathrm{ind}\, l}(S^0)$ in the following way. Let f be a bounded Fredholm map. Then the boundedness condition guarantees the existence of a disk D' with $f^{-1}(0) \subset \mathrm{int}\, D'$. Define

$$\Psi f = \gamma_{D'}(\Psi_{D'} f).$$

This definition is independent of D' in the following sense.

Lemma 2.17. *Let $f = l + c$ be a Fredholm map and assume that $D'_0, D'_1 \subset H'$ are disks with the property that $f^{-1}(0) \subset \mathrm{int}\, D'_0$ and $f^{-1}(0) \subset \mathrm{int}\, D'_1$. Then*

$$\gamma_{D'_0}(\Psi_{D'_0} f) = \gamma_{D'_1}(\Psi_{D'_1} f)$$

Proof. By Corollary 2.15 we can assume that $c(D'_0) \cup c(D'_1) \cup (\mathrm{im}\, l)^\perp$ is contained inside a finite dimensional subspace $V \subset H$. Set $V' = l^{-1}(V)$. Assume without loss of generality that $D'_0 \subset D'_1$. Let $S' = \partial D'_0$ and identify $\overline{D'_1 - D'_0}$ with $S' \times [0, 1]$ such that $S'_0 = S' \times \{0\}$ and $S'_1 = S' \times \{1\}$. Denote by $S = S(V)$ the unit sphere in V and define

$$F = \frac{f|_{(\overline{D'_1 - D'_0}) \cap V'}}{\left| f|_{(\overline{D'_1 - D'_0}) \cap V'} \right|} : S' \times [0, 1] \to S$$

Then F is a homotopy from $\gamma_{D_0}(\psi_{f|_{D'_0}, V})$ to $\gamma_{D_1}(\psi_{f|_{D'_1}, V})$. Since $\dim V$ can be arbitrarily large, we have $\gamma_{D'_0}(\Psi_{D'_0} f) = \gamma_{D'_1}(\Psi_{D'_1} f)$. □

Corollary 2.18. *Schwarz approximation induces an injective map*

$$\Psi : \pi_0(\mathcal{P}(H', H)) \to \pi^s_{\mathrm{ind}\, l}(S^0). \tag{2.10}$$

Proof. The only thing left to prove is that Ψ descends to a map on compact homotopy classes. Suppose that $f_t : [0, 1] \to \mathcal{P}_l(H', H)$ is a homotopy through bounded Fredholm maps. For each $t \in [0, 1]$, let $D'_t \subset H'$ be a disk such that $f_t^{-1}(0) \subset \mathrm{int}\, D'_t$. By continuity of the homotopy, for each $t \in [0, 1]$ there is an $\varepsilon_t > 0$ such that $f_s^{-1}(0) \subset \mathrm{int}\, D'_t$ for any $s \in (t - \varepsilon_t, t + \varepsilon_t)$. Thus by compactness of $[0, 1]$, we can choose a disk $D' \subset H'$ such that $f_t^{-1}(0) \subset \mathrm{int}\, D'$ for all $t \in [0, 1]$. Hence $f_t|_{D'}$ is a compact homotopy from $f_0|_{D'}$ to $f_1|_{D'}$ which shows that $\Psi_{D'} f_0 = \Psi_{D'} f_1$. Injectivity follows from Theorem 2.16. □

2.4 Equivalence

Let $f = l + c : H' \to H$ be a bounded Fredholm map. Recall that the Bauer-Furuta finite dimensional approximation φ_f is given by

$$\varphi_f = \rho_V f|_{S_{V'}} : (S_{V'}, \infty) \to (S_V, \infty)$$

for $V \subset H$ an admissible finite dimensional subspace with $V' = l^{-1}(V)$. Corollary 2.9 shows that Bauer-Furuta approximation defines a map

$$\begin{aligned} \Phi : \mathcal{P}_l(H', H) &\to \pi^s_{\operatorname{ind} l}(S^0) \\ f &\mapsto [\varphi_f]. \end{aligned}$$

Alternatively, let $D' \subset H'$ be a disk such that $f^{-1}(0) \subset \operatorname{int} D'$, which is guaranteed to exist by the boundedness condition. Let $S' = \partial D'$ and recall that $p_V : H \to V$ is the orthogonal projection. Assume for now that $p_V f$ does not vanish on $S' \cap V'$ and define

$$\psi_f = p_V f|_{D' \cap V'} : (D' \cap V', S' \cap V') \to (S_V, S_V \setminus \{0\}).$$

Then Schwarz approximation defines another map

$$\begin{aligned} \Psi : \mathcal{P}_l(H', H) &\to \pi^s_{\operatorname{ind} l}(S^0) \\ f &\mapsto \left[\frac{\psi_f|_{S' \cap V'}}{|\psi_f|_{S' \cap V'}|} \right]. \end{aligned}$$

The goal for this section is to show that Φ and Ψ are the same map. Consequently, this will show that Φ is invariant under compact homotopy.

We will show that φ_f and ψ_f are homotopic under some careful identifications. To simplify notation, let

$$\begin{aligned} D'_- &= D' \cap V' \\ D'_+ &= S_{V'} \setminus (V' \cap \operatorname{int}(D')) \\ S'_0 &= S' \cap V'. \end{aligned}$$

That is, $D'_\pm$ are the two hemispheres of $S_{V'}$ with S'_0 the equator. Define an intermediary map

$$\phi_f = \rho_V f|_{S_{V'}} : (S_{V'}, D'_+) \to (S_V, S_V \setminus \{0\}).$$

This definition of ϕ_f assumes that $\rho_V f$ does not vanish on D'_+. The following lemma shows that such a finite dimensional subspace V exists.

Lemma 2.19. *Let $f = l + c : H' \to H$ be a bounded Fredholm map. There exists a finite dimensional subspace $V \subset H$ such that:*

1. *V is an admissible subspace as in Definition 2.6,*
2. *$p_V f$ is non-vanishing on S'_0,*
3. *$\rho_V f|_{S_{V'}}$ is non-vanishing on D'_+.*

Any finite dimensional subspace $W \supset V$ also satisfies these three properties.

Proof. Since f is bounded, choose the closed disk D' to have the property that $|f(h)| > 1$ for $h \notin D'$. As explained in Remark 2.14, choose a $\delta > 0$ such that $|f(h)| > \delta$ for $h \in V' \setminus (V' \cap \operatorname{int}(D'))$. Let $\varepsilon = \min\{\frac{1}{4}, \frac{\delta}{2}\}$ and as in the proof of Lemma 2.13, cover the closure of $c(D')$ by finitely many ε-balls with centres $v_1, ..., v_n$. Then $V = \operatorname{span}\{v_i\} + (\operatorname{im} l)^{\perp}$ has the property that $|(1 - p_V)f(h)| < \varepsilon$ for all $h \in D' \cap V'$. Since $|f(h)| > \delta$ for $h \in S'$, it follows that $p_V f$ is non-vanishing on $S' \cap V'$. As shown in Proposition 2.7, V is an admissible subspace.

Suppose that $\rho_V f(h) = 0$ for some $h \in S_{V'}$. Notice from the definition of ρ_V in (2.4) that this implies that $f(h)$ is finite with $p_V f(h) = 0$ and $|(1 - p_V)f(h)| < 1$. This means that $|f(h)| < 1$ and $h \in D' \cap V'$. Therefore $|(1 - p_V)f(h)| < \varepsilon < \delta$ and $h \notin D'_+$ since $|f(h)| < \delta$. That is, $\rho_V f(h)$ is non-vanishing on D'_+. For any finite dimensional $W \supset V$, it is still the case that $|(1 - p_W)f(h)| < \varepsilon$ for $h \in D' \cap W'$ and the argument can be repeated. □

The above lemma ensures that V can be chosen so that φ_f, ψ_f and ϕ_f are each maps of pairs. Consider the following diagram:

$$\begin{tikzcd}
(S_{V'}, \infty) \arrow[r, "a"] \arrow[d, "\varphi_f"] & (S_{V'}, D'_+) \arrow[dr, "\phi_f"] & (D'_-, S'_0) \arrow[l, "b"'] \arrow[d, dashed, "\psi_f"] \\
(S_V, \infty) \arrow[rr, "c"] & & (S_V, S_V \setminus \{0\})
\end{tikzcd}$$

The maps a, b and c are the obvious inclusions of pairs. The dashed arrow ψ_f does not make the diagram commute, however we will show that it does commute up to homotopy. These inclusions induce functions between homotopy classes of maps of pairs:

$$\begin{array}{ccc} [(S_{V'},\infty)\,;(S_V,\infty)] & \xrightarrow{\quad c_* \quad} & [(S_{V'},\infty)\,;(S_V,S_V\setminus\{0\})] \\ & & \uparrow{\scriptstyle a^*} \\ \big[(D'_-,S'_0)\,;(S_V,S_V\setminus\{0\})\big] & \xleftarrow{\quad b^* \quad} & \big[(S_{V'},D'_+)\,;(S_V,S_V\setminus\{0\})\big] \end{array}$$

Proposition 2.20. *The maps a^*, b^* and c_* induced by inclusions are bijections. The composition*

$$b^*(a^*)^{-1}c_* : [(S_{V'},\infty)\,;(S_V,\infty)] \to \big[(D'_-,S'_0)\,;(S_V,S_V\setminus\{0\})\big]$$

identifies $[\varphi_f]$ with $[\psi_f]$.

Proof. Contracting D'_+ radially to ∞ defines a homotopy $F_t : S_{V'} \to S_{V'}$ with $F_0 = \mathrm{id}$ and $F_1 : (S_{V'}, D'_+) \to (S_{V'}, \infty)$ a map of pairs. The compositions $aF_1 : (S_{V'}, D'_+) \to (S_{V'}, D'_+)$ and $F_1 a : (S_{V'}, \infty) \to (S_{V'}, \infty)$ are both homotopy equivalent to the identity through maps of pairs, hence a is a homotopy equivalence of pairs and a^* is bijection.

It is easy to see that c_* is surjective. Any map $f : S_{V'} \to S_V$ with $f(\infty) \neq 0$ can be composed with a homotopy that moves $f(\infty)$ to ∞ along a path through $S_{V'} \setminus \{0\}$. For injectivity let $g_0, g_1 : S_{V'} \to S_V$ be maps with $g_i(\infty) = \infty$ and suppose that there is a homotopy g_t from g_0 to g_1 with $g_t(\infty) \neq 0$. Since $S_{V'} \times I$ is compact, there is an open neighbourhood $B \subset S_V$ of 0 such that $g_t(\infty) \in S_V \setminus B$ for all t. Thus $[g_0] = [g_1]$ as elements of $[(S_{V'}, \infty); (S_V, S_V \setminus B)]$. By the same reasoning as above, the inclusion $(S_V, \infty) \to (S_V, S_V \setminus B)$ is a homotopy equivalence of pairs. Hence $[g_0] = [g_1]$ as elements of $[(S_{V'}, \infty), (S_V, \infty)]$.

To see that b^* is surjective, suppose $f : D'_- \to S_V$ is a map with $f|_{S'_0}$ valued in $S_V \setminus \{0\}$. Notice that $S_{V'}$ is obtained from D'_- by attaching D'_+ over S'_0. Since $S_V \setminus \{0\}$ is contractible, $f|_{S'_0}$ can be extended to D'_+ by a null homotopy while remaining valued in $S_V \setminus \{0\}$. Thus f extends to $S_{V'}$ with $f(D'_+) \subset S_V \setminus \{0\}$.

For injectivity, let $b' : D'_- \to S_{V'}$ and $b'' : S'_0 \to D'_+$ be inclusions with mapping cones $C_{b'}$ and $C_{b''}$. Recall that the cofiber sequence $(D'_-, S'_0) \to (S_{V'}, D'_+) \to (C_{b'}, C_{b''})$ induces an exact sequence

$$[(C_{b'}, C_{b''}); (S_V, S_V \setminus \{0\})] \to [(S_{V'}, D'_+); (S_V, S_V \setminus \{0\})] \xrightarrow{b^*} [(D'_-, S'_0); (S_V, S_V \setminus \{0\})].$$

The cone $C_{b'}$ deformation retracts onto $C_{b''}$, hence

$$\begin{aligned}[(C_{b'}, C_{b''}); (S_V, S_V \setminus \{0\})] &\cong [(C_{b''}, C_{b''}); (S_V, S_V \setminus \{0\})] \\ &= [C_{b''}, S_V \setminus \{0\}].\end{aligned}$$

However $[C_{b''}, S_V \setminus \{0\}]$ is trivial since $S_V \setminus \{0\}$ is contractible. Thus b^* is injective by the exactness of the cofiber sequence.

It remains to show that $[\psi_f] = [b^*(a^*)^{-1}c_*(\varphi_f)]$. We have that $c_*\varphi_f = a^*\phi_f$, thus it is enough to show that $[\psi_f] = [b^*\phi_f]$. Note that both $\psi_f|_{S_0'}$ and $b^*\phi_f|_{S_0'}$ are valued in $V \setminus \{0\} \subset H \setminus V^\perp$. From Proposition 2.4, $\rho_V f|_{S_0'} = \lambda p_V f|_{S_0'}$ for some positive continuous function $\lambda : H \setminus V^\perp \to \mathbb{R}$. Hence the straight line homotopy from $b^*\phi_f$ to ψ_f defines a homotopy through maps of pairs. $\square$

Corollary 2.21. *The maps* $\Phi : \mathcal{P}(H', H) \to \pi^s_{\operatorname{ind} l}(S^0)$ *and* $\Psi : \mathcal{P}(H', H) \to \pi^s_{\operatorname{ind} l}(S^0)$ *agree. In particular,* Φ *descends to a bijection*

$$\Phi : \pi_0(\mathcal{P}(H', H)) \to \pi^s_{\operatorname{ind} l}(S^0). \tag{2.11}$$

Theorem 2.16 guarantees that Φ is injective and [8, Theorem 2.1] proves that Φ is surjective.

2.5 Parameterised case

Next, these methods of finite dimensional approximation are generalised to the setting of a Hilbert bundle over a finite CW-complex B. In particular, note that B is compact and Hausdorff.

Definition 2.22. *A Hilbert bundle* $H \to B$ *is a locally trivial fibre bundle with fibre an infinite dimensional separable Hilbert space* $\mathcal{U}$*. The structure group of* H *is the group of isometric bijections of* $\mathcal{U}$*.*

Fix two Hilbert bundles H, H' and a fibre preserving map $l : H' \to H$ that is fiberwise linear Fredholm.

Definition 2.23. *A Fredholm map* $f : H' \to H$ *(relative to* l*) is a continuous bundle map such that* $c = f - l$ *is compact.*

In this context, the map $c : H' \to H$ is compact if it maps bounded disk bundles in H' to pre-compact subsets of H. Here a disk bundle $D \subset H$ is a subbundle where each fibre is a disk, centred at zero, with constant radius.

Definition 2.24. *A Fredholm map $f : H' \to H$ is bounded if for any bounded disk bundle $D \subset H$, the preimage $f^{-1}(D)$ is contained in some disk bundle $D' \subset H'$.*

Let S_H denote the unit sphere subbundle $S(\mathbb{R}\oplus H) \subset \mathbb{R}\oplus H$. By the same argument as Lemma 2.3, a Fredholm map $f : H' \to H$ is bounded if and only if it extends continuously to a map $f : S_{H'} \to S_H$. This extension maps $\infty_b \in (S_{H'})_b$ to $\infty_{f(b)} \in (S_H)_{f(b)}$. Recall that the Thom space of H' is $TH' = D(H')/S(H')$ where $D(H')$ is the unit disk bundle in H' and $S(H')$ is the unit sphere bundle. The Thom space TH' has a natural basepoint $[S(H')]$. Equivalently, TH' can be defined as $S_{H'}/B_\infty$ where $B_\infty \subset S_{H'}$ denotes the image of the section at infinity. In this case, $[B_\infty]$ is the basepoint of TH'.

By a theorem of Kuiper [39], every infinite dimensional Hilbert bundle over a compact space is trivial and all choices of trivialisations are homotopic. Let $H \cong B \times \mathcal{U}$ be a trivialisation with projection map $p : H \to \mathcal{U}$ and assume $f : H' \to H$ is bounded Fredholm. Let $\hat{f} = pf$, $\hat{l} = pl$ and $\hat{c} = pc$. The map $\hat{f} : H' \to \mathcal{U}$ extends continuously to a map $\hat{f} : S_{H'} \to S_{\mathcal{U}}$. This factors through the Thom space TH', yielding a map

$$\hat{f} : TH' \to S_{\mathcal{U}}.$$

Theorem 2.25 ([10] Theorem 2.6)**.** *Let $f : H' \to H$ be a bounded Fredholm map between Hilbert bundles over B and fix a trivialisation $H \cong B\times\mathcal{U}$ with $p : B\times\mathcal{U} \to \mathcal{U}$ projection onto the second factor. For any admissible subspace $V \subset \mathcal{U}$, let $\varphi_f = \rho_V \hat{f}|_{S_{V'}} : S_{V'} \to S_V$. The stable cohomotopy class*

$$[\varphi_f] \in \pi^0_{\mathcal{U}}(B; \operatorname{ind} l)$$

is independent of the choice of admissible subspace $V \subset \mathcal{U}$ and presentation $f = l+c$ of f as a sum.

Proof. First, we find an admissible finite dimensional subspace V as in Proposition 2.7. The only difference is that an admissible subspace must surject onto $\operatorname{coker}(\hat{l}_b : H'_b \to \mathcal{U})$ for each $b \in B$. By [2, Proposition A5], there is a finite dimensional subspace $V_0 \subset \mathcal{U}$ containing $(\operatorname{im}\hat{l}_b)^\perp$ for each $b \in B$. Let $D' \subset H'$ be a disk bundle with $|f| > 1$ outside D'. The closure of $\hat{c}(D')$ can be covered by finitely many balls of radius $\varepsilon < \frac{1}{4}$ with centers $v_1, ..., v_n$. Let V be a finite dimensional subspace containing V_0 and $\{v_1, ..., v_n\}$. Now V is an admissible subspace and $V' = \hat{l}^{-1}(V)$ is a finite dimensional vector bundle of rank $\dim V + \operatorname{ind} l$. Further, the K-theory class $V' - V$ represents $\operatorname{ind} l$.

Let $T(\operatorname{ind} l)$ be the Thom spectrum of $\operatorname{ind} l$ and recall from Definition 1.10 that

$$\begin{aligned}\pi^0_{\mathcal{U}}(B; \operatorname{ind} l) &= \operatorname{Hom}(T(\operatorname{ind} l), \mathbb{S}^0)\\ &= \operatorname*{Colim}_{U \subset V^\perp}[S_U \wedge TV', S_U \wedge S_V].\end{aligned}$$

Here $U \subset \mathcal{U}$ is orthogonal to V and the connecting morphisms $[TV', S_V] \to [S_U \wedge TV', S_U \wedge S_V]$ are given by $f \mapsto \mathrm{id}_{S_U} \wedge f$. By the same argument as Proposition 2.7, $\varphi_f = \rho_V \hat{f}|_{S_{V'}}$ defines a stable cohomotopy class independent of the choice of admissible subspace V. Indeed, for any subspace $W = U \oplus V$ containing V with $U \perp V$, write $W' = \hat{l}^{-1}(W)$ as $W' = \widetilde{U} \oplus_B V'$ for $\widetilde{U}$ the fiberwise orthogonal complement of V' in W'. Note that $l|_{\widetilde{U}} : \widetilde{U} \to U \times B$ is a bundle isomorphism. Now $\rho_W \hat{f}|_{S_{W'}}$ is a suspension of $\rho_V \hat{f}|_{S_{V'}}$ by the following identifications

$$\begin{tikzcd} S_{W'} \arrow[r, "\rho_W \hat{f}|_{S_{W'}}"] \arrow[d, "="'] & S_W \arrow[d, "="] \\ S_{\widetilde{U}} \wedge_B S_{V'} \arrow[r, "\hat{l}|_{S_{\widetilde{U}}} \wedge \rho_V \hat{f}|_{S_{V'}}"] & S_U \wedge S_V . \end{tikzcd}$$

Note that since $\widetilde{U}$ is trivial, we can identify $TW' \cong S_U \wedge TV'$.

To see that $[\varphi_f]$ does not depend on the choice of decomposition $f = l + c$, let $f = l_i + c_i$ be two Fredholm decompositions for $i = 0, 1$. Let $F_t = l_t + c_t$ for $l_t = (1-t)l_0 + tl_1$ and $c_t = (1-t)c_0 + tc_1$ be a constant homotopy, noting that $F_t = f$ for all t. The map l_t is linear Fredholm since $l_t = l_0 + t(c_0 - c_1)$ and $t(c_0 - c_1)$ is compact. To check that the family of maps $\{c_t\}$ is compact, let $D' \subset \pi^*(H')$ be a bounded disk bundle where $\pi : B \times [0,1] \to B$ is projection onto the first factor. Let A and B be compact sets containing $\pi c_0(D')$ and $\pi c_1(D')$ respectively. Then $D = \{(t, (1-t)a + tb) \mid (a, b) \in A \times B\} \subset [0,1] \times \mathcal{U}$ is compact since it is the continuous image of $[0,1] \times A \times B$ under the map $(t, a, b) \mapsto (t, (1-t)a + tb)$.

Now F is a Fredholm map over $B \times [0,1]$ which is certainly bounded. Applying finite dimensional approximation to F gives a homotopy between finite dimensional approximations of f using the two different compositions $f = l_0 + c_0$ and $f = l_1 + c_1$. This also completes the proof of Corollary 2.9. $\square$

Definition 2.26. *Given a trivialisation $p : H \cong B \times \mathcal{U} \to \mathcal{U}$, the stable cohomotopy class $[\varphi_f] \in \pi^0_{\mathcal{U}}(B; \operatorname{ind} l)$ is called the Bauer-Furuta class of f.*

Let $\mathcal{P}_l(H', H)$ denote the set of Fredholm maps $f : H' \to H$, relative to l, topologised with the metric in (2.3). The above theorem shows that finite dimensional approximation determines a map $\Phi : \mathcal{P}_l(H', H) \to \pi^0_{\mathcal{U}}(B; \operatorname{ind} l)$. Further, Corollary 2.21 shows that Φ descends to an injective map $\Phi : \pi_0(\mathcal{P}(H', H)) \to \pi^0_{\mathcal{U}}(B; \operatorname{ind} l)$ on homotopy classes. This is actually a one-to-one correspondence.

Theorem 2.27 ([8] Theorem 2.1)**.** *A projection $p : H \cong B \times \mathcal{U} \to \mathcal{U}$ induces a natural bijection*

$$\Phi : \pi_0(\mathcal{P}_l(H', H)) \xrightarrow{\sim} \pi^0_{\mathcal{U}}(B; \operatorname{ind} l).$$

The map Φ sends the compact homotopy class $[f] \in \mathcal{P}_l(H', H)$ to the Bauer-Furuta class $[\varphi_f] \in \pi^0_{\mathcal{U}}(B; \operatorname{ind} l)$ determined by finite dimensional approximation.

2.6 Equivariant case

Let G be a compact lie group and $\mathcal{U}$ a G-universe as in Definition 1.11. In particular, this implies that $\mathcal{U}$ is infinite dimensional. For simplicity, we will assume that $\operatorname{Hom}_G(M, \mathcal{U})$ is only non-zero for finitely many isomorphism classes of irreducible G-modules M, which will certainly be the case in the situations we are interested in.

For any irreducible G-module M, let $\mathcal{U}_M = M \otimes_{k_M} \operatorname{Hom}_G(M, \mathcal{U})$ where $k_M = \operatorname{Hom}_G(M, M)$ is equal to $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ depending on whether M is a real, complex or quarternionic representation. In any case, $\mathcal{U}_M$ is identified as a real subspace of $\mathcal{U}$ by the map $m \otimes \varphi \mapsto \varphi(m)$. Note that this map is injective since M is irreducible. The G-action on $\mathcal{U}_M$ is given by $g \cdot (m \otimes \varphi) = (gm) \otimes \varphi = m \otimes \rho_g \varphi$. Now $\mathcal{U}$ decomposes as

$$\mathcal{U} = \oplus_M \mathcal{U}_M \tag{2.12}$$

where the direct sum is taken over isomorphism classes of irreducible representations M contained in $\mathcal{U}$. This is called the isotypical decomposition of H.

Lemma 2.28. *Any finite dimensional subspace $V \subset \mathcal{U}$ is contained in a finite dimensional G-invariant subspace.*

Proof. Let $\{v_1, ..., v_n\}$ be a basis of V. Write $v_i = \sum_M v_i^M$ with $v_i^M \in \mathcal{U}_M$, which is a finite sum by the assumptions on $\mathcal{U}$. Let $m_1, ..., m_k$ be a basis for M. Now

$v_i^M = \sum_j m_j \otimes \varphi_{i,j}^M$ for some $\varphi_{i,j}^M \in \mathrm{Hom}_G(M,\mathcal{U})$. Let $V_M = M \otimes \mathrm{span}\{\varphi_{i,j}\}$ for $1 \leq j \leq k$ and notice that V_M is finite dimensional and G-invariant. Then $\oplus_M V_M \subset \mathcal{U}$ is a finite dimensional G-invariant subspace that contains V. $\square$

Let B be a finite CW complex, which implies that B is compact and Hausdorff. We let G act on B trivially.

Definition 2.29. *A G-Hilbert bundle H is a locally trivial fibre-bundle over B with fibre a G-Hilbert space $\mathcal{U}$, along with a G-action on H that is fibre preserving and fiberwise orthogonal. Each fibre is required to be equivariantly isomorphic to $\mathcal{U}$.*

Fix G-Hilbert bundles H', H and a fibre preserving map $l : H' \to H$ which is fiberwise linear Fredholm and equivariant.

Definition 2.30. *An equivariant bundle map $f : H' \to H$ is Fredholm (relative to l) if $c = f - l$ is compact. A Fredholm map is bounded if the preimage of any disk bundle is contained in a disk bundle.*

Let $f = l + c : H' \to H$ be an equivariant Fredholm map between G-Hilbert bundles over B. By Kupier's theorem [39], there is an equivariant trivialisation $H \to \mathcal{U} \times B$. To see this, let $\underline{M} = M \times B$ be the trivial G-bundle associated to an irreducible G-module M and let $\mathrm{Hom}_G(\underline{M}, H)$ be the G-Hilbert bundle of equivariant bundle morphisms from $\underline{M}$ to H. Let $H_M = \underline{M} \otimes_{k_M} \mathrm{Hom}_G(\underline{M}, H)$ where $k_M = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ depending on whether M is a real, complex or quarternionic representation. Now $H = \oplus_M H_M$ is the isotypical decomposition of H as in (2.12), where the sum is taken over isomorphism classes of irreducible representations contained in H. Each module H_M is infinite dimensional, hence can be trivialised by Kupier's theorem. Further, all trivialisations are homotopic.

Fix an equivariant trivialisation $H \to \mathcal{U} \times B$ and let $p : \mathcal{U} \times B \to \mathcal{U}$ be projection onto the first factor. Assume that $f : H' \to H$ is bounded Fredholm. As in section 2.5, let $\hat{f} = pf, \hat{l} = pl$ and $\hat{c} = pc$ so that $\hat{f} = \hat{l} + \hat{c} : H' \to \mathcal{U}$. The spheres $S_{H'}$ and $S_{\mathcal{U}}$ have a natural G-action extending the action on H' and $\mathcal{U}$ respectively. Thus $\hat{f}$ extends equivariantly to a map from $S_{H'}$ to $S_{\mathcal{U}}$ and factors through the Thom space, yielding an equivariant map $\hat{f} : TH' \to S_{\mathcal{U}}$.

Definition 2.31. *A finite dimensional subrepresentation $V \subset \mathcal{U}$ is said to be admissible if it satisfies the following three conditions:*

1. *For each $b \in B$, the subspace V surjects onto* $\operatorname{coker}(\hat{l}_b : H'_b \to \mathcal{U})$. *Thus $V' = l^{-1}(\underline{V})$ is a vector bundle with $V' - \underline{V} = \operatorname{ind} l$, where $\underline{V} = p^{-1}(V)$.*

2. *For any finite dimensional subrepresentation $W = U + V$ with $U \subset V^{\perp}$, the map*
$$\hat{f}|_{TW'} = pf|_{TW'} : TW' \to S_{\mathcal{U}}$$
misses the unit sphere $S(W^{\perp})$.

3. *The maps $\rho_W \hat{f}|_{TW'}$ and $l_{S_{\widetilde{U}}} \wedge \rho_V \hat{f}|_{TV'}$ are G-homotopic as pointed maps*
$$TW' \cong S_{\widetilde{U}} \wedge TV' \to S_U \wedge S_V = S_W.$$

Proposition 2.32 ([10] Lemma 2.5)**.** *There exists an admissible finite dimensional subrepresentation $V \subset \mathcal{U}$.*

Proof. The argument is the same as Proposition 2.25, with a slight modification to ensure V is G-invariant. Use the boundedness condition to find a disk bundle D' such that $|f| > 1$ outside of D'. Cover the closure of $c(D')$ by finitely many $\varepsilon < \frac{1}{4}$ balls centred at $\{v_1, ..., v_n\}$. By [2, Proposition A5] there is a finite dimensional subspace $V_0 \subset \mathcal{U}$ with $(\operatorname{im} \hat{l}_b)^{\perp} \subset V_0$ for all $b \in B$. By Lemma 2.28, there is a finite dimensional G-invariant subspace V containing both V_0 and $\operatorname{span}\{v_1, ..., v_n\}$. As demonstrated before, V is an admissible subrepresentation. $\square$

Let V be an admissible subrepresentation and let $V' = \hat{l}^{-1}(V)$, which is a vector bundle with the property that $\operatorname{ind} l = V' - \underline{V}$ as an equivariant K-theory class. The equivariant stable cohomotopy group $\pi^n_{G,\mathcal{U}}(B; \operatorname{ind}(l))$ is defined by maps between G-spectra as in Definition 1.13.

Definition 2.33 ([10] Theorem 2.6)**.** *Let $f = l + c : H' \to H$ be an equivariant, bounded Fredholm map between G-Hilbert bundles over B. Fix an equivariant trivialisation $H \cong B \times \mathcal{U}$ with $p : B \times \mathcal{U} \to \mathcal{U}$ projection onto the second factor. The equivariant Bauer-Furuta class of f is the stable homotopy class*
$$[\varphi_f] \in \pi^0_{G,\mathcal{U}}(B; \operatorname{ind} l)$$
where $\varphi_f = \rho_V \hat{f}|_{S_{V'}} : TV' \to S_V$ for any admissible subrepresentation $V \subset \mathcal{U}$. This cohomotopy class is independent of the presentation of $f = l + c$ as a sum.

The proof that this definition is well defined is essentially the same as Theorem 2.25. As in Theorem 2.27, let $\mathcal{P}^G_l(H', H)$ denote the set of equivariant Fredholm maps $f : H' \to H$ relative to l. Topologise $\mathcal{P}^G_l(H', H)$ with the metric defined in (2.3).

Theorem 2.34 ([8] Theorem 2.1)**.** *A choice of equivariant projection* $p : H \cong B \times \mathcal{U} \to \mathcal{U}$ *induces a natural bijection*

$$\Phi : \pi_0(\mathcal{P}_l^G(H', H)) \xrightarrow{\sim} \pi^0_{G,\mathcal{U}}(B; \operatorname{ind} l).$$

The bijection maps the class $[f] \in \pi_0(\mathcal{P}_l^G(H', H))$ *to the equivariant Bauer-Furuta class* $[\varphi_f] \in \pi^0_{G,\mathcal{U}}(B; \operatorname{ind} l)$ *determined by equivariant finite dimensional approximation.*

Chapter 3

The Bauer-Furuta invariant

In [10], Bauer-Furuta developed a cohomotopy refinement of the integer valued Seiberg-Witten invariant called the Bauer-Furuta invariant. Their approach was to study the stable cohomotopy class of the Seiberg-Witten monopole map, rather than its solutions. The was achieved using finite dimensional approximation, which was carefully defined in Chapter 2.

In a subsequent paper [9], Bauer gave a formula for the Bauer-Furuta invariant of a connected sum of 4-manifolds. While the ideas used in the proof of his formula were sound, we were not able to reproduce some of the arguments he used and concluded that the proof was incomplete. In this chapter, we define the Bauer-Furuta invariant and revisit Bauer's connected sum formula. We carefully apply the ideas used in his argument to give a new, corrected proof of his result (see Theorem 3.8 and 4.13). This proof has the advantage that it immediately extends to the families setting, which will be discussed in Chapter 4.

3.1 The Seiberg-Witten monopole map

Let X be a closed, oriented, Riemannian 4-manifold equipped with a spin^c structure $\mathfrak{s}$. For now we assume that X is connected. Denote by $W^{\pm} \to X$ the associated positive and negative spinor bundles with $\Gamma : TX \to \mathrm{End}(W)$ spinor multiplication. Let ∇_A be a spin^c connection on W with $A = \frac{1}{4}\,\mathrm{Trace}(\nabla_A)$. That is, A is locally an imaginary-valued one-form with $2A$ the induced $U(1)$-connection on the determinant line bundle $\det(W^+)$. Let $D_A : C^\infty(X, W^+) \to C^\infty(X, W^-)$ be the Dirac operator associated to A and $F_A \in \Omega^2(X; i\mathbb{R})$ the curvature two-form of A, with F_A^+ the self-dual part.

Recall the map $\sigma : C^\infty(X, W^+) \to \Omega^2_+(X; i\mathbb{R})$ from (1.9) defined by

$$\sigma(\psi) = \rho^{-1}\left(\psi \otimes \psi^* - \frac{1}{2}|\psi|^2 \mathrm{Id}\right). \tag{3.1}$$

The endomorphism $\sigma(\psi)$ is skew-Hermitian, trace free, and identified as a self-dual imaginary valued 2-form. The unperturbed Seiberg-Witten equations for a connection A and positive spinor ψ are

$$\begin{aligned} D_A\psi &= 0 \\ i\sigma(\psi) &= iF_A^+. \end{aligned}$$

The factor of i is included so that the second line is an equation of real valued 2-forms. The objective of Seiberg-Witten theory is to find solutions to these equations, modulo gauge equivalence. Let $\mathcal{G} = \mathrm{Map}(X, S^1)$ denote the group of gauge transformations. Recall that a gauge transformation $u \in \mathcal{G}$ acts on connections by $u \cdot A = A + u^{-1}du$ and on spinors by pointwise multiplication with u^{-1}. The Seiberg-Witten equations are invariant under this action. Fix a reference connection A_0 so that $A = A_0 + ia$ for some 1-form $a \in \Omega^1(X)$. Note that $F^+_{A_0+ia} = F^+_{A_0} + id^+a$, so the Seiberg-Witten equations can be written as

$$\begin{aligned} D_{A_0}\psi + ia \cdot \psi &= 0 \\ d^+a - iF^+_{A_0} + i\sigma(\psi) &= 0. \end{aligned} \tag{3.2}$$

It is a theorem of Uhlenbeck [55, Theorem 7.14] that every gauge class of connection has a representative with $d^*a = 0$. Including this condition as a third equation is known as gauge-fixing, which is useful for the purpose of elliptical analysis. However, this third equation is not invariant under all gauge transformations, only harmonic gauge transformations. Recall that a gauge transformation u is harmonic if $d^*(u^{-1}du) = 0$ and let $\mathcal{G}_0$ denote the group of harmonic gauge transformations. Including gauge-fixing, our equations are now

$$\begin{aligned} D_{A_0}\psi + ia \cdot \psi &= 0 \\ d^+a - iF^+_{A_0} + i\sigma(\psi) &= 0 \\ d^*a &= 0. \end{aligned} \tag{3.3}$$

Solutions to (3.2) modulo $\mathcal{G}$ are in one-to-one correspondence with solutions to (3.3) modulo $\mathcal{G}_0$.

Let $\mathcal{H}^1(X)$ denote the space of harmonic 1-forms on X. Recall that each de Rahm cohomology class in $H^1(X;\mathbb{R})$ has a unique harmonic representative, hence identify $\mathcal{H}^1(X) = H^1(X;\mathbb{R})$. Let b_1 be the first Betti number of X and choose a homology basis $\{\alpha_1, ..., \alpha_{b_1}\}$ for $H_1(X;\mathbb{R})$. Define a projection map $\mathrm{pr} : \Omega^1(X) \to \mathcal{H}^1(X)$ by

$$(\mathrm{pr}(a))(\alpha_i) = \int_{\alpha_i} a. \tag{3.4}$$

Extend $\mathrm{pr}(a)$ linearly so that $\mathrm{pr}(a) \in \mathrm{Hom}(H_1(X), \mathbb{R}) = H^1(X;\mathbb{R})$. Using this harmonic projection and Hodge theory, any connection satisfying $d^*(A - A_0) = 0$ can be written uniquely as $A = A_0 + iw + ia$ with $w \in \mathcal{H}^1(X)$ and $a \in \ker d^*$ satisfying $\mathrm{pr}(a) = 0$. Additionally, a harmonic gauge transformation u acts on A by

$$u \cdot A = A_0 + i(w - iu^{-1}du) + ia.$$

This action only affects the $\mathcal{H}^1(X)$ component of A. To encapsulate this perspective, define trivial bundles

$$\begin{aligned} \tilde{\mathcal{A}} &= (A_0 + i\mathcal{H}^1(X)) \times \left(C^\infty(X, W^+) \oplus \Omega^1(X) \oplus H^0(X;\mathbb{R})\right) \\ \tilde{\mathcal{C}} &= (A_0 + i\mathcal{H}^1(X)) \times \left(C^\infty(X, W^-) \oplus \Omega^2_+(X) \oplus \Omega^0(X) \oplus \mathcal{H}^1(X)\right). \end{aligned}$$

Here $H^0(X;\mathbb{R}) = \mathbb{R}$ is viewed as the space of locally constant functions on X. For $w \in \mathcal{H}^1(X)$, we write $A = A_0 + iw$. Define a fibre preserving map $\tilde{\mu} : \tilde{\mathcal{A}} \to \tilde{\mathcal{C}}$ by

$$\tilde{\mu}^A(\psi, a, f) = (D_{A+ia}\psi, -iF^+_{A+ia} + i\sigma(\psi), d^*a + f, \mathrm{pr}(a)) \tag{3.5}$$

A zero $\tilde{\mu}^A(\psi, a, f) = 0$ corresponds to a Seiberg-Witten monopole $(A+ia, \psi)$, before dividing by harmonic gauge symmetry. Notice that $d^*a + f = 0$ if and only if $d^*a = 0$ and $f = 0$ since the image of d^* is L^2-orthogonal to the space of locally constant functions. Thus elements of $\tilde{\mu}^{-1}(0)$ are of the form $(\psi, a, 0)$ and for notational simplicity the third component is suppressed. Further, the homotopy class of $\tilde{\mu}$ is independent of the choice of projection map $\mathrm{pr} : \Omega^1(X) \to \mathcal{H}^1(X)$ since all choices are homotopic.

Let $u \in \mathcal{G}_0$ act on forms trivially, on spinors by pointwise multiplication with u^{-1} and on connections by addition with $u^{-1}du$. Let $\mathcal{G}'_0$ be the subgroup of based harmonic gauge transformations for some chosen basepoint $x_0 \in X$. That is, the group of harmonic gauge transformations $u : X \to S^1$ with $u(x_0) = 1$. The action of $\mathcal{G}'_0$ on $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{C}}$ is free and $\tilde{\mu}$ is $\mathcal{G}'_0$-equivariant. Hence the quotient spaces

$$\begin{aligned} \mathcal{A} &= \left((A_0 + i\mathcal{H}^1(X)) \times (C^\infty(X, W^+) \oplus \Omega^1(X) \oplus H^0(X;\mathbb{R})\right)/\mathcal{G}'_0 \\ \mathcal{C} &= \left((A_0 + i\mathcal{H}^1(X)) \times (C^\infty(X, W^-) \oplus \Omega^2_+(X) \oplus \Omega^0(X) \oplus \mathcal{H}^1(X))\right)/\mathcal{G}'_0 \end{aligned} \tag{3.6}$$

are trivial bundles over $(A_0 + i\mathcal{H}^1(X))/\mathcal{G}_0'$ with fibres $C^\infty(X, W^+) \oplus \Omega^1(X) \oplus H^0(X;\mathbb{R})$ and $C^\infty(X, W^-) \oplus \Omega^2_+(X) \oplus \Omega^0(X) \oplus \mathcal{H}^1(X)$ respectively.

Recall from [55, Proposition 5.30] that there is a split short exact sequence

$$1 \longrightarrow S^1 \longrightarrow \mathcal{G}_0 \longrightarrow H^1(X;\mathbb{Z}) \longrightarrow 1.$$

The S^1 term is the group of constant gauge transformations. Let $\mathcal{J}(X)$ denote the b_1-dimensional Jacobian torus

$$\mathcal{J}(X) = H^1(X;\mathbb{R})/2\pi H^1(X;\mathbb{Z}).$$

The above short exact sequence gives a natural identification

$$(A_0 + i\mathcal{H}^1(X))/\mathcal{G}_0' \cong \mathcal{J}(X).$$

The quotient spaces $\mathcal{A}$ and $\mathcal{C}$ are bundles over $\mathcal{J}(X)$. After dividing by $\mathcal{G}_0'$, there is a residual $\mathcal{G}_0/\mathcal{G}_0' = S^1$ action of the constant gauge transformations on $\mathcal{A}$ and $\mathcal{C}$. The constant gauge transformations act on spinors by multiplication and on forms trivially. Since $\tilde{\mu}$ is $\mathcal{G}_0$-equivariant it descends to an S^1-equivariant map

$$\mu = \tilde{\mu}/\mathcal{G}_0' : \mathcal{A} \to \mathcal{C}.$$

Fix an integer $k \geq 4$ and let $\mathcal{A}_k$ denote the L^2_k-Sobolev completion of $\mathcal{A}$. That is, the smooth sections of spinors and forms appearing in (3.6) are replaced with L^2_k-sections. Similarly, $\mathcal{C}_{k-1}$ denotes the L^2_{k-1}-Sobolev completion of $\mathcal{C}$. The $k \geq 4$ bound will be used in various applications of standard Sobolev embedding theorems. Now $\mathcal{A}_k$ and $\mathcal{C}_{k-1}$ are S^1-Hilbert bundles over $\mathcal{J}(X)$. Since $k \geq 4$, the fibres of $\mathcal{A}_k$ and $\mathcal{C}_{k-1}$ contain only continuous sections and $\mathrm{pr} : \Omega^1(X) \to \mathcal{H}^1(X)$ extends continuously to a map $\mathrm{pr} : L^2_k(X, T^*X) \to \mathcal{H}^1(X)$. Further, σ is continuous by Sobolev multiplication, hence the extension $\mu : \mathcal{A}_k \to \mathcal{C}_{k-1}$ is continuous. Moving forward, we will suppress the k and $k-1$ subscripts unless they need particular attention.

Definition 3.1. *The Seiberg-Witten monopole map of X is the S^1-equivariant bundle map*

$$\mu : \mathcal{A} \to \mathcal{C}$$

where $\mathcal{A}$ and $\mathcal{C}$ are Hilbert-bundles over $\mathcal{J}(X)$. The action of μ over a fibre $A \in \mathcal{J}(X)$ is defined by the equation

$$\mu^A(\psi, a, f) = (D_{A+ia}\psi, -iF^+_{A+ia} + i\sigma(\psi), d^*a + f, \mathrm{pr}(a)). \tag{3.7}$$

To show that μ is a Fredholm map as in Definition 2.23, for $A \in \mathcal{J}(X)$ define

$$\begin{aligned} l^A(\psi, a, f) &= (D_A\psi, d^+a, d^*a + f, \mathrm{pr}(a)) \\ c^A(\psi, a, f) &= (ia \cdot \psi, -iF_A^+ + i\sigma(\psi), 0, 0). \end{aligned} \tag{3.8}$$

Here l^A is the linearisation of μ^A and evidently $\mu^A = l^A + c^A$.

Lemma 3.2. *Let $H^2_+(X;\mathbb{R})$ denote the b^+-dimensional vector space of self-dual real harmonic 2-forms on X. The map $l : \mathcal{A} \to \mathcal{C}$ is fiberwise linear Fredholm with*

$$\begin{aligned} \ker l^A &= \ker D_A \\ \operatorname{coker} l^A &= \operatorname{coker} D_A \oplus H^2_+(X;\mathbb{R}). \end{aligned}$$

The map c is compact and consequently $\mu = l + c$ is Fredholm.

Proof. Since D_A and $d^+ + d^*$ are elliptic operators, the kernel and cokernel of l^A contains only smooth spinors and forms. Recall that for any one-from a, $(d^+ + d^*)a = 0$ implies that a is harmonic. If $\mathrm{pr}(a) = 0$ as well then a must be zero. This implies that the kernel of l^A is exactly $\ker D_A$. From Hodge theory, the cokernel of $d^+ + d^* : \Omega^1(X) \to \Omega^2_+(X) \oplus \Omega^0(X)$ is exactly $H^2_+(X;\mathbb{R}) \oplus H^0(X;\mathbb{R})$. Since the locally constant functions are in the image of l^A, we have

$$\operatorname{coker} l^A = \operatorname{coker} D_A \oplus H^2_+(X;\mathbb{R}).$$

To see that c is compact, consider the following commutative diagram.

$$\begin{tikzcd} \mathcal{A}_k \arrow[r, "c"] \arrow[d, "\Delta"'] & \mathcal{C}_{k-1} \\ \mathcal{A}_k \times \mathcal{A}_k \arrow[r, "m - iF_A^+"] & \mathcal{C}_k \arrow[u, "\iota"'] \end{tikzcd}$$

Here $\Delta : \mathcal{A}_k \to \mathcal{A}_k \times \mathcal{A}_k$ is the diagonal map and $\iota : \mathcal{C}_k \to \mathcal{C}_{k-1}$ is the inclusion, which is compact by the Rellich-Kondrachov theorem [49, 10.2.24]. The map $m : \mathcal{A}_k \times \mathcal{A}_k \to \mathcal{C}_k$ is a multiplication map defined by

$$m((\psi_1, a_1), (\psi_2, a_2)) = \frac{1}{2}(ia_1 \cdot \psi_2 + ia_2 \cdot \psi_1, i(\psi_1 \otimes \psi_2^* + \psi_2 \otimes \psi_1^* - |\psi_1||\psi_2|), 0, 0).$$

The endomorphism $\psi_1 \otimes \psi_2^* + \psi_2 \otimes \psi_1^* - |\psi_1||\psi_2|$ is trace-free and skew-hermitian, hence it is identified as an imaginary valued 2-form. The map iF_A^+ is just a constant map so that $(m - iF_A^+)(\Delta(\psi, a)) = (a \cdot \psi, -F_A^+ + i\sigma(\psi), 0, 0)$. The map m is continuous for $k > 2$ by standard Sobolev multiplication theorems [61]. Since m is bilinear and ι is compact, the composition c is compact. $\square$

The above lemma shows that μ defines a family of Fredholm operators over $\mathcal{J}(X)$. The spaces $\ker l^A$ and $\operatorname{coker} l^A$ do not define vector bundles over $\mathcal{J}(X)$ for varying A because their dimensions are not constant, however the virtual index bundle $\operatorname{ind} l$ is a well defined K-theory class [1]. Lemma 3.2 also illustrates that

$$\operatorname{ind} l = \operatorname{ind} D - H^+.$$

Here $\operatorname{ind} D$ is the complex virtual index bundle over $\mathcal{J}(X)$ defined by the Dirac operator D_A and $H^+ \to \mathcal{J}(X)$ is trivial with fibre $H^2_+(X;\mathbb{R})$. The based gauge group $\mathcal{G}_0$ acts on $\operatorname{ind} D$ by multiplication and on H^+ trivially.

Next we show that μ is bounded in the sense of Definition 2.30. The following result applies several standard techniques from ordinary Seiberg-Witten theory.

Proposition 3.3 ([10] Proposition 3.1)**.** *For any bounded disk bundle $B \subset \mathcal{C}$, the preimage $\mu^{-1}(B) \subset \mathcal{A}$ is contained in a bounded disk bundle.*

Proof. Fix a connection $A \in \mathcal{J}(X)$. Since $\mathcal{J}(X)$ is compact, it is enough to show that the preimage $(\mu^A)^{-1}(B)$ of a disk $B \subset \mathcal{C}^A$ is bounded in $\mathcal{A}^A$. Note that the S^1 action on $\mathcal{A}$ and $\mathcal{C}$ is orthogonal, so all disk bundles are S^1-invariant. From Hodge theory, we can write

$$\begin{aligned}\Omega^1(X) &= H^1(X;\mathbb{R}) \oplus d(\Omega^0(X)) \oplus d^*(\Omega^2(X))\\ \Omega^0(X) &= H^0(X;\mathbb{R}) \oplus d^*(\Omega^1(X)).\end{aligned}$$

Note that $d^*(\Omega^1(X)) = d^*d(\Omega^0(X))$. Thus the map

$$\begin{aligned}H^1(X;\mathbb{R}) \oplus d(\Omega^0(X)) \oplus H^0(X;\mathbb{R}) &\to H^1(X;\mathbb{R}) \oplus \Omega^0(X)\\ (\alpha, d\beta, f) &\mapsto (\mathrm{pr}(\alpha), d^*d\beta + f)\end{aligned}$$

is an isomorphism. Consequently we can restrict attention to the spaces

$$\begin{aligned}\mathcal{A}' &= L^2_k(X, W^+) \oplus d^*(L^2_k(X, \Lambda^2_+ T^*X))\\ \mathcal{C}' &= L^2_k(X, W^-) \oplus \Omega^2_+(X).\end{aligned} \tag{3.9}$$

That is, it is enough to show that for the map $\mu^A : \mathcal{A}' \to \mathcal{C}'$ defined by

$$\mu^A(\psi, a) = (D_A\psi, -iF_A^+ + d^+a + i\sigma(\psi)),$$

the preimage $(\mu^A)^{-1}(B)$ of a disk $B \subset \mathcal{C}'$ is bounded. In particular, we can assume that $\mathrm{pr}(a) = 0$ and $d^*a = 0$.

Let D be the elliptic operator $D = D_A \oplus (d^+ + d^*) : \mathcal{A}' \to \mathcal{C}'$ and let $D^* : \mathcal{C}' \to \mathcal{A}'$ be its formal L^2-adjoint. Define a map $\mathcal{D} : \mathcal{A}' \oplus \mathcal{C}' \to \mathcal{A}' \oplus \mathcal{C}'$ by

$$\mathcal{D} = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}.$$

Use $\mathcal{D}$ to define an L^2_k-inner product on $\mathcal{A}' \oplus \mathcal{C}'$ by

$$(\cdot,\cdot)_k = \sum_{i=0}^{k} \int_X \left\langle \mathcal{D}^i(\cdot), \mathcal{D}^i(\cdot) \right\rangle \mathrm{dvol}_g. \tag{3.10}$$

In particular this defines an explicit L^2_k-inner product on $\mathcal{A}'$ and $\mathcal{C}'$. Define L^p_k-norms in the same way. Note that the topologies of the L^p_k-completions of $\mathcal{A}'$ and $\mathcal{C}'$ are independent of the choice of Sobolev norm.

Let $\mu^A(\psi, a) = (\varphi, b) \in \mathcal{C}'$ with

$$\begin{aligned} \varphi &= D_{A'}\psi \\ b &= -iF^+_{A'} + i\sigma(\psi). \end{aligned} \tag{3.11}$$

Suppose that $\|(\varphi, b)\|_{L^2_k} \leq R$. We must show that $\|(\psi, a)\| \leq R'$ for some constant R' which depends on R. We will first bound $\|\psi\|_{L^\infty}$ and $\|a\|_{L^\infty}$, then we will obtain an L^2_k-bound using elliptic bootstrapping.

Let $A' = A + ia$ and recall the Weitzenböck formula (1.15)

$$D^*_{A'} D_{A'} = \nabla^*_{A'} \nabla_{A'} + \frac{1}{4} s_X + \frac{1}{2} F^+_{A'}. \tag{3.12}$$

Here s_X is the scalar curvature of X. It is a standard result in Seiberg-Witten theory that the Weitzenböck formula can be used to get a pointwise bound on $|\psi|$. To see this, let $\Delta_g : C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R})$ be the positive definite Laplace-Beltrami operator and recall the identity [55, Lemma 7.13]

$$\Delta_g |\psi|^2 = 2\mathrm{Re} \left\langle \nabla^*_{A'} \nabla_{A'} \psi, \psi \right\rangle - 2|\nabla_{A'}\psi|^2.$$

Applying the Weitzenböck formula yields

$$\begin{aligned} \Delta_g |\psi|^2 &\leq 2 \left\langle \nabla^*_{A'} \nabla_{A'} \psi, \psi \right\rangle \\ &= 2 \left\langle D^*_{A'} D_{A'} \psi - \frac{s_X}{4}\psi - \frac{1}{2} F^+_{A'}\psi, \psi \right\rangle \\ &= \left\langle 2D^*_{A'} D_{A'} \psi, \psi \right\rangle - \left\langle \frac{s_X}{2}\psi, \psi \right\rangle - \left\langle (b + \sigma(\psi))\psi, \psi \right\rangle \end{aligned} \tag{3.13}$$

Recall that $D_{A+ia}\psi = D_A\psi + ia \cdot \psi$ and $\sigma(\psi)\psi = \frac{1}{2}|\psi|^2\psi$. It follows that

$$\Delta_g|\psi|^2 + \frac{s}{2}|\psi|^2 + \frac{1}{2}|\psi|^4 \leq 2\left\langle D_A^*\varphi, \psi\right\rangle + 2\left\langle a \cdot \varphi, \psi\right\rangle - \left\langle b\psi, \psi\right\rangle .$$

Apply Cauchy-Schwarz to obtain

$$\begin{aligned} 2\Delta_g|\psi|^2 + s_X|\psi|^2 + |\psi|^4 &\leq 4\|D_A^*\varphi\|_{L^\infty}|\psi| + 4\|a\|_{L^\infty}\|\varphi\|_{L^\infty}|\psi| + 2\|b\|_{L^\infty}|\psi|^2 \\ &\leq C_1\left((1+\|a\|_{L^\infty})\|\varphi\|_{L^\infty}|\psi| + \|b\|_{L^\infty}|\psi|^2\right). \end{aligned}$$

Note that D_A^* is continuous in the L^∞-norm, so C_1 is a constant that depends on the L^∞-operator norm of D_A^*. Since $k \geq 3$, there is a continuous Sobolev embedding

$$L^2_{k-1}(X, W^- \oplus \Lambda^2_+(T^*X)) \subset L^\infty(X, W^- \oplus \Lambda^2_+(T^*X)).$$

Hence there exists a constant C_2 such that

$$\Delta_g|\psi|^2 + \frac{s_X}{2}|\psi|^2 + \frac{1}{2}|\psi|^4 \leq C_2\left((1+\|a\|_{L^\infty})\|\varphi\|_{L^2_{k-1}}|\psi| + \|b\|_{L^2_{k-1}}|\psi|^2\right). \quad (3.14)$$

Notice that $\|\varphi\|_{L^2_{k-1}}$ and $\|b\|_{L^2_{k-1}}$ are less than R by assumption. Since $\Delta_g|\psi|^2$ is non-negative at a maximum of ψ, it follows that

$$s_X\|\psi\|^2_{L^\infty} + \|\psi\|^4_{L^\infty} \leq C_2R\left((1+\|a\|_{L^\infty})\|\psi\|_{L^\infty} + \|\psi\|^2_{L^\infty}\right). \quad (3.15)$$

To proceed, we find a bound for $\|a\|_{L^\infty}$. Fix some $p \geq 4$ so that the Sobolev embedding $L^p_1(X, T^*X) \subset L^\infty(X, T^*X)$ gives a constant C_s with $\|a\|^\infty_L \leq C_s\|a\|_{L^p_1}$. Since $L = d^* + d^+ : C^\infty(X, \Lambda(T^*X)) \to C^\infty(X, \Lambda(T^*X))$ is a self-adjoint elliptic operator, there exists a parametrix $P : C^\infty(X, \Lambda(T^*X)) \to C^\infty(X, \Lambda(T^*X))$ with $PLa + \mathrm{pr}(a) = a$ [30, Theorem 4.12]. This bounded linear operator P extends continuously to a map $P : L^p(X, \Lambda T^*X) \to L^p_1(X, \Lambda T^*X)$, hence there exists a constant C_e such that

$$\begin{aligned} \|a\|_{L^\infty} &\leq C_s\|a\|_{L^p_1} \\ &\leq C_sC_e(\|d^+a\|_{L^p} + \|\,\mathrm{pr}(a)\|_{L^p}). \end{aligned}$$

Note that $a \in \operatorname{im} d^*$ which implies $\mathrm{pr}(a) = 0$ and $d^*a = 0$. Apply the Seiberg-Witten equation $d^+a = b - F^+_{A_0} + \sigma(\psi)$ to obtain

$$\|a\|_{L^\infty} \leq C_sC_e(\|b\|_{L^p} + \|F^+_A\|_{L^p} + \|\sigma(\psi)\|_{L^p}). \quad (3.16)$$

Notice that $\|F^+_A\|_{L^p}$ is constant since A is fixed. Using the Sobolev embeddings $L^2_{k-1}(X, \Lambda^2_+(T^*X)) \subset L^p(X, \Lambda^2_+(T^*X))$, $L^\infty(X, \Lambda^2_+(T^*X)) \subset L^p(X, \Lambda^2_+(T^*X))$ and the fact that $\mathcal{H}^1(X)$ is finite dimensional, there are constants C_3 and C_4 such that

$$\begin{aligned} \|a\|_{L^\infty} &\leq C_3(\|b\|_{L^2_{k-1}} + 1 + \|\sigma(\psi)\|^2_{L^\infty}) \\ &\leq C_3(1 + R + \frac{1}{2}\|\psi\|^2_{L^\infty}) \\ &\leq C_4R(1 + \|\psi\|^2_{L^\infty}). \end{aligned}$$

Combining this with equation (3.15) produces

$$\begin{aligned} s_X\|\psi\|^2_{L^\infty} + \|\psi\|^4_{L^\infty} &\leq C_2R\left((1 + C_4R(1 + \|\psi\|^2_{L^\infty}))\|\psi\|_{L^\infty} + \|\psi\|^2_{L^\infty}\right) \\ \|\psi\|^4_{L^\infty} &\leq C_5R^2\left(1 + \|\psi\|_{L^\infty} + \|\psi\|^2_{L^\infty}\right)\|\psi\|_{L^\infty} - s_X\|\psi\|^2_{L^\infty}, \end{aligned}$$

for some constant C_5. Thus $\|\psi\|^4_{L^\infty}$ is bounded by a degree three polynomial in $\|\psi\|_{L^\infty}$. This implies that there is a constant R_1 with $\|\psi\|_{L^\infty} \leq R_1'$ and by (3.16) we can assume $\|a\|_{L^\infty} \leq R_1'$ as well.

To obtain L^2_k-bounds on ψ and a, we use a standard bootstrapping argument. The idea is to show that if (ψ, a) is L^{2p}_{i-1}-bounded for some $1 \leq i \leq k$ with $p = 2^{k-i+1}$, then (ψ, a) is L^p_i-bounded. The chosen Sobolev norm in (3.10) gives

$$\begin{aligned} \|(\psi, a)\|^p_{L^p_i} - \|(\psi, a)\|^p_{L^p} &= \|(D_A\psi, d^+a)\|^p_{L^p_{i-1}} \\ &= \|(D_{A'}\psi - a\cdot\psi, b - F^+_A + \sigma(\psi))\|^p_{L^p_{i-1}} \\ &\leq \|(\varphi, b)\|^p_{L^p_{i-1}} + \|(a\cdot\psi, \sigma(\psi))\|^p_{L^p_{i-1}} + \|F^+_A\|^p_{L^p_{i-1}} \end{aligned}$$

Note that for the chosen p, there is a Sobolev embedding

$$L^2_{k-1}(X, W^- \oplus \Lambda^2_+(T^*X)) \subset L^p_{i-1}(X, W^- \oplus \Lambda^2_+(T^*X)).$$

Thus there is a bound

$$\begin{aligned} \|(\varphi, b)\|^p_{L^p_{i-1}} &\leq C_6\|(\varphi, b)\|^p_{L^2_{k-1}} \\ &\leq C_6R^p. \end{aligned}$$

The term $\|F^+_A\|^p_{L^p_{i-1}}$ is just a constant, and by Sobolev multiplication [61, Lemma B.3] there exists a constant C_{SM} such that

$$\begin{aligned} \|(a\cdot\psi, \sigma(\psi))\|_{L^p_{i-1}} &= \|(a\cdot\psi, \frac{1}{2}|\psi|^2)\|_{L^p_{i-1}} \\ &\leq C_{SM}(\|a\|_{L^{2p}_{i-1}}\|\psi\|_{L^{2p}_{i-1}} + \|\psi\|^2_{L^{2p}_{i-1}}). \end{aligned}$$

Thus there exists a constant C_7 such that

$$\|(\psi, a)\|_{L^p_i} \leq C_7R(1 + \|a\|_{L^{2p}_{i-1}}\|\psi\|_{L^{2p}_{i-1}} + \|\psi\|^2_{L^{2p}_{i-1}}) + \|(\psi, a)\|_{L^p}.$$

Starting with $i = k, p = 2$ and recursively applying this formula down to $i = 0, p = 2^{k+1}$, there is a polynomial f such that

$$\|(\psi, a)\|_{L^2_k} \leq f(\|(\psi, a)\|_{L^2}, \|(\psi, a)\|_{L^4}, ..., \|(\psi, a)\|_{L^{2^{k+1}}}).$$

The L^∞-bounds on ψ and a from earlier guarantees the existence of a constant C_8 such that $\|(\psi, a)\|_{L^p} \leq C_8$ for each $p = 2, 4, ..., 2^{k+1}$. Taking the supremum of $f(x_1, ..., x_k)$ with $|x_i| \leq C_8$ gives a constant R' with $\|(\psi, a)\|_{L^2_k} \leq R'$, as required. □

Remark 3.4: The argument simplifies immensely if we only aim to show that $\mu^{-1}(0)$ is bounded. The bounds in equation (3.14) and (3.16) become

$$\Delta_g|\psi|^2 + \frac{s_X}{2}|\psi|^2 + \frac{1}{2}|\psi|^4 \leq 0 \tag{3.17}$$

$$\|a\|_{L^\infty} \leq C_s C_e(\|F^+_{A_0}\|_{L^p} + \|\sigma(\psi)\|_{L^p}). \tag{3.18}$$

This gives $\|\psi\|^2_{L^\infty} \leq S$ where $S = \sup_X\{-s_X, 0\}$ and also a bound for $\|a\|_{L^\infty}$ in terms of S. The elliptic bootstrapping argument can be repeated with the relation

$$\|(\psi,a)\|^p_{L^p_i} - \|(\psi,a)\|^p_{L^p_0} = \|(a\cdot\psi, -F^+_{A_0} + \sigma(\psi))\|^p_{L^p_{i-1}}. \tag{3.19}$$

These ideas will be used in Section 3.2 to prove similar boundedness results.

To fit μ into the framework of Section 2.5, define an S^1-universe $\mathcal{U}$ by

$$\mathcal{U} = L^2_k(X, W^- \oplus \Lambda^2_+ T^*X \oplus \mathbb{R}) \oplus \mathcal{H}^1(X) \tag{3.20}$$

The circle S^1 acts on the spinor component of $\mathcal{U}$ by complex multiplication and trivially on the other components. The two irreducible S^1-representations contained in $\mathcal{U}$ are the trivial representation $\mathbb{R}$ and the regular representation $\mathbb{C}$. Since $\mathcal{U}$ contains infinitely many copies of both of these representations it satisfies Definition 1.11.

Let $p : \mathcal{C} \to \mathcal{J}(X) \times \mathcal{U}$ be a fixed equivariant trivialisation of $\mathcal{C}$. For the decomposition $\mu = l + c$ described in (3.8) we have $\operatorname{ind} l = \operatorname{ind} D - H^+$ as virtual bundles. Lemma 3.2 and Proposition 3.3 show that μ is a bounded equivariant Fredholm map over $\mathcal{J}(X)$ so that Theorem 2.33 applies.

Theorem 3.5 ([10] Corollary 3.2)**.** *Finite dimensional approximation applied to the monopole map μ defines an element of the equivariant stable cohomotopy group*

$$\begin{aligned}[\mu] &\in \pi^0_{S^1,\mathcal{U}}(\mathcal{J}(X); \operatorname{ind} l)\\ &= \pi^{b^+}_{S^1,\mathcal{U}}(\mathcal{J}(X); \operatorname{ind}(D)).\end{aligned}$$

The class $[\mu]$ is called the Bauer-Furuta invariant of X. It is independent of the decomposition $\mu = l + c$ and choice of projection map $\mathrm{pr} : \Omega^1(X) \to \mathcal{H}^1(X)$.

The cohomotopy groups $\pi^0_{S^1,\mathcal{U}}(\mathcal{J}(X); \operatorname{ind} l)$ and $\pi^{b^+}_{S^1,\mathcal{U}}(\mathcal{J}(X); \operatorname{ind}(D))$ are identical by suspension since H^+ is a trivial bundle of rank b^+. Recall that an orientation of $H^1(X;\mathbb{R}) \oplus H^2_+(X;\mathbb{R})$ is called a homology orientation of X. If $b^+ > b_1 + 1$, then a homology orientation determines a map $\pi^{b^+}_{S^1,\mathcal{U}}(\mathcal{J}(X); \operatorname{ind}(D)) \to \mathbb{Z}$ which sends $[\mu]$ to the integer-valued Seiberg-Witten invariant $SW_X(\mathfrak{s}) \in \mathbb{Z}$ [10, Proposition 3.3].

3.2 Connected sum formula

Two fundamental results for calculating Seiberg-Witten invariants are the vanishing theorem and the blowup formula. Proofs of both of these results can be found in Nicolaescu's book [48]. The vanishing theorem states that if $X = X_1 \# X_2$ with $b^+(X_1) > 0$ and $b^+(X_2) > 0$, then the integer-valued Seiberg-Witten invariant of X vanishes on all spin^c structures. The blowup formula describes how to calculate the Seiberg-Witten invariant of a connected sum $X \# \overline{\mathbb{CP}^2}$, however the techniques used in Nicolaescu's proof can be extended to obtain a general connected sum formula.

In this chapter, we derive a connected sum formula for the Bauer-Furuta invariant, which can be used to recover the Seiberg-Witten connected sum formula. The proof involves analysing the behaviour of monopoles on a tubular neck $S^3 \times [-L, L]$ with varying neck length $2L$. Thus great care is taken to determine how the necessary estimates depend on L.

3.2.1 Manifolds with separating necks

Fix some natural number $n \in \mathbb{N}$. For any $L > 2$, let $N(L)$ denote the disjoint union

$$N(L) = \coprod_{i=1}^{n} S^3 \times [-L, L].$$

Let $N(L)_i$ denote the ith connected component of $N(L)$ for any $1 \leq i \leq n$ and write $\partial N(L)^+ = \coprod_{i=1}^n S^3 \times \{L\}$ and $\partial N(L)^- = \coprod_{i=1}^n S^3 \times \{-L\}$ to denote the positive and negative boundary components. Equip $N(L)$ with a metric $g_{N(L)}$ that is the product of the standard round metric on S^3 and the standard interval metric on $[-L, L]$. Since $H^2(S^3; \mathbb{Z}) = 0$, there is a unique spin^c structure on S^3 with spinor bundles $W^{\pm}_{S^3}$. Let $p : N(L) \to S^3$ be projection onto the S^3 factor and define spinor bundles $W^{\pm}_{N(L)} = p^*(W^{\pm}_{S^3})$ on $N(L)$ by pullback. This defines a spin^c structure $\mathfrak{s}_{N(L)}$ on $N(L)$, which is also unique up to spin^c isomorphism.

Definition 3.6. *Let X be a closed, oriented 4-manifold with n connected components X_i for $i = 1, ..., n$. A separating neck of length $2L$ on X is a smooth embedding $\iota : N(L) \to X$ such that*

1. *For each $1 \leq i \leq n$, we have $\iota(N(L)_i) \subset X_i$*

2. *The neck complement $M = \overline{X - \iota(N(L-1))}$ decomposes as $M = M^- \coprod M^+$ where $\partial M^- = \iota(\partial N(L-1)^-)$ and $\partial M^+ = \iota(\partial N(L-1)^+)$, both with reversed orientations.*

Embedded in the above definition is a thickened up overlap between M and $N(L)$ which will be used for gluing purposes. Throughout this chapter we assume that X has a separating neck of length $2L > 4$ and identify $N(L)$ with its image $\iota(N(L)) \subset X$. Choose a metric g and a spin^c structure $\mathfrak{s}$ on X that extends $g_{N(L)}$ and $\mathfrak{s}_{N(L)}$. For $0 < R \leq L$, let $N(R)$ denote a segment of the neck given by

$$N(R) = \coprod_{i=1}^{n} (S^3 \times [-R, R]) \subset X.$$

Further, let $N(R_1, R_2) = \overline{N(R_2) - N(R_1)}$ for $R_2 \geq R_1$.

For each connected component $N(L)_i$ of $N(L)$, define collar neighbourhoods of the boundary $C_i^{\pm} \subset N(L)_i$ by

$$\begin{aligned} C_i^- &= S^3 \times [-L, -L+1] \\ C_i^+ &= S^3 \times [L-1, L]. \end{aligned}$$

Let $C = \coprod_i (C_i^- \cup C_i^+)$. The restriction $\iota|_C : C \to M$ identifies C as a collar neighbourhood of M, with reversed orientation. For any other neck length $L' > 2$, there is a natural isometric inclusion $C \to N(L')$ identifying C has a collar neighbourhood of $\partial N(L')$, again with reversed orientation. Let $X(L') = M \cup_C N(L')$, that is, $X(L')$ denotes the following pushout

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d, "\iota|_C"'] & N(L') \arrow[d, dashed] \\ M \arrow[r, dashed] & X(L'). \end{tikzcd}$$

Let $\tau \in S_n$ be an even permutation on n objects. Define a permuted inclusion map $\iota_\tau : C \to M$ such that $\iota_\tau|_{C_i^-} = \iota|_{C_i^-}$ and $\iota_\tau|_{C_i^+} = \iota|_{C_{\tau(i)}^+}$. That is, C_i^- is mapped to $\iota(C_i^-)$ but C_i^+ is mapped to $\iota(C_{\tau(i)}^+)$. Define the permuted 4-manifold X^τ by the following pushout

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d, "\iota_\tau"'] & N(L) \arrow[d, dashed] \\ M \arrow[r, dashed] & X^\tau. \end{tikzcd} \tag{3.21}$$

The point is that each boundary component of the form $\iota(C_i^-) \subset M$ has been connected by a cylinder $S^3 \times [-L, L]$ to $\iota(C^+_{\tau(i)})$. The permuted manifold X^τ inherits a metric g^τ and spin^c structure $\mathfrak{s}^\tau$ directly from X.

3.2.2 Permuting sections along the neck

Let $W^+|_{S^3\times[-L,L]}$ be the restriction of $W^+ \to N(L)$ to one of the connected components of $N(L)$. Define a bundle $F \to S^3 \times [-L, L]$ by $F = \oplus_{i=1}^n W^+|_{S^3\times[-L,L]}$. Since $N(L)$ has n connected components, a section $\psi : N(L) \to W^+$ can be identified with a vector of sections $\vec{\psi} : S^3 \times [-L, L] \to F$. That is, the restriction ψ_i to the ith component of $N(L)$ is identified with the ith factor of $\vec{\psi}$. Let $T : S^3 \times [-L, L] \to SO(n)$ denote a matrix valued function. For a section $\psi : N(L) \to W^+$ along $N(L)$, define an action by $T \cdot \psi = T\vec{\psi}$ where T acts pointwise on $\vec{\psi}$ and $T\vec{\psi}$ is identified with a section of $W^+ \to N(L)$. The same process defines an action on forms along the neck.

Let $\gamma : [0, 1] \to SO(n)$ be a smooth path from the identity to τ, which exists under the assumption that τ is even. Define a smooth map $\varphi : [-L, L] \to [0, 1]$ that vanishes on $[-L, 1]$ and is identically equal to 1 on $[1, L]$. Write $V : S^3 \times [-L, L] \to SO(n)$ to denote the matrix-valued function

$$V(x, t) = \gamma(\varphi(t)). \tag{3.22}$$

Note that V is constant along the S^3 factor. Let $(\psi, a) : N(L) \to W^+ \oplus T^*N(L)$ be a spinor-form pair along $N(L)$ and define $(\psi, a)^\tau = (V \cdot \psi, V \cdot a)$ by the action described above. The pair $(\psi, a)^\tau$ has the property that $(\psi, a)^\tau_i = (\psi, a)_i$ on C^- and $(\psi, a)^\tau_i = (\psi, a)_{\tau(i)}$ on C^+. Now given a section $(\psi, a) : X \to W^+ \oplus T^*X$ defined on all of X, this construction defines a section $(\psi, a)^\tau$ on X^τ with the property that (ψ, a) and $(\psi, a)^\tau$ agree away from $N(1)$. That is, $X - N(1)$ can be identified with $X^\tau - N(1)$ and (ψ, a) restricted to $X - N(1)$ agrees with $(\psi, a)^\tau$ restricted to $X^\tau - N(1)$.

For f a locally constant function on X, let f^τ be the locally constant function on X^τ that agrees with f on M^-. The map $f \in H^0(X; \mathbb{R}) \mapsto f^\tau \in H^0(X^\tau; \mathbb{R})$ is a bijection since X and X^τ have the same number of connected components. Further, the map $(\psi, a, f) \mapsto (\psi^\tau, a^\tau, f^\tau)$ defines an isomorphism $V_\mathcal{A} : \mathcal{A} \to \mathcal{A}^\tau$ of Hilbert bundles over $\mathcal{J}(X)$. Sobolev multiplication can be applied to show that $V_\mathcal{A}$ is continuous in the L^2_k-norm, as in Lemma 3.7 below. The inverse $V_\mathcal{A}$ is defined using τ^{-1}, hence it is also continuous by Sobolev multiplication. Similarly for $\mathcal{C}$, the action of V defines a map $V_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}^\tau$ that on the $\mathcal{H}^1(X)$ factor is just the identity, since $\mathcal{H}^1(X)$ can be identified with $\mathcal{H}^1(X^\tau)$. The map $V_{\mathcal{C}}$ is continuous with continuous inverse again by Sobolev multiplication. Since $V_{\mathcal{A}}$ and $V_{\mathcal{C}}$ are both given by the action of V, we will often suppress the subscripts.

Lemma 3.7. *Let (ψ, a) be a spinor-form pair on $X = X(L)$. There is a constant C_V, independent of L, such that*

$$\|V(\psi, a)\|_{L^2_k} \leq C_V \|(\psi, a)\|_{L^2_k}.$$

Proof. The matrix valued function $V = \gamma \circ \varphi : S^3 \times [-L, L] \to SO(n)$ is just a permutation outside of $S^3 \times [-1, 1]$, hence

$$\|V(\psi, a)\|_{L^2_k(X^\tau - N(1))} = \|(\psi, a)\|_{L^2_k(X - N(1))}.$$

Inside $N(1)$, we have that $V\psi$ along the i-th component of the neck is given by

$$(V\vec{\psi})_i = \sum_j V_{ij} \vec{\psi}_j.$$

Here V_{ij} are the component functions of V. The triangle inequality ensures that

$$\begin{aligned} \|V\vec{\psi}\|^2_{L^2_k(N(1))} &= \sum_i \left\| \sum_j V_{ij} \vec{\psi}_j \right\|^2_{L^2_k(N(1))} \\ &\leq \sum_{i,j} \left\| V_{ij} \vec{\psi}_j \right\|^2_{L^2_k(N(1))}. \end{aligned}$$

Since each function V_{ij} is smooth, the C^k-norm of V_{ij} over $N(1)$ provides a constant C such that, for any i and j,

$$\|V_{ij}\vec{\psi}_j\|^2_{L^2_k(N(1))} \leq C \|\vec{\psi}_j\|^2_{L^2_k(N(1))}.$$

The constant C does not depend on L. It follows that

$$\|V\vec{\psi}\|^2_{L^2_k(N(1))} \leq nC \|\psi\|^2_{L^2_k(N(1))}$$

Of course, the same calculation applies to a. Integrating over $N(1)$ and $X - N(1)$ separately gives

$$\begin{aligned} \|V(\psi, a)\|^2_{L^2_k(X^\tau)} &= \|V(\psi, a)\|^2_{L^2_k(X^\tau - N(1))} + \|V(\psi, a)\|^2_{L^2_k(N(1))} \\ &\leq \|(\psi, a)\|^2_{L^2_k(X - N(1))} + nC \|(\psi, a)\|^2_{L^2_k(N(1))} \\ &\leq (1 + nC) \|(\psi, a)\|^2_{L^2_k(X)}. \end{aligned}$$

Take $C_V = \sqrt{1 + nC}$, which is independent of L. $\square$

3.2.3 The Permutation Theorem

Let X be a 4-manifold with a separating neck of length $2L$. To define the monopole map for X as in Chapter 3.1, a few adjustments are necessary to ensure the map is compatible with the neck structure. To construct an appropriate reference connection, let $A_{N(L)}$ be a flat connection on the neck $N(L)$ that is identical on each neck component $N_i(L) = S^3 \times [-L, L]$. Such a connection exists since $H^2(N(L); \mathbb{R}) = 0$. To extend $A_{N(L)}$ to X, let A_X be some connection on X and define $\omega = A_X|_{N(L)} - A_{N(L)}$, which is a one form defined on the neck. Extend ω to a one form defined on all of X by, for instance, making ω zero outside a neighbourhood of $N(L)$. Then set $A_0 = A_X - \omega$, which is a connection on X that is flat on the neck and permutation invariant in the sense that $A_0|_{N_i(L)} = A_0|_{N_j(L)}$ for all i and j. Thus A_0 defines a reference connection on both X and X^τ.

Let $\ker d_{N(1)} \subset \ker d$ be the space of closed 1-forms which vanish on $N(1)$ so that $\omega \in \ker d_{N(1)}$ can be interpreted as a 1-form on both X and X^τ. Similarly, let $\mathcal{G}_{N(1)} \subset \mathcal{G}$ denote the subgroup of smooth gauge transformations that are identically equal to $1 \in S^1$ on $N(1)$. Again this means that $\mathcal{G}_{N(1)}$ acts on both X and X^τ. The natural map $(A_0 + i \ker d_{N(1)})/\mathcal{G}_{N(1)} \to (A_0 + i \ker d)/\mathcal{G}$ induced by inclusion is an isomorphism since $H^1(N(L); \mathbb{R}) = 0$. That is, we can identify $(A_0 + i \ker d_{N(1)})/\mathcal{G}_{N(1)} = \mathcal{J}(X)$ and $(A_0 + i \ker d_{N(1)})/\mathcal{G}_{N(1)} = \mathcal{J}(X^\tau)$. Once again fix $k \geq 4$ and define the bundles $\mathcal{A}$ and $\mathcal{C}$ by

$$\begin{aligned}
\mathcal{A} &= \left((A_0 + i \ker d_{N(1)}) \times (L^2_k(X, W^+ \oplus T^*X) \oplus H^0(X; \mathbb{R}))\right) / \mathcal{G}_{N(1)} \\
\mathcal{C} &= \left((A_0 + i \ker d_{N(1)}) \times (L^2_k(X, W^- \oplus \Lambda^2_+(T^*X) \oplus \mathbb{R}) \oplus \mathcal{H}^1(X))\right) / \mathcal{G}_{N(1)}.
\end{aligned}$$

Let $\mathbb{T}^n = (S^1)^{\times n}$ denote an n-dimensional torus where n is the number of connected components of X. There is a residual $\mathcal{G}/\mathcal{G}_{N(1)} = \mathbb{T}^n$ action by the locally constant gauge transformations on $\mathcal{A}$ and $\mathcal{C}$. These bundles are $\mathbb{T}^n$-equivariantly isomorphic to the bundles defined in (3.6). However, this modification has the advantage that $\mathcal{A}$ and $\mathcal{C}$ have been identified as bundles over $\mathcal{J}(X) = \mathcal{J}(X^\tau)$. Let $\mathcal{A}^\tau \to \mathcal{J}(X)$ and $\mathcal{C}^\tau \to \mathcal{J}(X)$ denote the same construction applied to X^τ.

The monopole map $\mu = \tilde{\mu}/\mathcal{G}_{N(1)} : \mathcal{A} \to \mathcal{C}$ is defined again using equation (3.5)

$$\mu^A(\psi, a, f) = (D_{A+ia}\psi, -iF^+_{A+ia} + i\sigma(\psi), d^*a + f, \mathrm{pr}(a)).$$

In this case, define the harmonic projection map $\mathrm{pr} : \Omega^1(X) \to H^1(X; \mathbb{R})$ in the following manner. By a Mayer-Vietoris argument, the first homology group of X splits as $H_1(X) = H_1(M^-) \oplus H_1(M^+)$. Choose a basis of curves $\{\alpha_1, ..., \alpha_{b_1(M^-)}, \beta_1, ..., \beta_{b_1(M^+)}\}$ for $H_1(X)$ with α_i smoothly embedded in M^- and β_j smoothly embedded in M^+.

Define $\mathrm{pr}(a) \in \mathrm{Hom}(H_1(X), \mathbb{R}) = H^1(X; \mathbb{R})$ by integration over this basis as in (3.4). This also shows that $H^1(X;\mathbb{R}) = H^1(X^\tau;\mathbb{R})$ and defines a projection map for X^τ. The $k \geq 4$ assumption guarantees that the fibres of $\mathcal{A}$ and $\mathcal{C}$ contain only continuous sections, which guarantees that pr is well defined and continuous in the L^2_k-norm. Sobolev multiplication ensures that σ is L^2_k-continuous, hence μ is continuous.

Recall, as defined in Theorem 3.5, that $\mathcal{U}$ is the $\mathbb{T}^n$-universe

$$\mathcal{U} = L^2_k(X, W^- \oplus \Lambda^2_+(T^*X) \oplus \mathbb{R}) \oplus \mathcal{H}(X). \tag{3.23}$$

Let l be the linearisation of μ defined in Lemma 3.2. Then the virtual bundle $\mathrm{ind}\, l \to \mathcal{J}(X)$ is given by $\mathrm{ind}\, l = \mathrm{ind}\, D_A - H^+$, where $\mathrm{ind}\, D_A \to \mathcal{J}(X)$ is the complex virtual index bundle and $H^+ \to \mathcal{J}(X)$ is trivial with fibre $H^2_+(X;\mathbb{R})$. Recall that $\mathcal{P}_l(\mathcal{A},\mathcal{C})^{\mathbb{T}^n}$ denotes the set of homotopy classes of bounded equivariant Fredholm maps $f : \mathcal{A} \to \mathcal{C}$ and is isomorphic to $\pi^{b^+}_{\mathbb{T}^n,\mathcal{U}}(\mathcal{J}(X); \mathrm{ind}\, D)$.

The monopole map μ_{X^τ} defines a class $[\mu_X] \in \pi_0(\mathcal{P}_l(\mathcal{A}^\tau, \mathcal{C}^\tau)^{\mathbb{T}^n})$ and the action of V determines an isomorphism

$$\begin{aligned} \mathcal{V} : \pi_0(\mathcal{P}_l(\mathcal{A},\mathcal{C})^{\mathbb{T}^n}) &\to \pi_0(\mathcal{P}_l(\mathcal{A}^\tau,\mathcal{C}^\tau)^{\mathbb{T}^n}) \\ [f] &\mapsto [VfV^{-1}]. \end{aligned} \tag{3.24}$$

Under this isomorphism, we can identify the Bauer-Furuta invariant of X^τ as an element of $\pi^{b^+}_{\mathbb{T}^n,\mathcal{U}}(\mathcal{J}(X); \mathrm{ind}\, D)$ corresponding to the bounded Fredholm map $V^{-1}\mu_{X^\tau}V$. The following theorem is a stronger result than the desired connected sum formula.

Theorem 3.8. *Let X be a closed, oriented 4-manifold that admits an n-component separating neck. Let $\tau \in S_n$ be an even permutation with X^τ the corresponding permuted manifold. Then*

$$[\mu_X] = [\mu_{X^\tau}] \tag{3.25}$$

as elements of $\pi^{b^+}_{\mathbb{T}^n,\mathcal{U}}(\mathcal{J}(X); \mathrm{ind}\, D)$

Let $D \subset \mathcal{A}$ be a disk bundle that contains the zeroes of μ_X and $V^{-1}\mu_{X^\tau}V$ with bounding sphere bundle $S \subset \mathcal{A}$. Recall that $C_l(D,\mathcal{C})^{\mathbb{T}^n}$ denotes the set of equivariant bounded Fredholm maps $f : D \to \mathcal{C}$ which are non-vanishing on S. In Section 2.6 it was shown that given a trivialisation $p : \mathcal{C} \to \mathcal{J}(X) \times \mathcal{U}$, the compact homotopy classes $\pi_0(C_l(S,\mathcal{C})^{\mathbb{T}^n})$ are naturally isomorphic to $\pi^0_{\mathbb{T}^n,\mathcal{U}}(\mathcal{J}(X); \mathrm{ind}\, l)$. Thus to prove Theorem 3.8, we will construct a homotopy from μ_X to $V^{-1}\mu_{X^\tau}V$ through compact perturbations of l.

3.3 Monopoles on the neck

The proof of Theorem 3.8 perturbs the monopole map μ_X using a series of homotopies. It is important to check that each homotopy is through compact perturbations of the linearised monopole map l as in Definition 2.10. To do this, standard techniques from the theory of Sobolev spaces, elliptic operators and monopoles on a cylinder are used to obtain estimates. However, at various stages of the proof, the length of the neck L is varied. Thus it is important to understand how these estimates depend on the length of the neck. This section deals with these issues and outlines the necessary techniques while paying careful attention to the variable neck length.

3.3.1 Sobolev estimates

Two fundamental theorems in the theory of Sobolev spaces are the Sobolev embedding theorem [14] and the Sobolev multiplication theorem [61]. These theorems give estimates that relate different Sobolev norms on a spinor-form pair (ψ, a) on X. For two neck lengths L_1 and L_2, we require estimates that apply to spinor-form pairs on both $X(L_1)$ and $X(L_2)$. The following results achieve this goal in the situations necessary for Theorem 3.8. The first lemma shows that bounds from Sobolev embeddings of the form $L^p_k \subset L^\infty$ can be chosen to be independent of the neck length L.

Lemma 3.9 ([9] Proposition 3.1)**.** *Let k and p be non-negative integers such that $k - \frac{4}{p} > 0$. There is a constant C_S such that, for any $L \geq 2$,*

$$\|(\psi, a)\|_{C^0(X)} \leq C_S \|(\psi, a)\|_{L^p_k(X)}$$

for any L^p_k-pair (ψ, a) on $X = X(L)$.

Proof. Fix a neck length $L \geq 2$ and let $X = X(L)$. For each $x \in X$ let $\delta_x : X \to [0, 1]$ be a smooth bump function in a small neighbourhood of x. Let $X_0 = X(2)$ and use the Sobolev embedding $L^p_k(X_0, W^+ \oplus T^*X_0) \subset L^\infty(X_0, W^+ \oplus T^*X_0)$ [61, Theorem B.2] to choose a constant C_1 with

$$\|(\psi', a')\|_{L^\infty(X_0)} \leq C_1 \|(\psi', a')\|_{L^p_k(X_0)}$$

for any L^p_k-pair (ψ', a') on X_0. Note that since $k - \frac{4}{p} > 0$, (ψ', a') is continuous and $\|(\psi', a')\|_{L^\infty(X_0)}$ agrees with $\|(\psi', a')\|_{C^0(X_0)}$. For any spinor ψ on X, $\delta_x \psi$ can be identified as a spinor on X_0. The same is true for a one-form a on X. Now for each $x \in X$,

$$\|(\delta_x \psi, \delta_x a)\|_{C^0(X)} \leq C_1 \|(\delta_x \psi, \delta_x a)\|_{L^p_k(X)}.$$

Since δ_x is smooth and defined locally, it has bounded C_k norm which is independent of L. Thus there exists a constant C_2 such that for all $x \in X$,

$$\|(\delta_x \psi, \delta_x a)\|_{L^p_k(X)} \leq C_2 \|(\psi, a)\|_{L^p_k(X)}.$$

It follows that

$$\begin{aligned} |(\psi, a)|_{C^0(X)} &= \sup_{x \in X} |(\delta_x \psi, \delta_x a)|_{C^0(X)} \\ &\leq C_1 \sup_{x \in X} \|(\delta_x \psi, \delta_x a)\|_{L^p_k(X)} \\ &\leq C_1 C_2 \|(\psi, a)\|_{L^p_k(X)}. \end{aligned}$$

Setting $C_S = C_1 C_2$ gives the result. □

The next lemma demonstrates that Sobolev multiplication bounds only depend linearly on the length of the neck.

Lemma 3.10. *Let $k \geq 0$ and $p \geq 1$ be integers. There is a constant C_{SM} such that, for any neck length $L \geq 2$,*

$$\|a \cdot \psi\|_{L^p_k} \leq C_{SM} L \|a\|_{L^{2p}_k} \|\psi\|_{L^{2p}_k}$$

for any L^{2p}_k-pair (ψ, a) on $X = X(L)$.

Proof. For notational simplicity, assume that X is connected. Recall that $M^{\pm}$ denotes the two halves of $M = \overline{X - N(L-1)}$ with tubular ends of the form

$$\begin{aligned} N(L)^- \cap M &= S^3 \times [-L, -L+1] \\ N(L)^+ \cap M &= S^3 \times [L-1, L]. \end{aligned}$$

The idea of the proof is to cut $N(L)$ into pieces that can be identified on $X_0 = X(2)$, then use Sobolev multiplication on X_0. Let $\phi : X \to [0,1]$ be a smooth function such that $\phi \equiv 1$ on $X - N(L - \frac{5}{4})$ and $\phi \equiv 0$ on $N(L-2)$. Define a function $\chi : \mathbb{R} \to [0,1]$ such that $\chi \equiv 1$ on $[0,1]$ and $\chi \equiv 0$ outside $[-\frac{1}{4}, \frac{5}{4}]$. Let χ_i be χ shifted by i so that $\chi_i \equiv 1$ on $[i, i+1]$ and $\chi_i \equiv 0$ outside $[i - \frac{1}{4}, i + \frac{5}{4}]$. Let $m = \lfloor L - \frac{5}{4} \rfloor$. For i an integer with $-(m+1) \leq i \leq m$, extend χ_i to $N(L) = S^3 \times [-L, L]$ by projection onto the interval factor. Let

$$\varphi = \sqrt{\phi^2 + \sum_{i=-m}^{m} \chi_i^2}.$$

Notice that φ is positive on X. Let $\varphi_i = \frac{\chi_i}{\varphi}$ for $-(m+1) \leq i \leq m$ with $\varphi_{m+1} = \frac{\phi}{\varphi}$. By construction,

$$\sum_{i=-(m+1)}^{m+1} \varphi_i^2 = 1.$$

For each $-(m+1) \leq i \leq m+1$, set $\psi_i = \varphi_i \psi$ and $a_i = \varphi_i a$. Both ψ_i and a_i can be identified as sections on X_0. For $-(m+1) \leq i \leq m$, this is accomplished by shifting the interval $[i - \frac{1}{4}, i + \frac{5}{4}]$ to $[-\frac{1}{4}, \frac{5}{4}]$. We can assume that the C^k norm of φ_i is bounded, which implies that there exist a constant C_1, independent of L, such that

$$\begin{aligned} \|\psi_i\|_{L^{2p}_k(X_0)} &\leq C_1 \|\psi\|_{L^{2p}_k(X)} \\ \|a_i\|_{L^{2p}_k(X_0)} &\leq C_1 \|a\|_{L^{2p}_k(X)}. \end{aligned} \tag{3.26}$$

In (3.10), the L^p_k-Sobolev norm on X_0 is defined as

$$\|(\psi_i, a_i)\|_{L^p_k(X_0)} = \sum_{j=0}^{k} \|(\mathcal{D}^j \psi_i, (d^* + d^+)^j a_i)\|_{L^p(X_0)}.$$

Equivalently, the L^p_k-norm can be instead defined by differentiating spinors with the spin^c connection ∇_A and forms with the Levi-Civita connection ∇. Thus there are constants $0 < c \leq C$ such that

$$c\|(\psi_i, a_i)\|_{L^p_k(X_0)} \leq \sum_{j=0}^{k} \|(\nabla_A^j \psi_i, \nabla^j a_i)\|_{L^p(X_0)} \leq C\|(\psi_i, a_i)\|_{L^p_k(X_0)}.$$

Calculating with repeated applications of the Leibniz rule gives

$$\begin{aligned} \|a_i \cdot \psi_i\|_{L^p_k(X_0)} &\leq \frac{1}{c} \sum_{j=0}^{k} \|\nabla_A^j (a_i \cdot \psi_i)\|_{L^p(X_0)} \\ &\leq \frac{1}{c} \sum_{j=0}^{k} \sum_{l=0}^{j} K_{j,l} \|\Gamma(\nabla^l a_i) \cdot (\nabla_A^{j-l} \psi_i)\|_{L^p(X_0)} \end{aligned}$$

for some non-negative constants $K_{j,l}$. Here $\Gamma(\nabla^l a_i) \in \mathrm{End}(W)$ is the matrix corresponding to spinor multiplication by the $(l+1)$-form $\nabla^l a_i$. The operator norm of $\Gamma(\nabla^l a_i)$ is equal to $|\nabla^l a_i|$, hence applying Sobolev multiplication [61, Lemma B.3] it follows that

$$\|a_i \cdot \psi_i\|_{L^p_k(X_0)} \leq C_2 \|a_i\|_{L^{2p}_k(X_0)} \|\psi_i\|_{L^{2p}_k(X_0)} \tag{3.27}$$

for some constant C_2. This constant depends on c, $K_{j,l}$ and Sobolev multiplication on X_0, hence is independent of L. Combining (3.26) and (3.27) produces the result.

$$\begin{aligned}
\|a \cdot \psi\|_{L^p_k(X)} &\leq \sum_{i=-m-1}^{m+1} \|a_i \cdot \psi_i\|_{L^p_k(X_0)} \\
&\leq C_2 \sum_{i=-m-1}^{m+1} \|a_i\|_{L^{2p}_k(X_0)} \|\psi_i\|_{L^{2p}_k(X_0)} \\
&\leq C_2 C_1^2 \sum_{i=-m-1}^{m+1} \|a\|_{L^{2p}_k(X)} \|\psi\|_{L^{2p}_k(X)} \\
&\leq C_{SM} L \|a\|_{L^{2p}_k(X)} \|\psi\|_{L^{2p}_k(X)}.
\end{aligned}$$

□

The same argument applied to $\sigma(\psi)$ instead gives the following result.

Lemma 3.11. *Let $k \geq 0$ and $p \geq 1$ be integers. There is a constant C_σ such that, for any neck length $L \geq 2$,*

$$\|\sigma(\psi)\|_{L^p_k} \leq C_\sigma L \|\psi\|^2_{L^{2p}_k}$$

for and $\psi \in L^{2p}_k(X(L), W^+)$.

3.3.2 Elliptic inequality

As noted in Remark 3.4, there is an estimate (3.18) which depends on a Sobolev embedding constant C_S and an elliptic operator constant C_E. As shown above, the Sobolev constant does not depend on the neck length L, however the elliptic constant C_E depends on the global geometry of $X(L)$. Now we focus attention on establishing a similar estimate for $|a|_{C^0}$ that does not depend on L.

To analyse the properties of monopoles on a neck of varying length, it is useful to apply Yang Mills theory on cylinders as in Chapter 2 of [20]. Most of the results and proofs in this section are adapted from this reference. Fix a neck length L with $X = X(L)$. For notational simplicity, assume that X only has one connected component. Recall that M^+ and M^- are the two halves of $M = \overline{X - N(L-1)}$. Attach infinite tubes to M^+ and M^- to get manifolds with tubular ends $Y^\pm$ of the form

$$\begin{aligned} Y^- &= M^- \cup S^3 \times [-L+1, \infty) \\ Y^+ &= S^3 \times (-\infty, L-1] \cup M^+. \end{aligned}$$

One-forms on the tubular component of $Y^\pm$ can be analysed by studying forms on the product $S^3 \times \mathbb{R}$. Let $\pi : S^3 \times \mathbb{R} \to S^3$ be projection onto the S^3 factor. All elements of $\Omega^1(S^3 \times \mathbb{R})$ are of the form $\omega_t + f dt$ for $\omega_t \in \Omega^1(S^3)$ a smooth family of one-forms on S^3 and $f : S^3 \times \mathbb{R} \to \mathbb{R}$ a smooth function. That is, we can identify

$$\Omega^1(S^3 \times \mathbb{R}) = \Omega^0(S^3 \times \mathbb{R}) \oplus C^\infty(S^3 \times \mathbb{R}, \pi^* T^* S^3).$$

Similarly, self-dual 2-forms $\Omega^2_+(S^3 \times \mathbb{R})$ can be identified with time-dependent 1-forms $\xi \in C^\infty(S^3 \times \mathbb{R}, \pi^* T^* S^3)$ by the isomorphism

$$\xi \mapsto \xi \wedge dt + *_3 \xi.$$

Here $*_3$ is the hodge star operator on S^3. Thus we can interpret the elliptic operator $d^* + d^+ : \Omega^1(S^3 \times \mathbb{R}) \to \Omega^0(S^3 \times \mathbb{R}) \oplus \Omega^2_+(S^3 \times \mathbb{R})$ as

$$d^* + d^+ : C^\infty(S^3 \times \mathbb{R}, \mathbb{R} \oplus \pi^* T^* S^3) \to C^\infty(S^3 \times \mathbb{R}, \mathbb{R} \oplus \pi^* T^* S^3). \tag{3.28}$$

Consider the operator $\mathcal{L} : \Omega^0(S^3) \oplus \Omega^1(S^3) \to \Omega^0(S^3) \oplus \Omega^1(S^3)$ defined by

$$\mathcal{L} = \begin{pmatrix} 0 & d^* \\ d & *d \end{pmatrix}. \tag{3.29}$$

This is a self-adjoint elliptic operator that squares to the Laplacian $\mathcal{L}^2 = dd^* + d^* d$ on $\Omega^0(S^3) \oplus \Omega^1(S^3)$. A direct calculation shows that under the identification (3.28),

$$d^* + d^+ = \frac{\partial}{\partial t} + \mathcal{L}$$

where $\frac{\partial}{\partial t}$ is the derivative in the $\mathbb{R}$ direction.

Since the tubular ends of Y are not compact, solutions to the operator $\frac{\partial}{\partial t} + \mathcal{L}$ will be studied in weighted Sobolev spaces. Weighted Sobolev spaces consist of functions that have a controlled exponential increase towards the tubular ends. To define them, fix a parameter $\alpha < 0$ and let f_α^- be a smooth function on $S^3 \times [-L, \infty)$ that is zero on $S^3 \times [-L, -L+2]$ and decreases with slope α on $S^3 \times [-L+3, \infty)$. Similarly, define f_α^+ on $S^3 \times (-\infty, L]$ to be zero on $S^3 \times [L-2, L]$ and decrease with slope α on $S^3 \times (-\infty, L-3]$. For instance, one possible choice of f_α^- for $t \geq -L+3$ (respectively f_α^+ for $t \leq L-3$) is

$$f_\alpha^-(s,t) = \alpha(t + L - 2)$$
$$f_\alpha^+(s,t) = -\alpha(t - L + 2).$$

Since $\alpha < 0$, both functions $f_\alpha^\pm$ are non-positive. Define the weighted Sobolev space $L_k^{p,\alpha}(Y^\pm)$ to be the completion of $L^p(Y^\pm)$ with respect to the norm

$$\|g\|_{L_k^{p,\alpha}} = \|\exp(f_\alpha^\pm)g\|_{L_k^p}.$$

Note that $\exp(f_\alpha^\pm)$ is decreasing exponentially towards the infinite end of $Y^\pm$. Moreover, the spaces $L_k^{p,\alpha}(Y^\pm)$ are independent of the original neck length L.

It is shown in [20] that $d^* + d^+ = \frac{\partial}{\partial t} + \mathcal{L}$ is a linear Fredholm operator on $L_1^{p,\alpha}$-forms if α is not in the spectrum of L. Since L is self-adjoint and elliptic it has discrete spectrum away from infinity, so choose $\alpha < 0$ to be greater than the maximal negative eigenvalue of L. As in (3.4), define a harmonic projection map $\mathrm{pr}^\pm : \Omega^1(Y^\pm) \to \Omega^1(Y^\pm)$ by integrating a homology basis of curves away from the neck. The image of $\mathrm{pr}^\pm$ is $H^1(X;\mathbb{R})$, identified as the space of harmonic forms $\mathcal{H}^1(Y^\pm) \subset \Omega^1(Y^\pm)$. Fix $p > 4$ so that $L_1^p(Y^\pm, T^*Y) \subset C^0(Y^\pm, T^*Y)$ by Sobolev embedding. Extend $\mathrm{pr}^\pm$ continuously to a map

$$\mathrm{pr}^\pm : L_1^{p,\alpha}(Y^\pm, T^*Y) \to L_1^{p,\alpha}(Y^\pm, T^*Y).$$

The operator

$$d^* + d^+ : L_1^{p,\alpha}(Y^\pm, T^*Y^\pm) \to L^{p,\alpha}(Y^\pm, \mathbb{R} \oplus \Lambda_+^2 T^*Y^\pm)$$

is Fredholm with kernel $\mathcal{H}^1(Y^\pm)$ and cokernel $H^0(Y^\pm;\mathbb{R}) \oplus H_+^2(Y^\pm;\mathbb{R})$. Let $H^\pm = \ker \mathrm{pr}^\pm$, which is a complement of $\ker(d^* + d^+)$. Thus the restriction $d^* + d^+ : H^\pm \to \mathrm{im}(d^* + d^+)$ is a linear bijection onto the closed subspace $\mathrm{im}(d^* + d^+)$. The bounded inverse theorem guarantees that the inverse is also continuous. Thus there are constants $C^\pm$ such that for $b \in L_1^{p,\alpha}(Y^\pm, \Lambda^1(Y^\pm))$, we have

$$\|b\|_{L_1^{p,\alpha}} \leq C^\pm \left(\|(d^* + d^+)b\|_{L^{p,\alpha}} + \|\mathrm{pr}^\pm(b)\| \right). \tag{3.30}$$

Importantly, the constants $C^\pm$ are independent of the neck length L. That is, for another choice of neck length L' and manifolds with tubular ends $(Y')^\pm$, there is an isometry from $L_k^{p,\alpha}(Y^\pm, T^*Y^\pm)$ to $L_k^{p,\alpha}((Y')^\pm, T^*(Y')^\pm)$ defined by shifting the interval component by $L' - L$.

To analyse the behaviour of forms away from the middle of the neck, define smooth cut-off functions $\beta^\pm : X \to [0,1]$ which vanish on $X^\mp \cup N(2)$ and are equal to 1 on $M^\pm$. For such β to exist, it will be necessary to assume that $L > 3$.

Lemma 3.12 ([9] Proposition 3.1)**.** *Let $\beta^\pm$ be cutaway functions as described above and fix $p > 4$. There exists a constant C such that, for any neck-length $L > 3$,*

$$|a|_{C^0(M)} \leq C \left(\|(d^* + d^+)\beta^+ a\|_{L^p(X)} + \|(d^* + d^+)\beta^- a\|_{L^p(X)} + \|\operatorname{pr}(a)\| \right)$$

for any $a \in L^p_1(X, T^*X)$ with $X = X(L)$.

Proof. The Sobolev embedding $L^p_1(X, T^*X) \subset L^\infty(X, T^*X)$ guarantees the existence of a constant C_S such that

$$|a|_{C^0(X)} \leq C_S \|a\|_{L^p_1(X)}. \tag{3.31}$$

Lemma 3.9 ensures that C_S can be chosen independently of L. To apply the elliptic bound, let $b_\pm = \beta^\pm a$ and notice that $e^{f^\pm_\alpha} b_\pm = a$ on $M^\pm$.

$$\begin{aligned} |a|_{C^0(M^\pm)} &= |e^{f^\pm_\alpha} b_\pm|_{C^0(M^\pm)} \\ &\leq |e^{f^\pm_\alpha} b_\pm|_{C^0(Y^\pm)} \\ &\leq C_S \|e^{f^\pm_\alpha} b_\pm\|_{L^p_1(Y^\pm)} \\ &= C_S \|b_\pm\|_{L^{p,\alpha}_1(Y^\pm)} \end{aligned} \tag{3.32}$$

Note that $b_\pm$ is compactly supported in $M^\pm \cup N(L-1) \subset Y^\pm$, so the Sobolev bound (3.31) applies to $|e^{f^\pm_\alpha} b_\pm|_{C^0(Y^\pm)}$. Now (3.30) gives

$$\begin{aligned} |a|_{C^0(M^\pm)} &\leq C_S C^\pm \left(\|(d^* + d^+) b_\pm\|_{L^{p,\alpha}(Y^\pm)} + \|\operatorname{pr}(b_\pm)\| \right) \\ &\leq C_S C^\pm \left(\|(d^* + d^+) b_\pm\|_{L^p(X)} + \|\operatorname{pr}(b_\pm)\| \right). \end{aligned}$$

This inequality follows since $f^\pm_\alpha \leq 0$ and $b^\pm$ is compactly supported on $M^\pm \cup N(L - 1) \subset Y^\pm$. Putting this together with $C = \max\{C^s C^+, C^s C^-\}$ yields

$$\begin{aligned} |a|_{C^0(M)} &\leq |a|_{C^0(M^+)} + |a|_{C^0(M^-)} \\ &\leq C \left(\|(d^* + d^+) b_+\|_{L^p(X)} + \|(d^* + d^+) b_-\|_{L^p(X)} + (\|\operatorname{pr}(b_+)\| + \|\operatorname{pr}(b_-)\|) \right). \end{aligned}$$

Recall that $\operatorname{pr}(b_\pm)$ is defined by integration over an orthonormal basis of curves contained in M. Since $b_\pm$ vanishes on $M^\mp$ we have

$$\begin{aligned} \|\operatorname{pr}(b_+)\| + \|\operatorname{pr}(b_-)\| &= \|\operatorname{pr}(b_+ + b_-)\| \\ &= \|\operatorname{pr}(a)\|. \end{aligned}$$

It follows that

$$|a|_{C^0(M)} \leq C\left(\|(d^* + d^+)b_+\|_{L^p(X)} + \|(d^* + d^+)b_-\|_{L^p(X)} + \|\operatorname{pr}(a)\|\right).$$

□

The following result resembles the elliptic bound (3.52) in the second step of the standard argument, however this bound is independent of the neck length of $X(L)$. Recall that $N(2, L-1)$ denotes $\overline{N(L-1) - N(2)}$.

Proposition 3.13 (Adapted from [9] Lemma 3.3)**.** *Fix $p > 4$. There exists a neck length L_0 and a constant C_E such that the following holds: For any $L \geq L_0$, set $X = X(L)$ and let $a \in L^p_1(X, T^*X)$ be a one-form such that $\operatorname{pr}(a) = 0$. If $(d^* + d^+)a$ vanishes on $N(L-1)$, then*

$$|a|_{C^0(M)} \leq C_E|(d^* + d^+)a|_{C^0(M)}.$$

Proof. Let $\beta^{\pm}$ be cut-off functions as described in Lemma 3.12. Assume without loss of generality that $|d\beta^{\pm}|_{C^0(X)} < \frac{2}{L}$, which is possible when $L > 6$. Lemma 3.12 gives a constant C_1, independent of L, such that

$$|a|_{C^0(M)} \leq C_1(\|(d^* + d^+)\beta^+ a\|_{L^p(X)} + \|(d^* + d^+)\beta^- a\|_{L^p(X)}). \tag{3.33}$$

Calculating with the Leibniz rule yields

$$\|(d^* + d^+)\beta^{\pm} a\|_{L^p(X)} \leq \|\beta^{\pm}(d^* + d^+)a\|_{L^p(X)} + \|d\beta^{\pm} \wedge a\|_{L^p(X)}.$$

The product $\beta^{\pm}(d^* + d^+)a$ is supported inside $M^{\pm}$ and since $\beta^{\pm} \equiv 1$ on $M^{\pm}$, we have

$$\|\beta^{\pm}(d^* + d^+)a\|_{L^p(X)} = \|(d^* + d^+)a\|_{L^p(M^{\pm})}.$$

Since $N(L-1)$ has non-negative Ricci curvature, the Weitzenböck formula [55, Ex 2.31] implies that $|a|$ is a harmonic function when restricted to $N(L-1)$. Thus the maximum principle holds and $\sup_{N(L-1)} |a| = \sup_{\partial N(L-1)} |a|$. Since $d\beta^{\pm}$ is supported inside $X^{\pm} \cap N(2, L-1)$, it follows that

$$\begin{aligned} \|d\beta^+ \wedge a\|_{L^p(X)} + \|d\beta^- \wedge a\|_{L^p(X)} &\leq \|d\beta^+ + d\beta^-\|_{L^p(N(L-1))} \sup_{N(2,L-1)} |a| \\ &\leq 4L^{\frac{1}{p}-1}\operatorname{vol}(S^3)^{\frac{1}{p}} \sup_{\partial N(L-1)} |a|. \end{aligned} \tag{3.34}$$

Putting this together with (3.33) gives

$$\begin{aligned}|a|_{C^0(M)} &\le C_1\|(d^*+d^+)a\|_{L^p(M)} + 4C_1L^{\frac{1}{p}-1}\mathrm{vol}(S^3)^{\frac{1}{p}}|a|_{C^0(\partial N(L))}\\ &\le C_1\mathrm{vol}(M)^{\frac{1}{p}}|(d^*+d^+)a|_{C^0(M)} + 4C_1L^{\frac{1}{p}-1}\mathrm{vol}(S^3)^{\frac{1}{p}}|a|_{C^0(M)}.\end{aligned}$$

Set $C_2 = C_1\mathrm{vol}(M)^{\frac{1}{p}}$ and $C_3 = 4C_1\mathrm{vol}(S^3)^{\frac{1}{p}}$ to obtain

$$|a|_{C^0(M)}(1 - C_3L^{\frac{1}{p}-1}) \le C_2|(d^*+d^+)a|_{C^0(M)}.$$

Since $p > 4$, $\frac{1}{p} - 1 < 0$ and $L \ge L_0$ implies $L^{\frac{1}{p}-1} \le L_0^{\frac{1}{p}-1}$. Set $L_0 = (2C_3)^{-\frac{p}{1-p}}$, which we can assume is larger than 6, so that $L \ge L_0$ implies

$$\begin{aligned}(1 - C_3L^{\frac{1}{p}-1}) &\ge (1 - C_3L_0^{\frac{1-p}{p}})\\ &\ge 1 - \frac{1}{2}\\ &= \frac{1}{2}.\end{aligned}$$

When $L \ge L_0$ it follows that

$$\begin{aligned}|a|_{C^0(M)} &\le C_2|(d^*+d^+)a|_{C^0(M)}(1 - C_3L^{\frac{1}{p}-1})^{-1}\\ &\le 2C_2|(d^*+d^+)a|_{C^0(M)}.\end{aligned}$$

Let $C_E = 2C_2$, which is independent of L. □

Remark 3.14: Suppose instead that $(d^*+d^+)a$ only vanishes on $N(2, L-1)$. Then the maximum of $(d^*+d^+)a$ could occur on $\partial N(2)$ instead of $\partial N(L-1)$. To overcome this, assume that there is a constant C, independent of L, such that

$$\sup_{N(2,L-1)} |a| \le C \sup_{\partial N(L-1)} |a|.$$

Since $(d^*+d^+)a = 0$ on $N(2, L-1)$ and $\beta^\pm$ is supported in $X^\pm - N(2)$, the product $\beta^\pm(d^*+d^+)a$ is still supported in $M^\pm$. We can still execute the above argument with (3.34) becoming

$$\begin{aligned}\|d\beta^+\wedge a\|_{L^p(X)} + \|d\beta^-\wedge a\|_{L^p(X)} &\le \|d\beta^+ + d\beta^-\|_{L^p(N(L-1))}\sup_{N(2,L-1)}|a|\\ &\le 4CL^{\frac{1}{p}-1}\mathrm{vol}(S^3)^{\frac{1}{p}}\sup_{\partial N(L-1)}|a|.\end{aligned}$$

Setting $C_3 = 4C_1C\mathrm{vol}(S^3)^{\frac{1}{p}}$, there still exists constants C_E and L_0 such that, when $L \ge L_0$,

$$|a|_{C^0(X)} \le C_E|(d^*+d^+)a|_{C^0(M)}.$$

3.3.3 Elliptic bootstrapping

For a fixed connection $A \in \mathcal{J}(X)$ on $X(L)$, the elliptic bootstrapping step of 3.4 gives an L^2_k-bound on a pair (ψ, a) of the form

$$\|(\psi, a)\|_{L^2_k} \leq C_B|(\psi, a)|_{C^0}.$$

The constant C_B depends on the curvature of A and the length of the neck L. To cooperate with neck stretching, we show that C_B only increases polynomially in L.

Lemma 3.15. *Let $A \in \mathcal{J}(X)$ be a connection on $X = X(L)$ and fix an integer $k \geq 2$. There are positive constants C_B and d such that, for any $L \geq 2$, if (ψ, a) is an L^2_k-pair with*

$$\begin{aligned} D_A\psi &= -ia \cdot \psi \\ d^+a &= iF_A^+ - i\sigma(\psi), \end{aligned}$$

then

$$\|(\psi, a)\|_{L^2_k} \leq C_B L^d(1 + |(\psi, a)|_{C^0})^d.$$

Proof. As in the elliptic bootstrapping step of Remark 3.4

$$\begin{aligned} \|(\psi, a)\|^p_{L^p_i} - \|(\psi, a)\|^p_{L^p} &= \|(D_A\psi, d^+a)\|^p_{L^p_{i-1}} \\ &\leq \|a \cdot \psi\|^p_{L^p_{i-1}} + (\|\sigma(\psi)\|_{L^p_{i-1}} + \|F_A^+\|_{L^p_{i-1}})^p \end{aligned}$$

for any $0 \leq i \leq k$ and $2 \leq p \leq 2^{k+1}$. By Lemma 3.10 and 3.11, there are constants C_{SM} and C_σ independent of L such that

$$\|(\psi, a)\|_{L^p_i} \leq C_{SM}L\|a\|_{L^{2p}_{i-1}}\|\psi\|_{L^{2p}_{i-1}} + C_\sigma L\|\psi\|^2_{L^p_{i-1}} + \|F_A^+\|_{L^p_{i-1}} + \|(\psi, a)\|_{L^p}.$$

Since A is flat on the neck, $\|F_A^+\|_{L^p_{i-1}}$ is a constant independent of L. Thus there is a constant C_1 such that

$$\|(\psi, a)\|_{L^p_i} \leq C_1L(\|(\psi, a)\|^2_{L^{2p}_{i-1}} + \|(\psi, a)\|_{L^p})$$

for all $0 \leq i \leq k$ and $2 \leq p \leq 2^{k+1}$. Starting with $i = k$ and $p = 2$, inductively applying this inequality gives a bound

$$\|(\psi, a)\|_{L^2_k} \leq L^{d_1}f(\|(\psi, a)\|_{L^2}, ..., \|(\psi, a)\|_{L^{2^{k+1}}})$$

for some natural number d_1 and polynomial f, both independent of L. Letting d_2 be the degree of f, there is a constant C_2 such that

$$|f(x_1, ..., x_k)| \leq C_2(1 + |x_1| + ... + |x_k|)^{d_2}.$$

Since $\mathrm{vol}(X(L))$ increases linearly with L, there is a bound

$$\begin{aligned}\|(\psi, a)\|_{L^p} &\leq \mathrm{vol}(X(L))^{\frac{1}{p}}|(\psi, a)|_{C^0} \\ &\leq C_3 L|(\psi, a)|_{C^0}.\end{aligned}$$

Here C_3 is a constant independent of L and p. Letting $d = d_1 + d_2$, it follows that

$$\begin{aligned}\|(\psi, a)\|_{L^2_k} &\leq C_2 L^{d_1}(1 + \|(\psi, a)\|_{L^2} + ... + \|(\psi, a)\|_{L^{2^{k+1}}})^{d_2} \\ &\leq C_2 L^{d_1} L^{d_2}(1 + C_3|(\psi, a)|_{C^0} + ... + C_3|(\psi, a)|_{C^0})^{d_2} \\ &\leq C_B L^d(1 + |(\psi, a)|_{C^0})^d\end{aligned}$$

for some constant C_B independent of L. □

Remark 3.16: Assume that there is a smooth function $\rho : X \to \mathbb{R}$ and constant C such that the pair (ψ, a) instead satisfies

$$\begin{aligned}D_A\psi &= -i\rho a \cdot \psi \\ \|d^+ a\|_{L^p_i} &\leq C(\|\sigma(\psi)\|_{L^p_i} + \|F^+_A\|_{L^p_i}) \\ \|\rho a\|_{L^p_i} &\leq C\|a\|_{L^p_i}\end{aligned}$$

for all $0 \leq i \leq k$, $2 \leq p \leq 2^{k+1}$. The same argument can be repeated, the only difference being that the constant C_1 now depends on C. Thus there still exists positive constants C_B and d such that

$$\|(\psi, a)\|_{L^2_k} \leq C_B L^d(1 + |(\psi, a)|_{C^0})^d.$$

These constants depend on C, but are independent of L so long as C is.

3.3.4 Exponential decay

Since $X(L)$ is compact, there are L^p-bounds on spinors and one-forms of the form

$$\|(\psi, a)\|_{L^p} \leq C_p|(\psi, a)|_{C^0} \tag{3.35}$$

with $C_p = \mathrm{vol}(X(L))^{\frac{1}{p}}$. However, this constant C_p grows linearly with the length of the neck. This creates issues when varying the neck length in the proof of Theorem

3.8. To circumvent this, we will show that monopoles decay exponentially towards the middle of the neck. This counteracts the linear growth of C_p so that (3.35) can still be applied near the middle of the neck. The following work is adapted from Chapter 3 of [20].

Let $E \to S^3$ be a vector bundle over S^3, equipped with a metric g_E and compatible connection ∇_E. For notational simplicity, we will assume that $N(L)$ has one connected component. Let $\pi : N(L) \to S^3$ be projection onto the S^3 component. Fix $k > 2$ and let $A : C^\infty(S^3, E) \to C^\infty(S^3, E)$ be a first order self-adjoint elliptic pseudo-differential operator on E. From spectral theory of elliptic operators, there is an orthonormal basis of eigenvectors $\{\phi_n\}_{n=-M}^\infty \subset L^2(S^3, E)$ for A with discrete real eigenvalues $\{\lambda_n\}$. Label the eigenvalues so that the non-zero eigenvalues have a positive index and the zero eigenvalues (of multiplicity $M+1$) have a non-positive index. Thus there is a $\delta > 0$ such that $|\lambda_n| > \delta$ for all $n \geq 1$. Also ensure that the labeling is chosen so that $|\lambda_n| \geq |\lambda_m|$ when $n \geq m$.

Let $f_0 \in C^\infty(S^3, E)$ be a smooth section with eigen decomposition $f_0 = \sum_n f_0^n \phi_n$ convergent in L^2 for $f_0^n \in \mathbb{R}$. Then Af_0 is also smooth and its eigen decomposition is $Af_0 = \sum_n \lambda_n f_0^n \phi_n$ due to the following calculation.

$$\begin{aligned}\langle Af_0, \phi_n \rangle_{L^2} &= \langle f_0, A\phi_n \rangle_{L^2} \\ &= \langle f_0, \lambda_n \phi_n \rangle_{L^2} \\ &= \lambda_n f_0^n.\end{aligned}$$

A smooth section $f \in C^\infty(S^3 \times [-L, L], \pi^* E)$ also has a decomposition $f_t = \sum_n f^n(t)\phi_n$ for some functions $f^n : [-L, L] \to \mathbb{R}$. The smoothness of f_t implies the smoothness of the component functions f^n by the following application of the Leibniz integral rule.

$$\begin{aligned}\langle (\partial_t^m f)_t, \phi_n \rangle_{L^2} &= \int_{S^3} (\partial_t^m f)_t \phi_n \mathrm{dvol}_{S^3} \\ &= \partial_t^m \int_{S^3} f_t \phi_n \mathrm{dvol}_{S^3} \\ &= \partial_t^m f^n(t).\end{aligned}$$

Define a pseudo-differential operator $D : C^\infty(S^3 \times [-L, L], \pi^* E) \to C^\infty(S^3 \times [-L, L], \pi^* E)$ by

$$D = \frac{\partial}{\partial t} + A. \tag{3.36}$$

Assume that D is elliptic and extend D to an operator on L^2 sections. Recall that $C^+ = S^3 \times [L-1, L]$ and $C^- = S^3 \times [-L, -L+1]$ denote collar neighbourhoods of the boundary of $S^3 \times [-L, L]$.

Proposition 3.17 (Adapted from [20] Lemma 3.2)**.** *Fix constants $M \geq 1$ and $L \geq 2M$. Suppose $f \in L^2(S^3 \times [-L, L], \pi^* E)$ such that f_t is orthogonal to* $\ker A$ *for all $t \in [-L, L]$. If $Df = (\partial_t + A)f = 0$ then*

$$\int_{S^3 \times [-2M, 2M]} |f|^2 \leq \left(\frac{e^{-2\delta(L-2M)}}{1 - e^{-2\delta}} \right) \left(\int_{C^-} |f|^2 + \int_{C^+} |f|^2 \right) \tag{3.37}$$

and

$$\sup_{S^3 \times [-M, M]} |f| \leq C_\delta e^{-\delta(L-2M)} \sup_{S^3 \times [-L, L]} |f|. \tag{3.38}$$

where δ and C_δ are positive constants independent of L and M.

Proof. Note that since D is assumed to be elliptic, $Df = 0$ implies that f is smooth by elliptic regularity. Write $Af_t = \sum_n \lambda_n f^n(t) \phi_n$ so that

$$\partial_t f + \sum_n \lambda_n f^n \phi_n = 0.$$

Taking the L^2-inner product with ϕ_n yields

$$\partial_t f^n(t) + \lambda_n f^n(t) = 0.$$

Since f_t is orthogonal to $\ker A$ it can be assumed that $n \geq 1$ and $\lambda_n \neq 0$ so that

$$f^n(t) = e^{-\lambda_n t} f^n(0).$$

Notice that if $\lambda_n > 0$ then f^n decays exponentially as t increases and if $\lambda_n < 0$ then f^n decays exponentially as t decreases. To capture this behaviour, split $f^n = f^n_+ + f^n_-$ defined by

$$f^n_+ = \begin{cases} f^n & \text{if } \lambda_n > 0 \\ 0 & \text{if } \lambda_n < 0 \end{cases} \qquad f^n_- = \begin{cases} 0 & \text{if } \lambda_n > 0 \\ f^n & \text{if } \lambda_n < 0. \end{cases}$$

Also let $f_+ = \sum_{n=1}^{\infty} f^n_+ \phi_n$ and $f_- = f - f_+$. Each half of $|f|^2$ is integrated separately.

$$\begin{aligned} \int_{S^3 \times [-2M, 2M]} |f_+|^2 &= \int_{-2M}^{2M} \|f_+(t)\|^2_{L^2} dt \\ &= \int_{-2M}^{2M} \sum_{n=1}^{\infty} e^{-2\lambda_n t} |f^n_+(0)|^2 dt \\ &= \sum_{n=1}^{\infty} \frac{\sinh(4M\lambda_n)}{\lambda_n} |f^n_+(0)|^2 \end{aligned} \tag{3.39}$$

Here the monotone convergence theorem has been used to swap the sum and the integral. Integrating instead over the band C^- gives

$$\int_{C^-} |f_+|^2 = \sum_{n=1}^{\infty} \left(\frac{e^{2\lambda_n L} - e^{2\lambda_n (L-1)}}{2\lambda_n} \right) |f_+^n(0)|^2 \tag{3.40}$$

Choose $\delta > 0$ such that $|\lambda_n| > \delta$ for $n \geq 1$ as explained above. When $\lambda_n > 0$, notice that

$$\begin{aligned} \frac{\sinh(4M\lambda_n)}{\lambda_n} &\leq \frac{e^{4M\lambda_n}}{2\lambda_n} \\ &= \frac{e^{2\lambda_n L}}{2\lambda_n} \left(\frac{e^{-2\lambda_n (L-2M)}}{1 - e^{-2\lambda_n}} \right) (1 - e^{-2\lambda_n}) \\ &= \left(\frac{e^{-2\lambda_n (L-2M)}}{1 - e^{-2\lambda_n}} \right) \left(\frac{e^{2\lambda_n L} - e^{2\lambda_n (L-1)}}{2\lambda_n} \right) \\ &\leq \left(\frac{e^{-2\delta (L-2M)}}{1 - e^{-2\delta}} \right) \left(\frac{e^{2\lambda_n L} - e^{2\lambda_n (L-1)}}{2\lambda_n} \right) \end{aligned} \tag{3.41}$$

The last line follows since $\lambda_n > \delta > 0$ and $L - 2M \geq 0$. Combining (3.39), (3.40) and (3.41) gives

$$\begin{aligned} \int_{S^3 \times [-2M, 2M]} |f_+|^2 &\leq \sum_{n=1}^{\infty} \left(\frac{e^{-2\delta (L-2M)}}{1 - e^{-2\delta}} \right) \left(\frac{e^{2\lambda_n L} - e^{2\lambda_n (L-1)}}{2\lambda_n} \right) |f_+^n(0)|^2 \\ &= \left(\frac{e^{-2\delta (L-2M)}}{1 - e^{-2\delta}} \right) \int_{C^-} |f_+|^2 \\ &\leq \left(\frac{e^{-2\delta (L-2M)}}{1 - e^{-2\delta}} \right) \int_{C^-} |f|^2. \end{aligned}$$

Similarly when $\lambda_n < 0$,

$$\begin{aligned} \int_{C^+} |f_-|^2 &= \sum_{n=1}^{\infty} \left(\frac{e^{-2\lambda_n (L-1)} - e^{-2\lambda_n L}}{2\lambda_n} \right) |f_-^n|^2 \\ &= \sum_{n=1}^{\infty} \left(\frac{e^{2|\lambda_n| L} - e^{2|\lambda_n| (L-1)}}{2|\lambda_n|} \right) |f_-^n|^2. \end{aligned}$$

Now (3.41) can be applied to get

$$\begin{aligned} \int_{S^3 \times [-2M, 2M]} |f_-|^2 &= \sum_{n=1}^{\infty} \frac{\sinh(4M|\lambda_n|)}{|\lambda_n|} |f_-^n(0)|^2 \\ &\leq \left(\frac{e^{-2\delta (L-2M)}}{1 - e^{-2\delta}} \right) \int_{C^+} |f_-|^2 \\ &\leq \left(\frac{e^{-2\delta (L-2M)}}{1 - e^{-2\delta}} \right) \int_{C^+} |f|^2. \end{aligned}$$

It follows that

$$\int_{S^3\times[-2M,2M]} |f|^2 \leq \left(\frac{e^{-2\delta(L-2M)}}{1-e^{-2\delta}}\right)\left(\int_{C^-}|f|^2 + \int_{C^+}|f|^2\right). \tag{3.42}$$

This proves the first inequality (3.37).

The supremum and essential supremum of $|f|$ agree because f is continuous. Since the sequence $(\sum_{i=1}^N f^n(t)\phi_n)_{N=1}^\infty$ converges to f_t in L^2 as $N\to\infty$, there is a subsequence that converges to f_t pointwise almost everywhere. Let $(x_0,t_0)\in S^3\times[-M,M]$ be any point such that

$$f_{t_0}(x_0) = \sum_{n=1}^\infty e^{-\lambda_n t_0} f^n(0)\phi_n(x_0).$$

Since $t_0\in[-M,M]$ it follows that

$$|f_{t_0}(x_0)| \leq \sum_{n=1}^\infty e^{M|\lambda_n|}|f^n(0)||\phi_n(x_0)|.$$

The Sobolev embedding $L^2_2(S^3,E)\to C^0(S^3,E)$ gives a constant C_S such that $|\phi_n|_{C^0}\leq C_S\|\phi_n\|_{L^2_2}$ for all n. Further the second order elliptic operator $A^2 : L^2_2(S^3,E)\to C^0(S^3,E)$ gives an elliptic inequality

$$\begin{aligned}\|\phi_n\|_{L^2_2} &\leq C_E(\|A^2\phi_n\|_{L^2} + \|\phi_n\|_{L^2})\\ &= C_E(\lambda_n^2+1)\|\phi_n\|_{L^2}.\end{aligned}$$

Note that C_S and C_E are independent of L. Since $\|\phi_n\|_{L^2}=1$, we have $|\phi_n|_{C_0}\leq C_EC_S(\lambda_n^2+1)$ and

$$|f_{t_0}(x_0)| \leq \sum_{n=1}^\infty C_EC_S(\lambda_n^2+1)e^{M|\lambda_n|}|f^n(0)|.$$

Lemma 3.20 provides a bound

$$\left(\sum_{n=1}^\infty(\lambda_n^2+1)e^{M|\lambda_n|}|f^n(0)|\right)^2 \leq C'\sum_{n=1}^\infty \frac{\sinh(4M|\lambda_n|)}{|\lambda_n|}|f^n(0)|^2$$

for some constant C' which depends only on $\{\lambda_n\}$. Combining this with (3.39) produces

$$\begin{aligned}|f_{t_0}(x_0)|^2 &\leq C\sum_{n=1}^\infty \frac{\sinh(4M|\lambda_n|)}{|\lambda_n|}|f^n(0)|^2\\ &= C\int_{S^3\times[-2M,2M]}|f|^2\end{aligned}$$

where $C = C'C_S^2C_E^2$. Applying (3.42) and taking the essential supremum over $S^3 \times [-M, M]$ yields

$$\begin{aligned}\sup_{S^3\times[-M,M]} |f|^2 &\leq C\left(\frac{e^{-2\delta(L-2M)}}{1-e^{-4\delta}}\right)\left(\int_{C^-}|f|^2 + \int_{C^+}|f|^2\right)\\ &\leq \left(\frac{2C\mathrm{vol}(S^3)}{1-e^{-4\delta}}\right)e^{-2\delta(L-2M)}\sup_{S^3\times[-L,L]}|f|^2.\end{aligned}$$

Let $C_\delta = \sqrt{\frac{2C\mathrm{vol}(S^3)}{1-e^{-4\delta}}}$ so that

$$\sup_{S^3\times[-M,M]}|f| \leq C_\delta e^{-\delta(L-2M)}\sup_{S^3\times[-L,L]}|f|.$$

□

Corollary 3.18. *Suppose that $a \in L^2(N(L-1), T^*N(L-1))$ is a 1-form such that $(d^* + d^+)a = 0$. Then for any $M \geq 1$ and $L \geq 2M + 1$,*

$$\sup_{N(M)}|a \wedge dt| \leq C_\delta e^{-\delta(L-2M)}\sup_{N(L-1)}|a| \tag{3.43}$$

for some positive constants δ and C_δ independent of L and M.

Proof. It is shown in (3.28) that $d^* + d^+$ can be identified as an operator on $C^\infty(N(L), \mathbb{R} \oplus \pi^*T^*S^3)$ and that $d^* + d^+ = \frac{\partial}{\partial t} + \mathcal{L}$. Here $\mathcal{L}$ is a self-adjoint, elliptic operator on $\Omega^0(S^3) \oplus \Omega^1(S^3)$ with $\mathcal{L}^2 = dd^* + d^*d$. Note that $d^* + d^+$ is also self-adjoint and elliptic. Since $b_1(S^3) = 0$, the kernel of $\mathcal{L}$ is one dimensional consisting of only constant functions. Thus there is an eigenbasis $\{\phi_n\}_{n=0}^\infty$ of $\mathcal{L}$ with eigenvalues $\{\lambda_n\}_{n=0}^\infty$ such that ϕ_0 is a non-zero constant function on S^3, $\lambda_0 = 0$ and $\lambda_n \neq 0$ for $n \geq 1$. Write

$$a_t = a_0(t)\phi_0 dt + \sum_{n=1}^\infty a_n(t)\phi_n$$

for some smooth functions $a_n : [-L+1, L-1] \to \mathbb{R}$. As in Proposition 3.17, $\partial_t a_0 + \lambda_0 a_0 = 0$ and therefore a_0 is a constant function. Now $a' = a - a_0\phi_0 dt$ is L^2-orthogonal to $\ker L$ for all t. Since $(d^* + d^+)(a_0\phi_0 dt) = 0$ we have $(d^* + d^+)a' = 0$. Proposition 3.17 gives constants C_1 and δ, independent of L and M, such that

$$\begin{aligned}\sup_{N(M)}|a'| &\leq C_1 e^{-\delta(L-2M-1)}\sup_{N(L-1)}|a'|\\ &\leq C_1 e^{-\delta(L-2M)}\left(\sup_{N(L-1)}|a| + \sup_{N(L-1)}|a_0\phi_0 dt|\right).\end{aligned}$$

Since a_0 and ϕ_0 are constants, we can calculate

$$\begin{aligned}\|a_0\phi_0 dt\|_{L^2}^2 &= \int_{N(L-1)} |a_0\phi_0 dt|^2 \\ &= 2\mathrm{vol}(S^3)(L-1)|a_0|^2|\phi_0|^2.\end{aligned}$$

The decomposition $a = a' + a_0\phi_0 dt$ is L^2-orthogonal, hence $\|a_0\phi_0 dt\|_{L^2}^2 = \|a\|_{L^2}^2 - \|a'\|_{L^2}^2$. It follows that

$$\begin{aligned}2\mathrm{vol}(S^3)(L-1)|a_0|^2|\phi_0|^2 &= \|a_0\phi_0 dt\|_{L^2}^2 \\ &\leq \|a\|_{L^2}^2 \\ &\leq 2(L-1)\mathrm{vol}(S^3)\sup_{N(L-1)}|a|^2.\end{aligned}$$

Thus $|a_0| \leq \frac{1}{|\phi_0|}\sup_{N(L-1)}|a|$ and there is a constant C_2 with

$$\sup_{N(M)}|a'| \leq C_2 e^{-\delta(L-2M)}\sup_{N(L-1)}|a|.$$

Finally, $|a\wedge dt| = |a'\wedge dt| \leq |a'|$, hence

$$\sup_{N(M)}|a\wedge dt| \leq C_2 e^{-\delta(L-2M)}\sup_{N(L-1)}|a|.$$

□

Corollary 3.19. *Let A_0 be a flat reference connection on $N(L)$ as in (3.46). Suppose $\psi \in L^2(N(L), W^+)$ is a spinor such that $D_{A_0}\psi = 0$. Then for any $M \geq 1$ and $L \geq 2M$,*

$$\sup_{N(M)}|\psi| \leq C_\delta e^{-\delta(L-2M)}\sup_{N(L)}|\psi| \tag{3.44}$$

for some positive constants δ and C_δ independent of L and M.

Proof. The spin^c structure on X is defined so that, on the neck, Clifford multiplication $\Gamma : TN(L) \to \mathrm{End}(W)$ is induced by the Clifford multiplication $\gamma : TS^3 \to \mathrm{End}(W_{S^3})$ on S^3.

$$\Gamma(\partial_{x_i}) = \begin{pmatrix} 0 & \gamma(\partial_{x_i}) \\ -\gamma(\partial_{x_i})^* & 0\end{pmatrix}, \qquad \Gamma(\partial_t) = \begin{pmatrix} 0 & \mathrm{id} \\ -\mathrm{id} & 0\end{pmatrix}. \tag{3.45}$$

Here $\{\partial_t, \partial_{x_1}, \partial_{x_2}, \partial_{x_3}\}$ is a basis for $TN(L)$ corresponding to local coordinates (x_1, x_2, x_3, t) of $N(L)$. The spin^c connection ∇_{A_0} for the reference connection A_0 is given by the formula

$$\nabla_{A_0} = dt \otimes \frac{\partial}{\partial t} + \nabla^{S^3}. \tag{3.46}$$

Here ∇^{S^3} is a spin^c connection on $W_{S^3} \to S^3$. Since $b_2(S^3) = 0$, it can be assumed that ∇^{S^3} is flat. This equation is understood by treating a spinor $\psi \in C^\infty(N(L), W^+)$ as a time-dependent family of spinors $\{\psi_t\}$ on S^3. Over the neck $N(L)$, the Dirac operator $D_{A_0} : C^\infty(X, W^+) \to C^\infty(X, W^-)$ takes the form

$$\begin{aligned} D_{A_0} &= \Gamma(\partial_t) \cdot \frac{\partial}{\partial t} + \sum_{i=1}^{3} \Gamma(\partial_{x_i}) \cdot \nabla^{S^3}_{x_i} \\ &= \Gamma(\partial_t) \cdot \frac{\partial}{\partial t} - \sum_{i=1}^{3} \Gamma(\partial_t) \cdot \gamma(\partial_{x_i}) \nabla^{S^3}_{x_i} \\ &= \Gamma(\partial_t) \left(\frac{\partial}{\partial t} - D^{S^3} \right). \end{aligned} \tag{3.47}$$

Here D^{S^3} is the self-adjoint Dirac operator associated to ∇^{S^3}. Note that both D_{A_0} and D^{S^3} are elliptic. Since A_0 is flat and S^3 has positive scalar curvature, the Weitzenböck formula implies that $\ker D^{S^3} = 0$. Therefore ψ is automatically orthogonal to $\ker D^{S^3}$ and the result follows from Proposition 3.17. $\square$

The two corollaries above are crucial results which will be applied in the proof of Theorem 3.8. To complete the proof of Proposition 3.17, it remains to prove the following lemma.

Lemma 3.20. *Let $A : C^\infty(S^3, E) \to C^\infty(S^3, E)$ be an elliptic, self-adjoint, pseudo-differential operator of positive order. Let $0 < |\lambda_1| \le |\lambda_2| \le \ldots$ denote the non-zero eigenvalues of A, ordered by magnitude. There exists a constant C such that, for any $M \ge 1$,*

$$\left(\sum_{n=1}^{\infty} (\lambda_n^2 + 1) e^{M|\lambda_n|} |a_n| \right)^2 \le C \sum_{n=1}^{\infty} \frac{\sinh(4M|\lambda_n|)}{|\lambda_n|} |a_n|^2 \tag{3.48}$$

for any real number sequence $\{a_n\}$.

Proof. First, apply the Cauchy-Schwarz inequality to obtain

$$\begin{aligned} \left(\sum_{n=1}^{\infty} (\lambda_n^2 + 1) e^{M|\lambda_n|} |a_n| \right)^2 &= \left(\sum_{n=1}^{\infty} \left(\frac{(\lambda_n^2 + 1)\sqrt{|\lambda_n|} e^{M|\lambda_n|}}{\sqrt{\sinh(4M|\lambda_n|)}} \right) \left(\frac{\sqrt{\sinh(4M|\lambda_n|)}}{\sqrt{|\lambda_n|}} |a_n| \right) \right)^2 \\ &\le \left(\sum_{n=1}^{\infty} \frac{(\lambda_n^2 + 1)^2 |\lambda_n| e^{2M|\lambda_n|}}{\sinh(4M|\lambda_n|)} \right) \left(\sum_{n=1}^{\infty} \frac{\sinh(4M|\lambda_n|)}{|\lambda_n|} |a_n|^2 \right) \end{aligned}$$

It suffices to bound $\sum_{n=1}^{\infty}\frac{(\lambda_n^2+1)^2|\lambda_n|e^{2M|\lambda_n|}}{\sinh(4M|\lambda_n|)}$. Fix $0<\delta<|\lambda_1|$. The function $\frac{e^{4x}}{\sinh(4x)}$ is bounded on $[\delta,\infty)$, therefore there is a constant C_1 such that, for all $x\geq\delta$,

$$\frac{e^{2x}}{\sinh(4x)}\leq C_1e^{-2x}$$

Apply this to $M|\lambda_n|$ to produce

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{e^{2M|\lambda_n|}(\lambda_n^2+1)^2|\lambda_n|}{\sinh(4M|\lambda_n|)}&\leq\sum_{n=1}^{\infty}C_1(\lambda_n^2+1)^2|\lambda_n|e^{-2M|\lambda_n|}\\&\leq\sum_{n=1}^{\infty}C_1(\lambda_n^2+1)^2|\lambda_n|e^{-2|\lambda_n|}.\end{aligned}$$

Similarly, there exists a constant C_2 such that $x(x^2+1)^2e^{-x}\leq C_2$ for all $x\geq 0$. It follows that

$$\sum_{n=1}^{\infty}C_1(\lambda_n^2+1)^2|\lambda_n|e^{-2|\lambda_n|}\leq\sum_{n=1}^{\infty}C_1C_2e^{-|\lambda_n|}. \tag{3.49}$$

Since A is elliptic and self-adjoint, Weyl's law [31, Lemma 1.6.3] implies that there exists a constant C_3 and an exponent $\alpha>0$ such that $|\lambda_n|\geq C_3n^{\alpha}$ for large enough n. Thus to show that (3.49) is finite, it is enough to show that

$$\sum_{n=1}^{\infty}e^{-n^{\alpha}}<\infty.$$

This follows from the integral test. Let $u=x^{\alpha}$ so that

$$\begin{aligned}\int_1^{\infty}e^{-x^{\alpha}}dx&=\frac{1}{\alpha}\int_1^{\infty}u^{\frac{1-\alpha}{\alpha}}e^{-u}du\\&\leq\frac{1}{\alpha}\Gamma\left(\frac{1}{\alpha}\right)\\&<\infty.\end{aligned}$$

Therefore $C=\sum_{n=1}^{\infty}C_1C_2e^{-|\lambda_n|}$ is a suitable constant. □

3.4 Proof of the Permutation Theorem

To prove the Theorem 3.8, it is enough to construct a compact homotopy between the bounded Fredholm maps μ_X and $V^{-1}\mu_{X^\tau}V$. Let $D\subset\mathcal{A}$ be a closed disk bundle that contains all the zeroes of μ_X and $V^{-1}\mu_{X^\tau}V$, with no zeroes on its bounding sphere bundle S. Write $\mu_X = l + c_0$ and $V_{\mathcal{C}}^{-1}\mu_{X^\tau}V = l + c_1$ as in Lemma 3.2 where l is the linearized monopole map with c_0 and c_1 compact. We will construct a homotopy c_t through compact maps such that $l + c_t$ does not vanish on S for any $t \in [0, 1]$. This will show that the Bauer-Furuta invariants $[\mu_X]$ and $[V_{\mathcal{C}}^{-1}\mu_{X^\tau}V] \in \pi^{b^+}_{\mathbb{T}^n,H}(\mathcal{J}(X); \mathrm{ind}(D))$ are equal by Theorem 2.16.

To accomplish this, a series of homotopies will be defined. First, it is shown that μ_X is homotopic to a map P which is a first order linear differential operator over the short neck $N(1)$. The next homotopy rotates P to $V_{\mathcal{C}}^{-1}PV$ over the short neck. Finally, reversing these deformations give $V_{\mathcal{C}}^{-1}PV \simeq V_{\mathcal{C}}^{-1}\mu_{X^\tau}V$. The idea to use these particular homotopies comes from Bauer's proof in [9]. However, we take great care to ensure that these homotopies satisfy necessary boundedness results and that these boundedness results are compatible with neck stretching. That is, the definition of the homotopies come from [9], but the proofs that they are homotopies through compact perturbations of l is original work.

3.4.1 The standard argument

A fundamental result of ordinary Seiberg-Witten theory is that the moduli space of solutions to the Seiberg-Witten equations, after suitable perturbation, is compact. It will be useful for the proof of Theorem 3.8 to unpack the details that go into this standard argument and loosen some of the requirements of each step. Recall that $k \geq 4$ is a fixed integer. Let $A \in \mathcal{J}(X)$ be a connection, $\psi \in L^2_k(X, W^+)$ a spinor and $a \in L^2_k(X, T^*X)$ a one-form. Writing $A' = A + ia$, the Seiberg-Witten equations with gauge fixing are:

$$\begin{aligned} D_{A'}\psi &= 0 \\ F^+_{A'} &= \sigma(\psi) \\ d^*a &= 0 \\ \mathrm{pr}(a) &= 0 \end{aligned} \tag{3.50}$$

The argument to bound the C^0 and L^2_k-norm of a monopole $(\psi, a) \in \mu^{-1}(0)$ proceeds as follows.

1. (Bounding $|\psi|_{C^0}$) Let $\rho^+ : \Omega^2_+(X; i\mathbb{R}) \to \mathrm{End}_0(W)$ be the map that identifies self-dual imaginary valued 2-forms with trace-free skew hermitian endomorphisms as described in (1.8). The Weitzenböck formula (3.13) for the connec-

tion A' implies that

$$\begin{aligned}\Delta_g|\psi|^2 &\leq 2\left\langle D^*_{A'}D_{A'}\psi - \frac{s_X}{4}\psi - \frac{1}{2}F^+_{A'}\psi, \psi\right\rangle \\ &= 2\left\langle D^*_{A'}D_{A'}\psi, \psi\right\rangle - \frac{s_X}{2}|\psi|^2 - \left\langle F^+_{A'}\psi, \psi\right\rangle.\end{aligned}$$

Here $s_X : X \to \mathbb{R}$ is the scalar curvature of X. The equation $D_{A'}\psi = 0$ implies

$$\Delta_g|\psi|^2 + \frac{s_X}{2}|\psi|^2 + \left\langle F^+_{A'}\psi, \psi\right\rangle \leq 0.$$

Using the equation $F^+_{A'} = \sigma(\psi)$ and the fact that $\sigma(\psi)\psi = \frac{1}{2}|\psi|^2\psi$, we obtain $\left\langle F^+_{A'}\psi, \psi\right\rangle = \frac{1}{2}|\psi|^4$. As noted in the proof of Proposition 3.3, $\Delta_g|\psi|^2$ is non-negative at a maximum of $|\psi|^2$ and it follows that

$$|\psi|^2_{C^0}(s_X(x) + |\psi|^2_{C^0}) \leq 0,$$

for some $x \in X$ where $|\psi|$ attains a maximum. This gives a bound $|\psi|^2_{C^0} \leq S$ where $S = \sup_X\{-s_X, 0\}$.

2. (Bounding $|a|_{C^0}$) The Sobolev embedding $L^p_1(X, T^*X) \subset L^\infty(X, T^*X)$ for $p > 4$ and an elliptic estimate for $d^* + d^+$ gives a C^0-bound on a.

$$|a|_{C^0} \leq C_S C_E(\|d^+a\|_{L^p} + \|d^*a\|_{L^p} + \|\operatorname{pr}(a)\|_{L^2}) \tag{3.51}$$

Here C_S comes from the Sobolev embedding and C_E is from the elliptic estimate. Given $d^*a = 0, \operatorname{pr}(a) = 0$ and $\|d^+a\|_{L^p} \leq \|\sigma(\psi)\|_{L^p} + \|F^+_A\|_{L^p}$ it follows that

$$|a|_{C^0} \leq C_S C_E(\|F^+_A\|_{L^p} + \|\sigma(\psi)\|_{L^p}). \tag{3.52}$$

Note that $\|F^+_A\|_{L^p}$ is a constant and $\|\sigma(\psi)\|_{L^p} = \frac{1}{2}\|\psi\|_{L^{2p}}$, which can be bounded by the C^0-bound for ψ.

3. (Bootstrapping) The Sobolev norm is chosen so that, for any $0 \leq i \leq k$ and $p \geq 2$,

$$\|(\psi, a)\|^p_{L^p_i} - \|(\psi, a)\|^p_{L^p} = \|(D_A\psi, d^+a)\|^p_{L^p_{i-1}}.$$

The standard Seiberg-Witten equations state that $D_A\psi = -ia \cdot \psi$ and $d^+a = iF^+_A - i\sigma(\psi)$ so that $\|d^+a\|_{L^p_i} \leq \|\sigma(\psi)\|_{L^p_i} + \|F^+_A\|_{L^p_i}$. However, for the bootstrapping argument it is enough to have $\|d^+a\|_{L^p_i} \leq C\|\sigma(\psi)\|_{L^p_i} + \|F^+_A\|_{L^p_i}$ for some constant C for each i and p. Then it follows that

$$\|(\psi, a)\|^p_{L^p_i} - \|(\psi, a)\|^p_{L^p} \leq \|a \cdot \psi\|^p_{L^p_{i-1}} + (C\|\sigma(\psi)\|_{L^p_{i-1}} + \|F^+_A\|_{L^p_{i-1}})^p.$$

Sobolev multiplication applied to $a \cdot \psi$ implies the existence of a constant C_{SM} such that

$$\|a \cdot \psi\|_{L^p_{i-1}} \leq C_{SM} \|a\|_{L^{2p}_{i-1}} \|\psi\|_{L^{2p}_{i-1}}.$$

Therefore there is a constant C' such that

$$\begin{aligned} \|(\psi, a)\|_{L^p_i} &\leq C_{SM} \|a\|_{L^{2p}_{i-1}} \|\psi\|_{L^{2p}_{i-1}} + C\|\psi\|^2_{L^{2p}_{i-1}} + \|F^+_A\|_{L^p_{i-1}} + \|(\psi, a)\|_{L^p} \\ &\leq C'(1 + \|(\psi, a)\|^2_{L^{2p}_{i-1}} + \|(\psi, a)\|_{L^p}). \end{aligned}$$

Starting with $i = k$ and $p = 2$, inductively applying this formula gives an inequality of the form

$$\|(\psi, a)\|_{L^2_k} \leq f(\|(\psi, a)\|_{L^2}, ..., \|(\psi, a)\|_{L^{2^{k+1}}}). \tag{3.53}$$

Here f is a polynomial which depends on F^+_A, the given constant C, and some constants from Sobolev multiplication. The C^0 bounds for ψ and a from steps 1 and 2 give L^p-bounds for (ψ, a) which produces an L^2_k-bound

$$\|(\psi, a)\|_{L^2_k} \leq C_B |(\psi, a)|_{C^0}.$$

This constant C_B depends on the curvature of the connection A and the neck length L.

The argument above proves that the solution set $(\mu^A)^{-1}(0)$ is L^2_k-bounded.

3.4.2 The first homotopy

Fix a neck length $2L \geq 4$ and set $X = X(L)$. Let $\rho_R : X \to [0, 1]$ for $R \leq L$ be a smooth function that cuts away $N(R - 1)$. That is, ρ_R vanishes on $N(R - 1)$, is identically 1 on $X - N(R)$ and along the neck depends only on the interval coordinate. By shifting the interval, we can assume that for any i, the C^i-norm of ρ_R is independent of R. For $s \in [0, 1]$, define a linear homotopy $\rho_{R,s}$ ending at ρ_R of the form

$$\rho_{R,s} = (1 - s) + s\rho_R. \tag{3.54}$$

Notice that $\rho_{R,s}$ is still identically 1 outside $N(R)$. This means that for $L' \geq L$, each function $\rho_{R,s}$ naturally extends to a function on $X(L')$ with the same properties.

Define the first homotopy F_s for $s \in [0,1]$ by

$$F_s^A(\psi, a, f) = (D_{A+ia}\psi, -iF^+_{A+ia} + i\rho_{L,s}\sigma(\psi), d^*a + f, \mathrm{pr}(a)). \tag{3.55}$$

Notice that $F_0 = \mu_X$ and that the quadratic term in the second factor of F_1 vanishes on $N(L-1)$. Since only the $\sigma(\psi)$ term is being modified, the standard argument 3.4.1 can be applied to bound solutions $(F_s^A)^{-1}(0)$.

Proposition 3.21. *Let $A \in \mathcal{J}(X)$ be a fixed connection on $X = X(L)$. The preimage $(F_s^A)^{-1}(0)$ is L^2_k-bounded. This bound is uniform in $s \in [0,1]$.*

Proof. A monopole $(\psi, a) \in (F_s^A)^{-1}(0)$ satisfies $D_{A+ia}\psi = 0$ and $F^+_{A+ia} = \rho_{L,s}\sigma(\psi)$. Following the standard argument, the Weitzenböck formula applied to the connection $A + ia$ gives a pointwise bound

$$\Delta_g|\psi|^2 + \frac{s}{2}|\psi|^2 + \left\langle F^+_{A+ia}\psi, \psi\right\rangle \le 2\left\langle D^*_{A+ia}D_{A+ia}\psi, \psi\right\rangle$$
$$\Delta_g|\psi|^2 + \frac{s}{2}|\psi|^2 + \frac{1}{2}\rho_{L,s}|\psi|^4 \le 0.$$

Recall that $S = \sup_X\{0, -s_X\}$ where s_X is the scalar curvature of X. Since s_X is positive along the neck, $\Delta_g|\psi|^2 \le 0$ on $N(L)$ and $|\psi|^2$ achieves a maximum on $M = \overline{X - N(L-1)}$. Since $\rho_{L,s} \equiv 1$ outside $N(L)$, it follows that $|\psi|^2_{C^0} \le S$.

Notice that $d^+a = -i\rho_{L,s}\sigma(\psi) + iF^+_A$ and

$$|d^+a| \le |\sigma(\psi)| + |F^+_A|.$$

Thus the elliptic estimate (3.51) still gives a C^0-bound for a as in (3.52)

$$|a|_{C^0} \le C_S C_E(\|\sigma(\psi)\|_{L^p} + \|F^+_A\|_{L^p}).$$

This shows that $|a|_{C^0}$ is bounded by a constant since $|\psi|_{C^0}$ is. For bootstrapping, $D_A\psi = -ia\cdot\psi$ and $\|d^+a\|_{L^p_i} = \|-\rho_{L,s}\sigma(\psi) + F^+_A\|_{L^p_i}$. The C^k-norm of $\rho_{L,s}$ gives a constant C such that, for all $0 \le i \le k$ and $2 \le p \le 2^{k+1}$,

$$\|d^+a\|_{L^p_i} \le C\|\sigma(\psi)\|_{L^p_i} + \|F^+_A\|_{L^p_i}.$$

Hence elliptic bootstrapping as in step 3 of the standard argument 3.4.1 gives an L^2_k-bound for (ψ, a).

$$\|(\psi, a)\|_{L^2_k} \le C_B|(\psi, a)|_{C^0}.$$

The norm $|(\psi, a)|_{C^0}$ is bounded by a constant, hence so is $\|(\psi, a)\|_{L^2_k}$. This bound is independent of s, but depends on the choice of connection $A \in \mathcal{J}(X)$ and neck length L. $\square$

Corollary 3.22. *The homotopy F_s is a homotopy through compact perturbations of the linearised monopole map l.*

Proof. For each connection $A \in \mathcal{J}(X)$, Proposition 3.21 provides a radius R^A such that

$$\|(\psi, a)\|_{L^2_k} \leq R^A.$$

This bound holds so long as $(\psi, a) \in (F^A_s)^{-1}(0)$ for any $s \in [0,1]$. Let R be the supremum of R^A over $\mathcal{J}(X)$, which exists since $\mathcal{J}(X)$ is compact. Let $D \subset \mathcal{A}$ be a subbundle with fibres L^2_k-disks of radius R. This proves a stronger result that in fact each pre-image $F^{-1}_s(0)$ is contained in a bounded disk bundle. □

3.4.3 The second homotopy

The second homotopy G_s for $s \in [0,3]$ is constructed in three stages. For $s \in [0,1]$ define

$$G^A_s(\psi, a, f) = (D_{A+i\rho_{(M,s)}a}\psi, -iF^+_{A+ia} + i\rho_L\sigma(\psi), d^*a + f, \mathrm{pr}(a)) \tag{3.56}$$

This homotopy eliminates the other quadratic term $ia \cdot \psi$ from $N(M-1)$. Here $M \geq 3$ is a constant, independent of L, to be defined later in Proposition 3.27. We can assume without loss of generality that $L \geq 2M$.

To define the second stage of G_s for $s \in [1,2]$, let $P = G_1$. This stage will transform P to $P^\tau = V^{-1}PV$. Recall from (3.22) that over the neck $N(L)$, a section $\psi : N(L) \to W^+$ can be identified with a vector of sections $\vec{\psi} : S^3 \times [-L, L] \to \oplus^n_{i=1} F$ where $F = W^+ \to S^3 \times [-L, L]$. The matrix-valued function $V : S^3 \times [-L, L] \to SO(n)$ is defined by $V(x,t) = \gamma(\varphi(t))$ where γ is a path from id to τ in $SO(n)$ and $\varphi : S^3 \times [-L, L] \to [0,1]$ is a parametrisation that increase from 0 to 1 along $N(1)$. The action of V on ψ is given by $V \cdot \psi = V\vec{\psi}$, identified as a section of $W^+|_{N(L)}$. The same process defines an action of V on forms over $N(L)$.

Restricting to $N(M-1)$, P is given by the formula

$$P^A(\psi, a, f) = (D_A\psi, d^+a, d^*a + f, \mathrm{pr}(a)).$$

That is, inside $N(M-1)$, the quadratic terms of μ_X have been eliminated and $\mu_X = l$. Note that $F^+_A = 0$ inside $N(M-1)$ since A is flat on the neck. Define another map $Q : \mathcal{A} \to \mathcal{C}$ by

$$Q^A(\psi, a) = V^{-1}\partial_t V(dt \cdot \vec{\psi}, (dt \wedge \vec{a})^+, *(*\vec{a} \wedge dt), 0).$$

Here $V^{-1}\partial_t V$ is a matrix functions which acts on each vector $dt \cdot \vec{\psi}$, $(dt \wedge \vec{a})^+$ and $*(*\vec{a} \wedge dt)$. Notice that Q vanishes outside of $N(1)$ since $\partial_t V = 0$ away from the short neck. Applying the Leibniz rule, it follows that

$$V^{-1}P^A V(\psi, a, f) = P^A(\psi, a, f) + Q^A(\psi, a).$$

For $s \in [1, 2]$, define a homotopy of matrices V_s by

$$V_s(x, t) = \gamma((s-1) \cdot \varphi(t)) : S^3 \times [-L, L] \to SO(n).$$

Notice that $V_1 = \mathrm{id}$ and $V_2 = V$. When $s \in (1, 2)$, the vector $V_s(\psi, a)$ is only defined on $N(L)$. As before, we can define

$$Q^A_s(\psi, a) = V_s^{-1}\partial_t V_s(dt \cdot \vec{\psi}, (dt \wedge \vec{a})^+, *(\vec{a} \wedge dt), 0).$$

For $s \in [1, 2]$, define G_s by

$$G_s = P + Q_s. \tag{3.57}$$

Each Q_s still has the property that $Q_s = 0$ outside of $N(1)$, hence this formula is well defined globally. Restricted to the neck $N(L-1)$, equation (3.57) is equivalent to $G_s = V_s^{-1}PV_s$. For the final stage $s \in [2, 3]$, let $G_s = V^{-1}G_{3-s}V$. Now G is a homotopy from $G_0 = F_1$ to $G_3 = V^{-1}F_1V$.

Since G alters the $D_{A+ia}\psi = 0$ equation, step 1 and 3 of the standard argument do not directly apply to bound $G_s^{-1}(0)$. However to show that G is a compact homotopy, it is only necessary to find an L^2_k-disk bundle containing $G_0^{-1}(0)$ and $G_3^{-1}(0)$ with a bounding sphere bundle that does not intersect $G_s^{-1}(0)$. The following results help accomplish this by proving a similar result for the C^0-norm of a zero of G_s.

Lemma 3.23. *Let $(\psi, a) \in G_s^{-1}(0)$ for some $s \in [0, 3]$ on $X = X(L)$. If $\sup_X |\psi|$ is achieved at some $x \in M$, then $|\psi|^2_{C^0(X)} \le S$ where $S = \sup_X\{0, -s_X\}$.*

Proof. Restricted to M, the pair (ψ, a) satisfies $D_{A+ia}\psi = 0$ and $F^+_{A+ia} = \sigma(\psi)$. As in Proposition 3.21, the Weitzenböck formula on M gives

$$\Delta_g|\psi|^2 + \frac{s_X}{2}|\psi|^2 + \frac{1}{2}|\psi|^4 \le 0$$

Since X has no boundary, $\Delta_g|\psi|^2 \ge 0$ at x. Since $x \in M$, it follows that

$$|\psi(x)|^2(s_X(x) + |\psi(x)|^2) \le 0.$$

Therefore $|\psi|^2 \le S$ since $|\psi(x)| = |\psi|_{C^0(X)}$. □

Lemma 3.24. *Let (ψ, a) be a spinor-from pair along the n-component neck $N(L)$. For any $0 \leq R \leq L$, we have*

$$\begin{aligned} \sup_{N(R)} |\psi| \leq n \sup_{N(R)} |V_s\psi| \leq n^2 \sup_{N(R)} |\psi| \\ \sup_{N(R)} |a| \leq n \sup_{N(R)} |V_s a| \leq n^2 \sup_{N(R)} |a|. \end{aligned}$$

Proof. We prove only the spinor case. Let $\vec{\psi}$ be the vectorised version of ψ over the neck as in Section 3.2.2. That is, $\vec{\psi} : S^3 \times [-L, L] \to \oplus_{i=1}^n W^+$ with the ith component $\vec{\psi}_i$ corresponding to the restriction of ψ to the ith connected component of $N(L)$. The restriction of $V_s\psi$ to the ith connected component of $N(L)$ is given by the ith component of $V_s\vec{\psi}$. Inside $N(R)$, we have

$$\begin{aligned} |(V_s\vec{\psi})_i| &= \left| \sum_j (V_s)_{ij} \vec{\psi}_j \right| \\ &\leq \sum_j |(V_s)_{ij}||\vec{\psi}_j| \\ &\leq \left(\sum_j |(V_s)_{ij}| \right) \sup_{N(R)} |\psi| \\ &= n \sup_{N(R)} |\psi|. \end{aligned}$$

The last line follows since V_s is valued in $SO(n)$, hence the absolute value of each of its entries is less than 1. Therefore $\sup_{N(R)} |V_s\psi|^2 \leq n \sup_{N(R)} |\psi|^2$. The same calculation shows that $\sup_{N(R)} |\psi|^2 = \sup_{N(R)} |V_s^{-1} V_s\psi|^2 \leq \sup_{N(R)} n|V_s\psi|^2$. $\square$

Remark 3.25: For any $R \leq L$, the same calculation can be used to show that

$$\begin{aligned} \sup_{\partial N(R)} |\psi| \leq n \sup_{\partial N(R)} |V_s\psi| \leq n^2 \sup_{\partial N(R)} |\psi| \\ \sup_{\partial N(R)} |a| \leq n \sup_{\partial N(R)} |V_s a| \leq n^2 \sup_{\partial N(R)} |a|. \end{aligned}$$

Lemma 3.26. *There exists positive constants L_0, C_E, δ and C_δ such that the following holds. For any $s \in [0, 3]$, let $(\psi, a) \in (G_s^A)^{-1}(0)$ for some $A \in \mathcal{J}(X)$ on $X = X(L)$. If $L \geq L_0$, then*

$$\begin{aligned} |a|_{C^0(X)} \leq C_E |(d^* + d^+)a|_{C^0(M)} \\ \sup_{N(M)} |a \wedge dt| \leq C_\delta e^{-\delta(L-2M)} \sup_{N(L-1)} |a|. \end{aligned} \tag{3.58}$$

Proof. For $s \in [0,1]$, we have

$$\begin{aligned} d^+a &= iF_A^+ - i\rho_L\sigma(\psi) \\ d^*a &= 0 \\ \mathrm{pr}(a) &= 0. \end{aligned}$$

Along $N(L-1)$, $-iF_A^+ + d^+a = 0$ and therefore $d^+a = 0$ since A is flat on the neck. Thus $(d^* + d^+)a$ vanishes on $N(L-1)$. Recall that $k \geq 4$, so Sobolev embedding implies that the L_k^2-form a is L_1^p for any p. Hence Proposition 3.13 gives constants C'_E and L_1 such that, if $L \geq L_1$ then

$$|a|_{C^0(X)} \leq C'_E|(d^* + d^+)a|_{C^0(M)}.$$

Further, Corollary 3.18 applies to $a \wedge dt$ yielding, for some $\delta > 0$ and C'_δ independent of L,

$$\sup_{N(M)} |a \wedge dt| \leq C'_\delta e^{-\delta(L-2M)}|a|_{C^0(N(L-1))}.$$

If $s \in [1,2]$, the condition $\mathrm{pr}(a) = 0$ still holds. Restricting to $N(L-1)$ we have $V_s^{-1}PV_s(\psi, a) = 0$ and therefore $V_s(\psi, a)$ is a solution to P. Note that $V_s(\psi, a)$ is only defined on $N(L-1)$ and that $(d^* + d^+)V_s a = 0$ on $N(L-1)$. This means that $\sup_{N(L-1)} |V_s a| = \sup_{\partial N(L-1)} |V_s a|$ by the maximum principle. Lemma 3.24 implies that

$$\begin{aligned} \sup_{N(L-1)} |a| &\leq n \sup_{N(L-1)} |V_s a| \\ &= n \sup_{\partial N(L-1)} |V_s a| \\ &\leq n^2 \sup_{\partial N(L-1)} |a|. \end{aligned} \tag{3.59}$$

Thus $|a|_{C^0(X)} \leq n^2|a|_{C^0(M)}$. Restricting to $X - N(1)$ instead, we have $P(\psi, a) = 0$. This means that $(d^* + d)a = 0$ along $N(2, L)$. Now (3.59) with Remark 3.14 implies the existence of constants L_2 and C''_E such that, if $L \geq L_2$,

$$\begin{aligned} |a|_{C^0(X)} &\leq n^2|a|_{C^0(M)} \\ &\leq n^2 C''_E|(d^* + d^+)a|_{C^0(M)}. \end{aligned} \tag{3.60}$$

To obtain the exponential bound on $a \wedge dt$, note that $V_s(a \wedge dt) = (V_s a) \wedge dt$. We have $(d^* + d^+)V_s a = 0$ on $N(L-1)$ and Corollary 3.18 applies to $V_s a \wedge dt$, yielding

$$\sup_{N(M)} |V_s a \wedge dt| \leq C'_\delta e^{-\delta(L-2M)} \sup_{N(L-1)} |V_s a|.$$

By Lemma 3.24, it follows that

$$\begin{aligned} \sup_{N(M)} |a \wedge dt| &\leq n \sup_{N(M)} |V_s a \wedge dt| \\ &\leq nC'_\delta e^{-\delta(L-2M)} \sup_{N(L-1)} |V_s a| \\ &\leq n^2 C'_\delta e^{-\delta(L-2M)} \sup_{N(L-1)} |a|. \end{aligned} \tag{3.61}$$

For the third stage $s \in [2,3]$, we have $V^{-1}G_{3-s}V(\psi, a) = 0$. Thus $V(\psi, a)$, which is defined globally, is a solution of G_{3-s}. The argument for the second stage can be repeated to establish (3.60) and (3.61). Setting $C_E = \max\{C'_E, n^2 C''_E\}$, $L_0 = \max\{L_1, L_2\}$ and $C_\delta = n^2 C'_\delta$ ensures that (3.58) is satisfied for any $s \in [0,3]$. $\square$

Proposition 3.27. *Let $A \in \mathcal{J}(X)$ be a connection on $X = X(L)$. There exists positive constants U_0, L_0, C, δ and M such that the following holds. If $L \geq L_0$, then for any $s \in [0,3]$, there are no solutions $(\psi, a) \in (G_s^A)^{-1}(0)$ with C^0-norm in the interval $[U_0, U(L)]$, where*

$$U(L) = Ce^{\delta(L-2M)}. \tag{3.62}$$

Proof. Let $(\psi, a) \in (G_s^A)^{-1}(0)$ for some $s \in [0,3]$. Notice that for any stage of G_s, on $X - N(M)$ the pair (ψ, a) satisfies

$$\begin{aligned} D_{A+ia}\psi &= 0 \\ d^+ a &= iF_A^+ - i\rho_L \sigma(\psi) \\ d^* a &= 0 \\ \mathrm{pr}(a) &= 0. \end{aligned}$$

Lemma 3.26 gives constants C_E and L_0 such that, for $L \geq L_0$,

$$|a|_{C^0(X)} \leq C_E |(d^* + d^+)a|_{C^0(M)}.$$

Applying the Seiberg-Witten style equations above gives

$$\begin{aligned} |a|_{C^0(X)} &\leq C_E(|F_A^+|_{C^0} + |\sigma(\psi)|_{C^0(M)}) \\ &= C_E(|F_A^+|_{C^0} + \frac{1}{2}|\psi|^2_{C^0(M)}). \end{aligned}$$

Recall that $S = \sup_X\{-s_X, 0\}$ where s_X is the scalar curvature of X. Let

$$U'_0 = 1 + \sqrt{S} + C_E(|F_A^+|_{C^0} + \frac{1}{2}S).$$

Note that $|F_A^+|_{C^0}$ and S do not depend on L. To show that $|(\psi,a)|_{C^0(X)} < U_0'$ it is enough to show that $|\psi|^2_{C^0(X)} \leq S$. By Lemma 3.23, it suffices to show that $\sup_X |\psi| = \sup_M |\psi|$.

For now assume that $s \in [0,1]$ so that ψ satisfies $D_{A+i\rho_{(M,s)}a}\psi = 0$ and $d^+a = iF_A^+ - i\rho_L\sigma(\psi)$. Inside $N(L-1)$, the Weitzenböck formula applied to the connection $A' = A + i\rho_{(M,s)}a$ gives

$$\Delta_g|\psi|^2 \leq \left\langle D_{A'}^* D_{A'}\psi - \frac{s_N}{2}\psi - F_{A'}^+\psi, \psi \right\rangle.$$

Here s_N is the scalar curvature of the neck, which is a positive constant. Since A is flat on the neck, $F_{A'}^+ = id^+(\rho_{(M,s)}a)$. Since $d^+a = 0$ on $N(L-1)$, it follows that

$$\begin{aligned}\Delta_g|\psi|^2 &\leq -\frac{s_N}{2}|\psi|^2 + \|(d\rho_{(2,s)} \wedge a)^+\||\psi|^2 \\ &= |\psi|^2\left(\sqrt{2}|(d\rho_{(M,s)} \wedge a)^+| - \frac{s_N}{2}\right). \end{aligned} \tag{3.63}$$

Here $\|(d\rho_{(M,s)} \wedge a)^+\|$ is the operator norm of $d^+(\rho_{M,s}a) = (d\rho_{(M,s)} \wedge a)^+$ identified as an element of $\mathrm{End}_0(W^+)$ and $|(d\rho_{(M,s)} \wedge a)^+|$ is the norm of $(d\rho_{(M,s)} \wedge a)^+$ as a 2-form. The relation $\|(d\rho_{(M,s)} \wedge a)^+\| = \sqrt{2}|(d\rho_{(M,s)} \wedge a)^+|$ is shown in [55, Lemma 7.4].

Since $d\rho_{(M,s)}$ is supported in $N(M)$, (3.63) guarantees that $\Delta_g|\psi|^2 < 0$ on $N(L-1) - N(M)$. It remains to show that $\Delta_g|\psi|^2 < 0$ on $N(M)$. Since $\rho_{M,s}$ is constant on spheres, $d\rho_{(M,s)} = \partial_t\rho_{(M,s)}dt$. Define

$$R = \sqrt{2}\sup_{s\in[0,1]}|\partial_t\rho_{(M,s)}|_{C^0(X)}.$$

If follows that

$$\Delta_g|\psi|^2 \leq |\psi|\left(R|a \wedge dt| - \frac{s_N}{2}\right). \tag{3.64}$$

Lemma 3.26 provides constants δ, C_δ such that if $L \geq L_0$, then

$$\sup_{N(M)}|a \wedge dt| \leq C_\delta e^{-\delta(L-2M)}\sup_{N(L)}|a|.$$

Define the constant $C > 0$ by

$$C = \frac{s_N}{4RC_\delta}. \tag{3.65}$$

This is positive since s_N, R and C_δ are. Define $U'(L)$ by

$$U'(L) = Ce^{\delta(L-2M)}.$$

Note that the definition of C is independent of L and the chosen connection $A \in \mathcal{J}(X)$. Further, it can be assumed that L is large enough to ensure that $U'(L) > U_0'$. When $|(\psi, a)|_{C^0} \leq U'(L)$ and $L \geq L_0$, inside $N(M)$ we have

$$\begin{aligned} R|a \wedge dt| &\leq RC_\delta e^{-\delta(L-2M)} \sup_{N(L)} |a| \\ &\leq RC_\delta e^{-\delta(L-2M)} U'(L) \\ &\leq \frac{s_N}{4}. \end{aligned} \tag{3.66}$$

From (3.64) it follows that $\Delta_g|\psi|^2 < 0$ on all of $N(L-1)$. Therefore $\sup_{N(L-1)} |\psi| = \sup_{\partial N(L-1)} |\psi|$ because $\Delta_g|\psi|^2$ is non-negative at an interior local maximum. Consequently $\sup_X |\psi| = \sup_M |\psi|$, thus $|\psi|_{C^0} \leq S$ and $|(\psi, a)|_{C^0} < U_0$. It remains to shows that $|\psi|_{C^0} \leq S$ for $s \in [1, 3]$.

Suppose $(\psi, a) \in G_s^{-1}(0)$ for some $s \in [1, 2]$ with $|(\psi, a)|_{C^0} \leq U'(L)$. This means that on $X - N(1)$ we have $P(\psi, a) = 0$. However on $N(L)$, $G_s = V_s^{-1} P V_s$ and $V_s(\psi, a)$ is a solution to P. Again we show that $|\psi|^2_{C^0} \leq S$ by showing that $\sup_X |\psi| = \sup_M |\psi|$.

Restricting to $N(1, L-1) = N(L-1) - N(1)$, the Weitzenböck formula as before for the connection $A' = A + i\rho_M a$ gives

$$\Delta_g|\psi|^2 \leq |\psi| \left(R|a \wedge dt| - \frac{s_N}{2} \right).$$

For $L \geq L_0$, Lemma 3.26 still applies to (ψ, a) yielding

$$\sup_{N(M)} |a \wedge t| \leq C_\delta e^{-\delta(L-2M)} \sup_{N(L)} |a|. \tag{3.67}$$

Thus the calculation in (3.66) guarantees $\Delta_g|\psi|^2 < 0$ on $N(1, L-1)$. This implies that

$$\sup_X |\psi| = \max\{\sup_{N(1)} |\psi|, \sup_M |\psi|\}. \tag{3.68}$$

Notice that $D_A V_s \psi = 0$ on $N(M-1)$. Thus Corollary 3.19 implies the existence of constants $\delta', C_\delta' > 0$ such that

$$\sup_{N(1)} |V_s \psi| \leq C_\delta' e^{-\delta'(M-3)} \sup_{N(M-1)} |V_s \psi|. \tag{3.69}$$

Fix a large enough M to ensure that

$$C_\delta' e^{-\delta'(M-3)} \leq \frac{1}{n^2}. \tag{3.70}$$

Note that this definition of M is independent of L, and we can assume that $L_0 \geq 2M$. Since $V_s\psi$ is a solution to P along $N(L-1)$, we have that

$$\sup_{N(L-1)} |V_s\psi| = \sup_{\partial N(L-1)} |V_s\psi|.$$

This follows from the the argument presented in the $s \in [0,1]$ case. It follows from Lemma 3.24, (3.69) and (3.70) that

$$\begin{aligned}
\sup_{N(1)} |\psi| &\leq n \sup_{N(1)} |V_s\psi| \\
&\leq nC'_\delta e^{-\delta'(M-3)} \sup_{N(M-1)} |V_s\psi| \\
&\leq \frac{1}{n} \sup_{N(M-1)} |V_s\psi| \\
&\leq \frac{1}{n} \sup_{\partial N(L-1)} |V_s\psi| \\
&\leq \sup_{\partial N(L-1)} |\psi|.
\end{aligned}$$

That is, $\sup_{N(1)} |\psi| \leq \sup_M |\psi|$ and therefore $\sup_X |\psi| = \sup_M |\psi|$ by (3.68). Thus Lemma 3.23 guarantees $|\psi|^2 \leq S$ and $|(\psi, a)| < U'_0$.

For the third stage $s \in [2,3]$, we have $G_s(\psi, a) = V^{-1}G_{3-s}V(\psi, a) = 0$. Note that $V(\psi, a)$ is defined globally and thus $V(\psi, a)$ is a solution of G_{3-s}. Further, by the same calculation as Lemma 3.24, $|V(\psi, a)|_{C^0} \leq n|(\psi, a)|_{C^0} \leq n^2|V(\psi, a)|_{C^0}$. This implies that if $|(\psi, a)|_{C^0} \leq \frac{1}{n}U'(L)$, then $|V(\psi, a)|_{C^0} \leq U'(L)$ and $|(\psi, a)| \leq nU'_0$. The result follows by taking $U(L) = \frac{1}{n}U'(L)$ and $U_0 = nU'_0$, ensuring that L_0 is large enough so that $U(L) > U_0$ for $L \geq L_0$. $\square$

The above lemma shows that given a neck length L and a connection A, there are no elements of $(G_s^A)^{-1}(0)$ with C^0-norm in the interval $[U_0, U(L)]$. This will be used to find an L^2_k-disk in $\mathcal{A}^A$ with boundary that does not intersect $(G_s^A)^{-1}(0)$ for any $s \in [0,3]$. The L^2_k-norm of a pair $(\psi, a) \in (G_s^A)^{-1}(0)$ can be bounded by its C^0-norm using elliptic bootstrapping, although these bounds increase polynomially with L. The exponential increase of $U(L)$ counteracts this.

First we show that the endpoints $(G_0^A)^{-1}(0)$ and $(G_3^A)^{-1}(0)$ are contained in an L^2_k-disk with radius that increases polynomially with L.

Lemma 3.28. *Let $A \in \mathcal{J}(X)$ be a connection on X. There exists positive constants*

C, d and L_0 such that, for any $L \geq L_0$,

$$\|(\psi, a)\|_{L^2_k} \leq CL^d$$

for any solution $(\psi, a) \in G_0^{-1}(0) \cup G_3^{-1}(0)$ on $X(L)$.

Proof. For $(\psi, a) \in G_0^{-1}(0)$ we have

$$\begin{aligned} D_{A+ia}\psi &= 0 \\ d^+a &= iF_A^+ - i\rho_L\sigma(\psi) \\ d^*a &= 0. \end{aligned}$$

As in Proposition 3.21, the Weitzenböck formula gives

$$|\psi|^2_{C^0} \leq S.$$

Since $(d + d^*)a = 0$ on $N(L-1)$, Proposition 3.13 provides constants L_0 and C' such that $L \geq L_0$ implies

$$\begin{aligned} |a|_{C^0} &\leq C'|(d^* + d^+)a|_{C^0} \\ &\leq C'(|F_A^+|_{C^0} + \frac{1}{2}S). \end{aligned}$$

Let $U = 1 + \sqrt{S} + C'(|F_A^+|_{C^0} + \frac{1}{2}S)$ so that $|(\psi, a)|_{C^0} < U$. Notice that $|\rho_L\sigma(\psi)| \leq |\sigma(\psi)|$ and that $d\rho_L$ is supported on $N(L) - N(L-1)$. Therefore the C^k-norm of ρ can be used to obtain a constant C_ρ such that $\|\rho_L\sigma(\psi)\|_{L^p_i} \leq C_\rho\|\sigma(\psi)\|_{L^p_i}$ with C_ρ independent of L. Now applying elliptic bootstrapping as in Remark 3.16, there are constants C_B and d such that

$$\begin{aligned} \|(\psi, a)\|_{L^2_k} &\leq C_B L^d (1 + U)^d \\ &\leq C_1 L^d. \end{aligned}$$

The constant C_1 is independent of L since C_B, d and U are.

The argument for $(\psi, a) \in G_3^{-1}(0)$ is similar. Recall $G_3 = V^{-1}G_0V$ so that $V(\psi, a)$ is a solution to G_0 and therefore

$$\|V(\psi, a)\|_{L^2_k} \leq C_1 L^d.$$

Applying V^{-1} gives

$$\begin{aligned} \|(\psi, a)\|_{L^2_k} &= \|V^{-1}V(\psi, a)\|_{L^2_k} \\ &\leq C_{V^{-1}}\|V(\psi, a)\|_{L^2_k} \\ &\leq C_1 C_{V^{-1}}(1 + L)^d. \end{aligned}$$

Here $C_{V^{-1}}$ is a constant from Lemma 3.7 that is independent of L. The result follows with $C = \max\{C_1, C_{V^{-1}}C_1\}$. $\square$

It remains to find disk bundle D such that its bounding sphere bundle S does not intersection $G_s^{-1}(0)$ for any $s \in [0,3]$. This is done by combining Proposition 3.27 with the following elliptic bootstrapping result.

Lemma 3.29. *Let $A \in \mathcal{J}(X)$ be a connection on $X = X(L)$. There are constants C_B and d such that, for any $L \geq 2$, if $(\psi, a) \in (G_s^A)^{-1}(0)$ for some $s \in [0,3]$ then*

$$\|(\psi, a)\|_{L^2_k} \leq C_B L^d (1 + |(\psi, a)|_{C^0})^d.$$

Proof. First assume that $s \in [0,1]$ so that $(\psi, a) \in (G_s^A)^{-1}(0)$ implies

$$\begin{aligned} D_A\psi &= -i\rho_{M,s}a \\ d^+a &= iF_A^+ - i\rho_L\sigma(\psi). \end{aligned}$$

For any $0 \leq i \leq k$ and $2 \leq p \leq 2^{k+1}$, there is a constant C_1 such that

$$\|\rho_{M,s}a\|_{L^p_i} \leq C_1\|a\|_{L^p_i}. \tag{3.71}$$

This constant comes from the C^k-norm of $\rho_{M,s}$. Since a and $\rho_{M,s}a$ only differ on $N(M) - N(M-1)$, C_1 is independent of L. Taking the supremum over $s \in [0,1]$, we can assume that (3.71) holds for any s. Similarly,

$$\begin{aligned} \|d^+a\|_{L^p_i} &\leq \|F_A^+\|_{L^p_i} + \|\rho_L\sigma(\psi)\|_{L^p_i} \\ &\leq C_2(\|F_A^+\|_{L^p_i} + \|\sigma(\psi)\|_{L^p_i}). \end{aligned}$$

Once again C_2 can be chosen independent of L. Now apply bootstrapping as in Remark 3.16 to obtain

$$\|(\psi, a)\|_{L^2_k} \leq C_B' L^d (1 + |(\psi, a)|_{C^0})^d$$

for some constants $C_B' > 0$ and $d \geq 1$, both independent of L. This proves the result for $s \in [0,1]$.

If $s \in [1,2]$, we have $P(\psi, a) = 0$ on $X - N(1)$ and $PV_s(\psi, a) = 0$ on $N(1)$. In particular, $D_A V_s\psi = 0$ and $d^+V_s a = 0$ on $N(1)$. This means that

$$\begin{aligned} \|V_s(\psi, a)\|^2_{L^2_k(N(1))} &= \|V_s(\psi, a)\|^2_{L^2(N(1))} \\ &= \int_{N(1)} |V_s(\psi, a)|^2 \mathrm{dvol} \\ &\leq \sup_{N(1)} |V_s(\psi, a)|^2 \cdot \mathrm{vol}(N(1)) \end{aligned}$$

From Lemma 3.24 is follows that

$$\begin{aligned}\|V_s(\psi,a)\|^2_{L^2_k(N(1))} &\leq n^2\mathrm{vol}(N(1))\cdot\sup_{N(1)}|(\psi,a)|^2\\ &\leq C_4|(\psi,a)|^2_{C^0}.\end{aligned}\tag{3.72}$$

By the same argument used in Lemma 3.7, there is a constant C_5 such that

$$\|(\psi,a)\|^2_{L^2_k(N(1))}\leq C_5\|V_s(\psi,a)\|^2_{L^2_k(N(1))}.$$

The elliptic bootstrapping argument of Lemma 3.15 can be applied to (ψ,a) over $X-N(1)$ to obtain

$$\begin{aligned}\|(\psi,a)\|^2_{L^2_k} &= \|(\psi,a)\|^2_{L^2_k(X-N(1))}+\|(\psi,a)\|^2_{L^2_k(N(1))}\\ &\leq C_6L^d(1+|(\psi,a)|_{C^0})^d+C_4C_5|(\psi,a)|^2_{C^0}\\ &\leq C''_BL^d(1+|(\psi,a)|_{C^0})^d.\end{aligned}$$

For $s\in[2,3]$, we have $G_s(\psi,a)=V^{-1}G_{3-s}V(\psi,a)=0$. Thus $G_{3-s}V(\psi,a)=0$ globally and Lemma 3.15 applies to $V(\psi,a)$. Calculating with Lemma 3.7 and Lemma 3.24 gives

$$\begin{aligned}\|(\psi,a)\|_{L^2_k} &\leq C_{V^{-1}}\|V(\psi,a)\|_{L^2_k}\\ &\leq C_{V^{-1}}C_7L^d(1+|V(\psi,a)|_{C^0})^d\\ &\leq C'''_BL^d(1+|(\psi,a)|_{C^0})^d.\end{aligned}$$

Hence the result follows with $C_B=\max\{C'_B,C''_B,C'''_B\}$. □

Proposition 3.30. *Fix a neck length L for $X=X(L)$. There is a threshold neck length L_0 such that if $L\geq L_0$, then there is an L^2_k-disk bundle $D\subset\mathcal{A}$ with bounding sphere bundle $S\subset D$ such that*

1. $G_0^{-1}(0)\cup G_3^{-1}(0)\subset D$
2. $G_s^{-1}(0)\cap S=\emptyset$ *for all* $s\in[0,3]$.

That is, G is a homotopy through compact perturbations of the linearised monopole map l.

Proof. For each connection $A\in\mathcal{J}(X)$, choose constants C_1^A and d from Lemma 3.28 so that, for solutions $(\psi,a)\in(G_0^A)^{-1}(0)\cup(G_3^A)^{-1}(0)$

$$\|(\psi,a)\|_{L^2_k}\leq C_1^AL^d$$

for large enough L. Define C_1 to be the supremum of C_1^A over $\mathcal{J}(X)$. The constant C_1 will be used to establish (1).

To ensure the chosen disk bundle satisfies (2), let U_0^A, C, M and δ be constants from Proposition 3.27 with

$$U(L) = Ce^{-\delta(L-2M)}.$$

Note that C, M and δ can be chosen independently of A and that M is now fixed. For large enough L, we have

$$|(\psi, a)|_{C^0} \leq U(L) \Rightarrow |(\psi, a)|_{C^0} < U_0^A$$

for solutions $(\psi, a) \in (G_s^A)^{-1}(0)$. Let U_0 be the supremum of U_0^A over $\mathcal{J}(X)$. It can be assumed that L_0 is large enough so that $U(L) > U_0$ when $L \geq L_0$. It follows that

$$|(\psi, a)|_{C^0} \leq U(L) \Rightarrow |(\psi, a)|_{C^0} < U_0$$

for solutions $(\psi, a) \in G_s^{-1}(0)$ for any $s \in [0, 3]$.

From Lemma 3.9, let C_S be a Sobolev constant independent of L such that, for (ψ, a) in any fibre of $\mathcal{A}$,

$$|(\psi, a)|_{C^0} \leq C_S \|(\psi, a)\|_{L_k^2}$$

Set $R(L) = \frac{U(L)}{C_S}$. Let $D \subset \mathcal{A}$ be the L_k^2-disk sub-bundle with fibres of radius $R(L)$ with bounding sphere bundle S. Since $R(L)$ increases exponentially, ensure L_0 is large enough so that, for all $L \geq L_0$,

$$C_1 L^d \leq R(L)$$

This guarantees that $G_0^{-1}(0) \cup G_3^{-1}(0) \subset D$. To show that there are no solutions on S, let $A \in \mathcal{J}(X)$ be a connection and suppose that $(\psi, a) \in (G_s^A)^{-1}(0) \cap D$. Then $\|(\psi, a)\|_{L_k^2} \leq R(L)$ and by the Sobolev embedding theorem, $|(\psi, a)|_{C^0} \leq U(L)$. Proposition 3.27 implies

$$|(\psi, a)|_{C^0} < U_0.$$

Applying elliptic bootstrapping as in Lemma 3.29 gives

$$\|(\psi, a)\|_{L_k^2} \leq C_B^A L^d (1 + U_0)^d$$

for some constants C_B^A and d both independent of L. Let $C_B = \sup_{\mathcal{J}(X)} C_B^A$ and ensure L_0 is large enough so that

$$C_B L^d (1 + U_0)^d \leq \frac{1}{2} \frac{U(L)}{C_S}$$

for any $L \geq L_0$. Again this is possible since $U(L)$ increases exponentially with L. Now

$$\|(\psi, a)\|_{L^2_k} \leq \frac{1}{2}R(L)$$

Thus there are no solutions to $G_s^{-1}(0)$ in S when $L \geq L_0$. □

3.4.4 The third homotopy

The third homotopy H_s for $s \in [0,1]$ is given by

$$H_s = V^{-1}F_{1-s}V.$$

This homotopy starts at $H_0 = G_3 = V^{-1}F_1V$ and ends at $H_1 = V^{-1}\mu_{X^\tau}V$. In the same manor as the first homotopy, the solutions $(H_s^A)^{-1}(0)$ are uniformly L^2_k-bounded.

Proposition 3.31. *The homotopy H_s for $s \in [0,1]$ is a homotopy through compact perturbations of the linearised monopole map l.*

Proof. A solution $(\psi, a) \in (H_s)^{-1}(0)$ satisfies $F^A_{1-s}V(\psi, a) = 0$ for some $A \in \mathcal{J}(X)$. Proposition 3.21 provides a constant R, independent of s and A, such that

$$\|V(\psi, a)\|_{L^2_k} \leq R.$$

Applying Lemma 3.7 to V^{-1}, there is a constant $C_{V^{-1}}$ such that

$$\begin{aligned}\|(\psi, a)\|_{L^2_k} &= \|V^{-1}V(\psi, a)\|_{L^2_k}\\ &\leq C_{V^{-1}}R.\end{aligned}$$

The sub-bundle $D \subset \mathcal{A}$ with fibres L^2_k-disks of radius $C_{V^{-1}}R$ contains $H_s^{-1}(0)$ for all $s \in [0,1]$. □

Finally, we can use the three homotopies F, G and H to prove Theorem 3.8

Proof of Theorem 3.8. The concatenation $F \cdot G \cdot H$ is a homotopy from μ_X to $V^{-1}\mu_{X^\tau}V$. Proposition 3.22, 3.30 and 3.31 show that each of these is a homotopy through compact perturbations of the linearised monopole map l. These homotopies are defined on different sized disks, but Lemma 2.17 guarantees that they can be concatenated. The equivalence proved in Corollary 2.21 shows that the Bauer-Furuta classes $[\mu_X]$ and $[\mu_{X^\tau}]$ are equal in $\pi^{b^+}_{\mathbb{T}^n,\mathcal{U}}(\mathcal{J}(X), \operatorname{ind} D)$, where the class $[\mu_{X^\tau}]$ is represented by the bounded Fredholm map $V^{-1}\mu_{X^\tau}V$. $\square$

Theorem 3.8 can be used to derive a connected sum formula for the Bauer-Furuta invariant. This discussion is deferred to Chapter 4.4.1 where we extend Theorem 3.8 to the families setting.

Chapter 4

The families Bauer-Furuta invariant

As described in Chapter 2, the Bauer-Furuta invariant naturally extends to the setting of families of 4-manifolds. In this chapter, we define the families Bauer-Furuta invariant and generalise Theorem 3.8 to 4-manifold families. This extension (Theorem 4.3) is a new result and leads to a new connected sum formula (Theorem 4.15) for the families Bauer-Furuta invariant.

Let B be a compact, connected, oriented smooth manifold. For our purposes, a family of 4-manifolds is a smooth, locally trivial, oriented fibre bundle $\pi : E \to B$ with each fibre diffeomorphic to a closed, oriented 4-manifold X. In particular, $E \to B$ has transition functions valued in $\mathrm{Diff}^+(X)$. For $b \in B$, denote the fibres of E as $X_b = \pi^{-1}(b)$.

Let $T(E/B) \to E$ be the vertical tangent bundle $T(E/B) = \ker \pi_*$, which is a 4-dimensional real vector bundle over E. Let g be a metric on $T(E/B)$ with ∇ the associated Levi-Civita connection. One can think of g and ∇ as a smoothly varying family of metrics $\{g_b\}_{b \in B}$ and connections $\{\nabla_b\}_{b \in B}$ on the fibres X_b. Let $\mathfrak{s}_E$ be a spin^c structure on $T(E/B)$ with associated spinor bundles $W^\pm \to E$. This induces a smoothly varying family of spin^c structures $\{\mathfrak{s}_b\}_{b \in B}$ on the fibres of E. Let $\mathcal{L} = \det(W^+)$ be the determinant line bundle of W^+, which is a family of $U(1)$-bundles over B. A $U(1)$-connection $2A$ on $\mathcal{L}$ defines a family of spin^c connections ∇^A on W^+.

Let $\Lambda^i T^*(E/B) \to E$ denote the i-th exterior power of $T^*(E/B)$. A section of $\Lambda^i T^*(E/B)$ is a family of i-forms on the fibres X_b. Write $\Omega^i_B(E) = C^\infty(E, \Lambda^i T^*(E/B))$ to denote the set of families of smooth i-forms, which has the structure of a vector bundle $\Omega^i_B(E) \to B$. Similarly, $C^\infty(E, W^+) \to B$ denotes the bundle of families of smooth spinors over B.

4.1 Families with separating necks

Let $V_0 \to B$ be a rank 4 oriented Riemannian vector bundle equipped with a spin^c structure $\mathfrak{s}_{V_0}$. Denote by $S(V_0) \subset V_0$ the unit sphere sub-bundle of V_0. When performing a families connected sum, $S(V_0)$ will be obtained as the normal bundle of a section of the vertical tangent bundle of one of the summands. Fix a natural number $n \geq 1$ and for each $1 \leq i \leq n$, let $N_B(L)_i$ denote the family

$$N_B(L)_i = S(V_0) \times [-L, L].$$

Set $N_B(L) = \coprod_{i=1}^n N_B(L)_i$. We write $N_b(L)$ to denote the fibre of $N_B(L) \to B$ over $b \in B$. Denote the families of the positive and negative fiberwise boundary components by

$$\partial N_B(L)^+ = \coprod_{i=1}^n S(V_0) \times \{L\}$$
$$\partial N_B(L)^- = \coprod_{i=1}^n S(V_0) \times \{-L\}.$$

Since the transition maps of V_0 are valued in $SO(4)$, the vertical tangent bundle $T(S(W)/B)$ can be equipped with a metric $g_{S(W)}$ that restricts to the standard round metric on each fibre. Equip the vertical tangent bundle of $N_B(L)$ with the metric $g_{N_B(L)} = g_{S(W)} + dt^2$ which on each fibre is the product of the standard round metric on S^3 and the standard interval metric on $[-L, L]$. The spin^c structure $\mathfrak{s}_{V_0}$ determines a 3-dimensional spin^c structure on the vertical tangent space of $S(V_0)$. Pulling this back to $N_B(L)$ defines a spin^c structure $\mathfrak{s}_{N_B(L)}$ on $T(N_B(L)/B)$ such that on each fibre, the spinor multiplication is given by (3.45).

For each $1 \leq i \leq n$, let $E_i \to B$ be a connected family of closed, oriented 4-manifolds X_i. Set $E = \coprod_i E_i$ with fibre $X = \coprod_i X_i$.

Definition 4.1. *An n-component separating neck of length $2L$ on E is a smooth embedding $\iota : N_B(L) \to E$ covering the identity such that the restriction to each fibre $\iota_b : N_b(L) \to E_b$ gives E_b the structure of a separating neck of length $2L$.*

Given a 4-manifold family $E \to B$ with a separating neck of length $2L$, we identify $N_B(L)$ with its image $\iota(N_B(L))$. Assume for convenience that $L > 2$. For each connected component $N_B(L)_i$ of $N_B(L)$, define collar subbundles $C_i^{\pm} \subset N_B(L)_i$ by

$$\begin{aligned} C_i^- &= (S(V_0) \times [-L, -L+1]) \times B \\ C_i^+ &= (S(V_0) \times [L-1, L]) \times B. \end{aligned}$$

Let $C = \coprod_i (C_i^- \cup C_i^+)$. Each fibre C_b is a collar neighbourhood of the boundary of $N_b(L)$. Removing $N_B(L-1)$ from E gives a family of manifolds $M = \overline{E - N_B(L-1)}$ with fibres $\overline{X_b - N_b(L-1)}$ and a natural inclusion $\iota : C \to M$. For any other neck length $L' > 2$, there is a natural isometric inclusion $C \to N(L')$ identifying C has a collar neighbourhood of $\partial N(L')$. Let $E(L') = M \cup_C N_B(L')$. That is, $E(L')$ is defined by the following pushout

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d, "\iota"'] & N_B(L') \arrow[d, dashed] \\ M \arrow[r, dashed] & E(L'). \end{tikzcd}$$

Let $\tau \in S_n$ be an even permutation on n objects. Define a permuted inclusion map $\iota_\tau : C \to M$ such that $\iota_\tau|_{C_i^-} = \iota|_{C_i^-}$ and $\iota_\tau|_{C_i^+} = \iota|_{C^+_{\tau(i)}}$. That is, C_i^- is mapped to $\iota(C_i^-)$ but C_i^+ is mapped to $\iota(C^+_{\tau(i)})$. Define the permuted family E^τ by the following pushout

$$\begin{tikzcd} C \arrow[r, hook] \arrow[d, "\iota_\tau"'] & N_B(L) \arrow[d, dashed] \\ M \arrow[r, dashed] & E^\tau. \end{tikzcd}$$

Fiberwise, each boundary component of the form $\iota(C_i^-)_b \subset M_b$ has been connected by a cylinder $S^3 \times [-L, L]$ to $\iota(C^+_{\tau(i)})_b$. We write X^τ to denote the standard fibre of E^τ.

4.2 The families Seiberg-Witten monopole map

Fix a reference spin^c connection A_0 on E. Any other connection A can be written as $A = A_0 + ia$ for some family of one-forms $a \in C^\infty(E, T^*(E/B))$. Let n be the number of connected components of X and fix an integer $k \geq 4$.

To define the families monopole map, we follow the construction in [7, Example 2.1 and 2.4]. For now assume that $b_1(X) = 0$. Define Hilbert space bundles $\mathcal{A}$ and $\mathcal{C}$

over B by

$$\begin{aligned}\mathcal{A} &= L^2_k(E, W^+ \oplus T^*(E/B)) \oplus \mathbb{R}^n \\ \mathcal{C} &= L^2_{k-1}(E, W^- \oplus \Lambda^2_+ T^*(E/B) \oplus \mathbb{R}).\end{aligned} \tag{4.1}$$

The $\mathbb{R}^n$ term in $\mathcal{A}$ is identified with the space of locally constant functions $H^0(X;\mathbb{R})$ on X. Denote by $\mathbb{T}^n = (S^1)^{\times n}$ the group of locally constant gauge transformations. Let $\mathbb{T}^n$ act on $\mathcal{A}$ and $\mathcal{C}$ in the usual manner, on spinors by multiplication and on forms trivially. This action is fibre-preserving and orthogonal. The monopole map $\mu : \mathcal{A} \to \mathcal{C}$ is the $\mathbb{T}^n$-equivariant bundle map given by the formula

$$\mu(\psi, a, f) = (D_{A_0+ia}\psi, -iF^+_{A_0+ia} + i\sigma(\psi), d^*a + f).$$

There is a decomposition $\mu = l + c$ with

$$\begin{aligned}l(\psi, a, f) &= (D_{A_0}\psi, d^+a, d^*a + f) \\ c(\psi, a, f) &= (ia \cdot \psi, -iF^+_{A_0} + i\sigma(\psi), 0).\end{aligned} \tag{4.2}$$

The map l is linear Fredholm and c is compact, hence μ is a Fredholm map. Since B is compact, the argument presented in Proposition 3.3 extends to this setting and shows that μ is a bounded Fredholm map.

In the case that $b_1(X) > 0$, fix a smooth section $x : B \to E$. Let $\mathcal{H}^1(\mathbb{R}) \subset C^\infty(E, T^*(E/B))$ denote the subbundle of real harmonic forms. That is, $\mathcal{H}^1(\mathbb{R}) \to B$ is a vector bundle with fibre $\mathcal{H}^1(X_b;\mathbb{R})$ over $b \in B$. Now pullback the bundles defined in (4.1) to bundles over $\mathcal{H}^1(\mathbb{R})$:

$$\begin{aligned}\tilde{\mathcal{A}} &= L^2_k(E, W^+ \oplus T^*(E/B)) \oplus \mathbb{R}^n \to \mathcal{H}^1(\mathbb{R}) \\ \tilde{\mathcal{C}} &= L^2_{k-1}(E, W^- \oplus \Lambda^2_+ T^*(E/B) \oplus \mathbb{R}) \oplus \mathcal{H}^1(\mathbb{R}) \to \mathcal{H}^1(\mathbb{R}).\end{aligned}$$

The tilde notation is used because we are yet to quotient out by harmonic gauge transformations. Let $A_\theta = A_0 + i\theta$ denote the connection associated to $\theta \in \mathcal{H}(\mathbb{R})$. Note that since θ is harmonic, $F_{A_\theta} = F_{A_0}$. Define $\tilde{\mu} : \tilde{\mathcal{A}} \to \tilde{\mathcal{C}}$ by

$$\tilde{\mu}_\theta(\psi, a, f) = (D_{A_\theta+ia}\psi, -iF_{A_0+ia} + i\sigma(\psi), d^*a + f, \mathrm{pr}(a)). \tag{4.3}$$

This is the monopole map with gauge fixing, before dividing out by the harmonic gauge transformations. The projection bundle map $\mathrm{pr} : L^2_k(E, T^*(E/B)) \to \mathcal{H}^1(\mathbb{R})$ in the families setting takes a little more care to define. Let $\{U_\beta\} \subset B$ be a trivialising open cover of B with $E|_{U_\beta} \cong U_\beta \times X$. Choose cycles $\alpha_1, ..., \alpha_{b_1(X)}$ that restrict to a homology basis on each fibre of $E|_{U_\beta}$. Define a map $\mathrm{pr}_\beta : \Omega^1_B(E)|_{U_\beta} \to \mathcal{H}^1(\mathbb{R})|_{U_\beta}$ fibrewise using equation (3.4). Let $\{\rho_\beta\}$ be a partition of unity subordinate to $\{U_\beta\}$ and define $\mathrm{pr} : \Omega_B(E) \to \mathcal{H}^1(\mathbb{R})$ by $\mathrm{pr} = \sum_\beta \rho_\beta \,\mathrm{pr}_\beta$. This map has the property that if $a \in \Omega^1_B(E)$ is a family of closed one forms, then $\mathrm{pr}(a) \in \mathcal{H}^1(\mathbb{R})$ is the cohomology class of a in each fibre. Further, pr extends continuously to a map $\mathrm{pr} : L^2_k(E, T^*(E/B)) \to \mathcal{H}^1(\mathbb{R})$.

To account for the harmonic gauge transformations, let $\mathcal{H}(2\pi\mathbb{Z}) \to B$ be the bundle of groups over B with fibre $H^1(X_b; 2\pi\mathbb{Z})$. That is, $\mathcal{H}(2\pi\mathbb{Z}) \subset \mathcal{H}(\mathbb{R})$ is the subbundle of harmonic 1-forms on X_b with periods that are integral multiples of 2π. For each $\omega \in \mathcal{H}(2\pi\mathbb{Z})$ and $b \in B$, define a map $g_{\omega,b} : X_b \to S^1$ by

$$g_{\omega,b}(y) = \exp\left(i \int_{x(b)}^{y} \omega \right).$$

This map is well defined since the periods of ω are multiples of 2π. Further, $g_{\omega,b}$ is the unique harmonic gauge transformation with the property that $g^{-1}_{\omega,b} dg_{\omega,b} = i\omega$ and $g_{\omega,b}(x(b)) = 1$. The gauge transformation g_ω acts on a connection A by $g_\omega \cdot A = A + i\omega$.

Let the bundle of groups $\mathcal{H}(2\pi\mathbb{Z})$ act on $\mathcal{H}(\mathbb{R})$ fiberwise by $\omega \cdot \theta = \theta + \omega$. The quotient bundle $\mathcal{J} = \mathcal{H}(\mathbb{R})/\mathcal{H}(2\pi\mathbb{Z})$ is the $b_1(X)$-dimensional Jacobian torus bundle over B. That is, each fibre $\mathcal{J}_b$ is the Jacobian torus $\mathcal{J}(X_b)$ of X_b. Define an action of $\mathcal{H}(2\pi\mathbb{Z})$ on elements $(\psi, a, f) \in \tilde{\mathcal{A}}_\theta$ and $(\phi, \eta, g, \alpha) \in \tilde{\mathcal{C}}_\theta$ by

$$\begin{aligned} \omega \cdot (\theta, (\psi, a, f)) &= (\theta + \omega, (g^{-1}_\omega \psi, a, f)) \\ \omega \cdot (\theta, (\phi, \eta, g, \alpha)) &= (\theta + \omega, (g^{-1}_\omega \phi, \eta, g, \alpha)). \end{aligned}$$

This is the free action of the based harmonic gauge transformations g_ω. Under this action, $\tilde{\mu}$ is equivariant. The fiberwise quotients $\mathcal{A} = \tilde{\mathcal{A}}/\mathcal{H}(2\pi\mathbb{Z})$ and $\mathcal{C} = \tilde{\mathcal{C}}/\mathcal{H}(2\pi\mathbb{Z})$ are Hilbert bundles over $\mathcal{J}$ with a residual $\mathbb{T}^n$-action of the constant gauge transformations. The map $\tilde{\mu}$ descends to a $\mathbb{T}^n$-equivariant Fredholm map $\mu : \mathcal{A} \to \mathcal{C}$ over $\mathcal{J}$. This is the families monopole map in the setting $b_1(X) > 0$. In a similar fashion to (4.2), $\mu = l + c$ is a Fredholm map with

$$\begin{aligned} l_\theta(\psi, a, f) &= (D_{A_\theta}\psi, d^+ a, d^* a + f, \mathrm{pr}(a)) \\ c_\theta(\psi, a, f) &= (ia \cdot \psi, -iF^+_{A_0} + i\sigma(\psi), 0, 0). \end{aligned} \tag{4.4}$$

Define a $\mathbb{T}^n$ universe $\mathcal{U}$ by

$$\mathcal{U} = L^2_{k-1}(X, W|^-_X \oplus \Lambda^2_+(T^*X) \oplus \mathbb{R}) \oplus \mathcal{H}^1(X; \mathbb{R}). \tag{4.5}$$

This is a $\mathbb{T}^n$-universe that can be identified with each fibre of $\mathcal{C}$. The map l defines a family of linear Fredholm maps over $\mathcal{J}$, so let $\mathrm{ind}_{\mathcal{J}}\, l$ denote the corresponding virtual index bundle. Let $H^+ \to \mathcal{J}$ denote the rank $b^+(X)$ trivial bundle with fibre $H^2_+(X; \mathbb{R})$ so that we have the relation $\mathrm{ind}_{\mathcal{J}}\, l = \mathrm{ind}_{\mathcal{J}}\, D - H^+$.

Definition 4.2. *The families Bauer-Furuta invariant of a 4-manifold family $E \to B$ is the cohomotopy class*

$$\begin{aligned}[\mu] &\in \pi^0_{\mathbb{T}^n,\mathcal{U}}(\mathcal{J}, \operatorname{ind}_{\mathcal{J}} l) \\ &= \pi^{b^+}_{\mathbb{T}^n,\mathcal{U}}(\mathcal{J}, \operatorname{ind}_{\mathcal{J}} D). \end{aligned} \tag{4.6}$$

Now suppose that E is a family of 4-manifolds with necks. As in Chapter 3.2.3, there are some slight modifications needed for the monopole map to be compatible with the arguments used in Theorem 3.8. First, we need the reference connection A_0 to be flat on the neck and permutation invariant. To achieve this, choose a partition of unity $\{\rho_\beta\}$ subordinate to a trivialising open cover $\{U_\beta\}$ of B. On $E|_{U_\beta} \cong U_\beta \times X(L)$, define A_0^β as in the unparameterised case. Then $A_0 = \sum_\beta \rho_\beta A_0^\beta$ is a suitable reference connection.

Moreover, let $\mathcal{G}_{N(1)} \to B$ be the bundle of Gauge groups with fibre maps $(\mathcal{G}_{N(1)})_b = C^\infty(X_b, S^1)$ that fixes the short neck $N(1)_b$ and let $\ker d_{N(1)} \subset \Omega^1_B(E)$ be the subset of families of forms $a \in \ker d$ that vanish on $N(1)$. The inclusion $(A_0 + i \ker d_{N(1)})/\mathcal{G}_{N(1)} \to \mathcal{J}_E$ is a smooth bundle map over B that restricts to a diffeomorphism on each fibre, hence we can identify $(A_0 + i\ker d_{N(1)})/\mathcal{G}_{N(1)} = \mathcal{J}_E$. Now for any even permutation τ, $\mathcal{J}_E = \mathcal{J}_{E^\tau}$ which means that μ_E and μ_{E^τ} can be treated as bundle maps over the same space $\mathcal{J} = \mathcal{J}_E$.

4.2.1 Permuting families of sections

Let E be a family of manifolds $X(L)$ with necks of length $2L$. Denote by $\widehat{W}^+ \to S(V_0) \times [-L, L]$ the restriction of $W^+ \to N_B(L)$ to one of the connected components of $N_B(L)$. Define $F = \oplus_{i=1}^n \widehat{W}^+ \to S(V_0) \times [-L, L]$ to be the direct sum of n-copies of $\widehat{W}^+$ over $S(V_0) \times [-L, L]$. Since $N_B(L)$ has n connected components, a section $\psi : N_B(L) \to W^+$ can be identified with a vector of sections $\vec{\psi} : S(V_0) \times [-L, L] \to F$. That is, the restriction ψ_i to the ith component of $N_B(L)$ is identified with the ith component of $\vec{\psi}$. Let $T : S(V_0) \times [-L, L] \to SO(n)$ denote a matrix valued function. For a section $\psi : N_B(L) \to W^+$ along $N_B(L)$, define an action by $T \cdot \psi = T\vec{\psi}$ where T acts pointwise on $\vec{\psi}$ and $T\vec{\psi}$ is identified with a section of $W^+ \to N_B(L)$. The same process defines an action on forms along the neck $a : N_B(L) \to \Lambda^i(T^*(N_B(L)/B))$.

As in (3.22), let $\gamma : [0, 1] \to SO(n)$ be a smooth path from the identity to τ, which exists under the assumption that τ is even, and let $\varphi : [-L, L] \to [0, 1]$ be a smooth map that vanishes on $[-L, 1]$ and is identically equal to 1 on $[1, L]$. Let $V : S(V_0) \times [-L, L] \to SO(n)$ denote the matrix-valued function

$$V(x, t) = \gamma(\varphi(t)). \tag{4.7}$$

Note that V is constant along the $S(V_0)$ factor. Let $(\psi, a) : N_B(L) \to W^+ \oplus T^*(N_B(L)/B)$ be a spinor-form pair along $N_B(L)$ and define $(\psi, a)^\tau = (V \cdot \psi, V \cdot a)$ by the action described above. The pair $(\psi, a)^\tau$ has the property that $(\psi, a)^\tau_i = (\psi, a)_i$ on C^- and $(\psi, a)^\tau_i = (\psi, a)_{\tau(i)}$ on C^+. Now given a section $(\psi, a) : E \to W^+ \oplus T^*(E/B)$ defined on all of E, this permutation process defines a section $(\psi, a)^\tau$ on E^τ with the property that (ψ, a) and $(\psi, a)^\tau$ agree away from the neck.

This process defines an isomorphism $V_\mathcal{A} : \mathcal{A}_E \to \mathcal{A}_{E^\tau}$ of Hilbert bundles over $\mathcal{J}$. Similarly for $\mathcal{C}$, the action of V defines a map $V_\mathcal{C} : \mathcal{C}_E \to \mathcal{C}_{E^\tau}$ that on the $\mathcal{H}^1(X; \mathbb{R})$ factor is just the identity. The proof of Lemma 3.7 extends to these maps and so they are continuous in the L^2_k-norm. The maps $V_\mathcal{A}$ and $V_\mathcal{C}$ identify $\pi(\mathcal{P}_l(\mathcal{A}, \mathcal{C})^{\mathbb{T}^n})$ and $\pi(\mathcal{P}_l(\mathcal{A}^\tau, \mathcal{C}^\tau)^{\mathbb{T}^n})$ as in (3.24), both of which can be identified with $\pi^{b^+}_{\mathbb{T}^n, \mathcal{U}}(\mathcal{J}, \operatorname{ind} D)$. With this understanding, the statement of Theorem 3.8 immediately extends to the families setting. This extension is a new result.

Theorem 4.3 (Families Permutation Theorem). *Let $E \to B$ be a family of closed 4-manifolds that admits an n-component separating neck. Let $\tau \in S_n$ be a permutation with E^τ the corresponding permuted family. Then*

$$[\mu_E] = [\mu_{E^\tau}] \tag{4.8}$$

as elements of $\pi^{b^+}_{\mathbb{T}^n, \mathcal{U}}(\mathcal{J}, \operatorname{ind} D)$.

Let $D \subset \mathcal{A}$ be a disk bundle that contains the zeroes of both μ_E and μ_{E^τ} with boundary sphere bundle S. Recall from Section 2.3 that$C_l(D, \mathcal{C})^{\mathbb{T}^n}$ denotes the set of $\mathbb{T}^n$-equivariant bounded Fredholm maps $f : \mathcal{A} \to \mathcal{C}$ which differ from the linearised monopole map l by a compact map, and are non-vanishing on the boundary sphere bundle S. Let $\mathcal{U}, H^+$ and $\operatorname{ind}_\mathcal{J} l$ be defined as in equation 4.6. In Section 2.6 it was shown that given a trivialisation $p : \mathcal{C} \to \mathcal{J} \times U$, the compact homotopy classes $\pi_0(C_l(D, \mathcal{C})^{\mathbb{T}^n})$ are naturally isomorphic to $\pi^0_{\mathbb{T}^n, \mathcal{U}}(\mathcal{J}; \operatorname{ind} l)$. Once again, to prove the theorem it is enough to show that μ_E is homotopic to $V^{-1} \mu_{E^\tau} V$ through compact perturbations of l.

4.3 Proof of the Families Permutation Theorem

Most of the ground work needed to prove Theorem 4.3 was handled in Chapter 3.2. In this chapter, we will lift these results to the families setting, taking care to ensure that the relevant arguments carry over smoothly.

Fix $L > 2$ and let $E = E(L) \to B$ be a family of closed 4-manifolds with a separating neck of length $2L$. Fix a reference connection A_0, which can be assumed to be flat on the neck $N_B(L)$. Recall that for $\theta \in \mathcal{J}$, A_θ denotes the associated connection $A_0 + i\theta$. Note that A_θ is also flat on the neck. As in Section 3.4, we construct three compact homotopies F, G and H that when concatenated give a homotopy from μ_E to $V^{-1}\mu_{E^\tau}V$. Let $\mu_E = l + c$ as in (4.2) where l is the linearised monopole map and c is a compact map.

For a given $R \leq L$, let $\rho_R : E \to [0,1]$ be a smooth function that vanishes on $N_B(R-1)$ and is identically 1 on $E - N_B(R)$. Along $N_B(R) - N_B(R-1)$, we require ρ_R to only depend on the interval coordinate. For $s \in [0,1]$, let $\rho_{R,s}$ be a linear homotopy ending at ρ_R of the form

$$\rho_{R,s} = (1-s) + s\rho_R.$$

4.3.1 The first homotopy

To define the first homotopy $F : \mathcal{A} \to \mathcal{C}$ fiberwise, let $\theta \in H^1(X_b;\mathbb{R})$ for some $b \in B$. As in (3.55), set

$$F_s^\theta(\psi, a) = (D_{A_\theta+ia}\psi, -iF^+_{A_\theta+ia} + i\rho_{L,s}\sigma(\psi), d^*a, \mathrm{pr}(a)).$$

Proposition 4.4. *The map $F_s : \mathcal{A} \to \mathcal{C}$ is a homotopy through compact perturbations of l.*

Proof. Fix a class $[\theta] \in \mathcal{J}$ with a representative $\theta \in \mathcal{H}^1(X_b;\mathbb{R})$ for some $b \in B$. The fibre X_b is a closed 4-manifold with separating neck $N_b(L)$ of length $2L$. For any $s \in [0,1]$, applying Proposition 3.21 to X_b with the connection A_θ gives a radius R^θ such that

$$\|(\psi, a)\|_{L^2_k} \leq R^\theta$$

for any $(\psi, a) \in (F_s^\theta)^{-1}(0)$. This bound does not depend on $s \in [0,1]$. Let R be the supremum of R^θ over $\mathcal{J}$, which exists since $\mathcal{J}$ is compact. Further, let $D \subset \mathcal{A}$ be a disk bundle over $\mathcal{J}$ with L_k^2-radius R. This shows in fact that each preimage $F_s^{-1}(0)$ is contained in a bounded disk bundle, a stronger result than required. □

4.3.2 The second homotopy

The second homotopy G_s for $s \in [0,3]$ is constructed in three stages. For $s \in [0,1]$ define

$$G_s^\theta(\psi, a) = (D_{A_\theta + i\rho_{(M,s)}a}\psi, -iF^+_{A_\theta + ia} + i\rho_L\sigma(\psi), d^*a, \mathrm{pr}(a))$$

This homotopy eliminates the other quadratic term $ia \cdot \psi$ from $N_B(M-1)$. The constant $M \geq 3$ will be defined later. It is assumed without loss of generality that $L \geq 2M$.

To define the second stage of G, let $P = G_1$. This stage will transform P to $P^\tau = V^{-1}PV$ where the action of V was defined in equation 4.7. Restricting to $N_B(1)$, P is a first order linear differential operator given by the formula

$$P^\theta(\psi, a) = (D_{A_\theta}\psi, d^+a, d^*a, \mathrm{pr}(a)).$$

Note that $F^+_{A_\theta} = 0$ since A_θ is flat on the neck. For $s \in [0,1]$, let

$$V_s(x,t) = \gamma((s-1)\cdot\varphi(t)) : S(V_0) \times [-L, L] \to SO(n).$$

As before, define $Q_s : \mathcal{A} \to \mathcal{C}$ by

$$Q_s^\theta(\psi, a) = V_s^{-1}\partial_t V_s(dt \cdot \psi, (dt \wedge a)^+, *(*\vec{a} \wedge dt), 0).$$

Note that Q vanishes away from $N(1)$. The Leibniz rule shows that

$$V_s^{-1}PV_s(\psi, a) = P(\psi, a) + Q_s(\psi, a).$$

For $s \in [1,2]$, define G_s by

$$G_s = P + Q_s. \tag{4.9}$$

Restricted to the neck $N_B(L-1)$, we have $G_s = V_s^{-1}PV_s$ so that $G_2 = V^{-1}PV$.

For the third stage $s \in [2,3]$, define $G_s = V^{-1}G_{(3-s)}V$ which is a homotopy from $G_2 = V^{-1}PV$ to $G_3 = V^{-1}F_1V$. Thus G is a homotopy from F_1 to $V^{-1}F_1V$.

Proposition 4.5. *There are constants M and L_0 such that, if $L \geq L_0$, then $G_s : \mathcal{A} \to \mathcal{C}$ is a homotopy through compact perturbations of l.*

Proof. The argument is similar to Proposition 3.30. For any $b \in B$ and $\theta \in \mathcal{H}^1(X_b;\mathbb{R})$, apply Lemma 3.28 to X_b to get constants C_1^θ and d such that, for large enough L,

$$\|(\psi_b, a_b)\|_{L^2_k} \leq C_1^\theta L^d$$

for any $(\psi_b, a_b) \in (G_0^\theta)^{-1}(0) \cup (G_3^\theta)^{-1}(0)$. The constant d from the bootstrapping argument only depends on k, hence the same d can be used for each θ. Let $C_1 = \sup_{\theta\in\mathcal{J}} C_1^\theta$ so that

$$\|(\psi, a)\|_{L^2_k} \leq C_1 L^d \tag{4.10}$$

for (ψ, a) in any fibre of $G_0^{-1}(0) \cup G_3^{-1}(0)$.

Again for each $b \in B$ and $\theta \in \mathcal{H}^1(X_b;\mathbb{R})$, apply Proposition 3.27 on X_b to get constants U_0^θ, C, δ and M such that, for large enough L,

$$|(\psi_b, a_b)|_{C^0} \leq U(L) \Rightarrow |(\psi_b, a_b)|_{C^0} < U_0^\theta$$

so long as $(\psi_b, a_b) \in (G_s^\theta)^{-1}(0)$ for some $s \in [0, 3]$. Recall that $U(L) = Ce^{-\delta(L-2M)}$. The constant δ is chosen based on the eigenvalues of the first order elliptic operator $\mathcal{L}$ on S^3 defined in (3.29). Thus the same δ can be used for any θ on any fibre X_b of E. Further, from (3.65) we can see that C only depends on δ, the scalar curvature of $S^3 \times [-L, L]$, and the derivative of ρ. Hence C is also independent of θ and b. By similar reasoning, M can also be chosen independently from θ and b by (3.70).

Letting $U_0 = \sup_{\theta\in\mathcal{J}} U_0^\theta$, it follows that

$$|(\psi, a)|_{C^0} \leq U(L) \Rightarrow |(\psi, a)|_{C^0} < U_0 \tag{4.11}$$

so long as (ψ, a) is an element of some fibre of $G_s^{-1}(0)$ for some $s \in [0, 3]$.

For each $b \in B$, apply Lemma 3.9 to X_b to get Sobolev embedding constants C_S^b, independent of L. Let $C_S = \sup_{b\in B} C_S^b$ so that, for any L^2_k-pair (ψ, a) on any fibre X_b,

$$|(\psi, a)|_{C^0} \leq C_S \|(\psi, a)\|_{L^2_k}. \tag{4.12}$$

Finally, to facilitate bootstrapping, for each $b \in B$ and $\theta \in \mathcal{H}^1(X_b;\mathbb{R})$, use Lemma 3.29 to choose a constant C_B^θ such that

$$\|(\psi_b, a_b)\|_{L^2_k} \leq C_B^\theta L^d (1 + |(\psi_b, a_b)|_{C^0})^d$$

This holds so long as $(\psi_b, a_b) \in (G_s^\theta)^{-1}(0)$ for some $s \in [0,3]$. Once again let $C_B = \sup_{\theta \in \mathcal{J}} C_B^\theta$ so that

$$\|(\psi, a)\|_{L^2_k} \leq C_B L^d (1 + |(\psi, a)|_{C^0})^d \tag{4.13}$$

so long as (ψ, a) is an element of some fibre of $G_s^{-1}(0)$ for some $s \in [0,3]$.

Set $R(L) = \frac{U(L)}{C_S}$ and let $D \subset \mathcal{A}$ be a disk bundle with L^2_k-radius $R(L)$. Let S denote the bounding sphere bundle of D. Choose L_0 large enough so that $L \geq L_0$ implies

$$R(L) \geq \max\{C_1 L^d, 2C_B L^d (1 + U_0)^d\}.$$

This is achievable since $R(L)$ increases exponentially. By (4.10), $R(L)$ contains $G_0^{-1}(0) \cup G_3^{-1}(0)$. Further, suppose $\theta \in \mathcal{H}^1(X_b;\mathbb{R})$ for some $b \in B$ with $(\psi, a) \in (G_s^\theta)^{-1}(0) \cap D$ for some $s \in [0,3]$. Then $\|(\psi, a)\|_{L^2_k} \leq R(L)$ and by (4.12), $|(\psi, a)|_{C^0} \leq U(L)$. Thus $|(\psi, a)|_{C^0} < U_0$ by (4.11) and (4.13) implies that

$$\begin{aligned}\|(\psi, a)\|_{L^2_k} &\leq C_B L^d (1 + U_0)^d \\ &\leq \frac{1}{2} R(L).\end{aligned}$$

That is, $(G_s^\theta)^{-1}(0)$ does not intersect S for any $\theta \in \mathcal{J}$. □

4.3.3 The third homotopy

The third homotopy H_s for $s \in [0,1]$ is given by

$$H_s = V^{-1} F_{1-s} V.$$

This homotopy starts at $H_0 = G_3 = V^{-1} F_1 V$ and ends at $H_1 = V^{-1} \mu_{E^\tau} V$.

Proposition 4.6. *The homotopy H_s is a homotopy through compact perturbations of l.*

Proof. A solution $(\psi, a) \in (H_s)^{-1}(0)$ satisfies $F^{\theta}_{1-s}V(\psi, a) = 0$ for some $b \in B$ and $\theta \in \mathcal{H}^1(X_b; \mathbb{R})$. Proposition 4.4 provides a constant R, independent of s and θ, such that

$$\|V(\psi, a)\|_{L^2_k} \leq R.$$

Lemma 3.7 gives a constant $C_{V^{-1}}$ such that

$$\begin{aligned}\|(\psi, a)\|_{L^2_k} &= \|V^{-1}V(\psi, a)\|_{L^2_k} \\ &\leq C_{V^{-1}}R.\end{aligned}$$

The constant $C_{V^{-1}}$ can be chosen independently of $\theta \in \mathcal{J}$. The disk bundle $D \subset \mathcal{A}_k$ with fibres of L^2_k-radius $C_{V^{-1}}R$ contains $H_s^{-1}(0)$ for all $s \in [0, 1]$. □

Proof of Theorem 4.3. By the same argument used in the proof of Theorem 3.8, the concatenation $F \cdot G \cdot H$ is a homotopy from μ_E to $V^{-1}\mu_{E^\tau}V$ through compact perturbations of l. Thus the Bauer-Furuta classes $[\mu_E]$ and $[\mu_{E^\tau}]$ are equal in $\pi^{b^+}_{\mathbb{T}^n,\mathcal{U}}(\mathcal{J}_E, \operatorname{ind} D)$, where the class $[\mu_{E^\tau}]$ is represented by the bounded Fredholm map $V^{-1}\mu_{E^\tau}V$. □

Remark 4.7: The definition of the separating neck $N_B(L)$ asked that the fibres of the neck components are of the form $S^3 \times [-L, L]$, with the application to connected sums in mind. However in Chapter 3.3, we did not use anything particularly special to S^3. We only used that fact that S^3 has a positive scalar curvature metric and that $b_1(S^3) = 0$. Thus Theorem 3.8 will extend to the case that the neck is a product $M \times [-L, L]$ with M any spherical 3-manifold.

4.4 Connected sums of families

For $j \in \{1, 2\}$, let $E_j \to B$ be a family of closed, oriented 4-manifolds X_j. To define the families connected sum, it is necessary to have sections $i_j : B \to E_j$ with normal bundles $V_j \to B$ and an orientation reversing isomorphism $\varphi : V_1 \to V_2$. Since the fibre of E_j is 4-dimensional, V_j is a real 4-dimensional vector bundle. Fix a metric on V_j and identify the open unit disk bundle $D(V_j)$ as a tubular neighbourhood of i_j with $S(V_j)$ the bounding unit sphere bundle. Let $U_j = \overline{E_j - D(V_j)}$ so that

$$\begin{aligned}E_1 &= U_1 \cup_{S(-V_1)} D(V_1) \\ E_2 &= D(V_2) \cup_{S(V_2)} U_2.\end{aligned} \tag{4.14}$$

Here we are interpreting $S(-V_1)$ as the ingoing boundary of $D(V_1)$, hence the negative sign, and $S(V_2)$ as the outgoing boundary of $D(V_2)$. Thus φ identifies $S(-V_1)$ with $S(V_2)$. Topologically the families connected sum $E = E_1 \#_B E_2$ is defined as

$$E = U_1 \cup_{S(-V_1)} U_2. \tag{4.15}$$

We write $S(V) \subset E$ to denote $S(-V_1) \subset U_1$, which has been identified with $\varphi(S(-V_1)) = S(V_2) \subset U_2$. To define a metric on E, attach cylinders to E_1 and E_2 to get

$$\begin{aligned} \hat{E}_1 &= U_1 \cup_{S(-V_1)} (S(V_1) \times [0, \infty)) \\ \hat{E}_2 &= (S(V_2) \times (\infty, 0]) \cup_{S(V_2)} E_2. \end{aligned}$$

Let g_1 be the metric on $S(V_1) \times [0, \infty)$ which restricts to a product of the standard round metric and interval metric on the fibres. The metric g_1 can be smoothly extended to $\hat{E}_1$ using a collar neighbourhood. Repeat the same process to get a metric g_2 on $\hat{E}_2$. For $L > 0$, let

$$\begin{aligned} \hat{E}_1(L) &= \hat{E}_1 - (S(V_1) \times (L+1, \infty)) \\ \hat{E}_2(L) &= \hat{E}_2 - (S(V_2) \times (-\infty, -L-1)) \,. \end{aligned}$$

For gluing along the cylindrical ends, define a smooth map

$$\begin{gathered} f : S^3 \times [L-1, L+1] \to S^3 \times [-L-1, -L+1] \\ f(x,t) = (x, t - 2L). \end{gathered}$$

Now let $E(L) = E_1(L) \cup_f E_2(L)$ with metric $g_{E(L)} = g_1 \cup_f g_2$. By construction $E(L)$ is a 4-manifold family with standard fibre $X(L) = X_1 \# X_2$ with a separating neck of length $2L$. Up to diffeomorphism, the families connected sum $E(L)$ depends only on the given sections i_1 and i_2 and the orientation reversing diffeomorphism of the normal bundles φ.

To get a spin^c structure on $E = E(L)$, let $\mathfrak{s}_j$ be a spin^c structure on the vertical tangent bundle $T(E_j/B)$ for $j \in \{1, 2\}$. Write $\mathcal{S}(E)$ to denote the set of isomorphism classes of spin^c structures on E. There is a restriction map defined by

$$\begin{gathered} r : \mathcal{S}(E) \to \mathcal{S}(E_1) \times \mathcal{S}(E_2) \\ r(\mathfrak{s}) = (\mathfrak{s}|_{E_1}, \mathfrak{s}|_{E_2}) \end{gathered}$$

Lemma 4.8. *The restriction map $r : \mathcal{S}(E) \to \mathcal{S}(E_1) \times \mathcal{S}(E_2)$ is a bijection onto the subset $T \subset \mathcal{S}(E_1) \times \mathcal{S}(E_2)$ defined by*

$$T = \{(\mathfrak{s}_1, \mathfrak{s}_2) \in \mathcal{S}(E_1) \times \mathcal{S}(E_2) \mid \mathfrak{s}_1|_{S(V)} \cong \mathfrak{s}_2|_{S(V)}\}.$$

Proof. From (4.15) is it clear that the image of r is contained in T. Given $(\mathfrak{s}_1, \mathfrak{s}_2) \in T$, a spin^c structure $\mathfrak{s}$ on E can be obtained from gluing, hence r is surjective. It remains to prove injectivity.

Suppose $\mathfrak{s}, \mathfrak{s}'$ are spin^c structures on E with $r(\mathfrak{s}) = r(\mathfrak{s}')$. That is, there are isomorphisms $\varphi_j : \mathfrak{s}|_{E_j} \to \mathfrak{s}'|_{E_j}$ for $j \in \{1, 2\}$. If $\varphi_1|_{S(V)} = \varphi_2|_{S(V)}$, then φ_1 and φ_2 would glue to give an isomorphism $\mathfrak{s} \to \mathfrak{s}'$.

Let $\psi = \varphi_1^{-1}|_{S(-V)} \circ \varphi_2|_{S(V)}$ so that $\varphi_2|_{S(V)} = \varphi_1|_{S(V)} \circ \psi$. The map ψ is an automorphism of spin^c structures over $S(V)$ and therefore is determined by a smooth map $f : S(V) \to S^1$. We claim that f extends to a smooth map $\tilde{f} : E_1 \to S^1$. Assuming this claim implies that ψ extends to an automorphism $\tilde{\psi}$ of $\mathfrak{s}|_{E_1}$. Setting $\varphi_1' = \varphi_1 \circ \tilde{\psi} : \mathfrak{s}|_{E_1} \to \mathfrak{s}'|_{E_1}$ gives an isomorphism of spin^c structures with the property that $\varphi_1'|_{S(V)} = \varphi_2|_{S(V)}$ and the result follows by gluing.

To prove the claim, recall that the set of homotopy class of maps $[S(V), S^1]$ are in bijection with $H^1(S(V); \mathbb{Z})$. The Serre spectral sequence implies that $H^1(S(V); \mathbb{Z})$ is isomorphic to $H^1(B; \mathbb{Z})$ by pullback. That is, the homotopy class of f corresponds to the pullback of an element $\alpha \in H^1(B; \mathbb{Z})$. Pulling back α to $H^1(E_1; \mathbb{Z})$ corresponds to a homotopy class of $[E_1, S^1]$ and we can choose a representative $\tilde{f}$ that restricts to f on $S(V)$. □

Corollary 4.9. *For $j \in \{1, 2\}$, let $E_j \to B$ be a 4-manifold family equipped with a spin^c structure $\mathfrak{s}_j$ on the vertical tangent bundle. Let $i_j : B \to E_j$ be a section with normal bundle V_j and assume that an orientation reversing isomorphism $\varphi : V_1 \to V_2$ is given. An extension of $\mathfrak{s}_1$ and $\mathfrak{s}_2$ to the families connected sum $E = E_1 \#_B E_2$ exists if and only if*

$$\varphi(i_1^*(\mathfrak{s}_{E_1})) \cong i_2^*(\mathfrak{s}_{E_2}).$$

4.4.1 Families Bauer-Furuta formula

The families Bauer-Furuta connected sum formula follows from the Theorem 4.3 by the following observations. For a disjoint union of families $E = \coprod_{i=1}^n E_i$ the monopole map $\mu_E : \mathcal{A} \to \mathcal{C}$ is the direct sum

$$\mu_E = \bigoplus_{i=1}^n \mu_{E_i} : \bigoplus_{i=1}^n \mathcal{A}_{E_i} \to \bigoplus_{i=1}^n \mathcal{C}_{E_i}.$$

Assume that each E_i is connected and let $\mathcal{U}_i$ be an S^1-universe for E_i as in (4.5). Then $\mathcal{U} = \oplus_i \mathcal{U}_i$ is a $\mathbb{T}^n$-universe with $\mathbb{T}^n$ acting component-wise and the Bauer-Furuta class of μ_E is an element of $\pi_{\mathbb{T}^n,\mathcal{U}}(\mathcal{J}; \mathrm{ind}(l))$.

Proposition 4.10. *If $E = \coprod_{i=1}^n E_i$ is a disjoint union of families of 4-manifolds over B, then the Bauer-Furuta class $[\mu_E] \in \pi_{\mathbb{T}^n,\mathcal{U}}(\mathcal{J}; \mathrm{ind}(l))$ is given by the fibrewise smash product*

$$[\mu_E] = [\mu_{E_1}] \wedge_{\mathcal{J}} \cdots \wedge_{\mathcal{J}} [\mu_{E_n}].$$

The above proposition follows directly from the definition of $[\mu_E]$ outlined in Definition 2.33. The next observation demonstrates a method for calculating the Bauer-Furuta invariant in the simplest cases. Recall that $H^+ \to \mathcal{J}$ is the rank $b^+(X)$ trivial bundle with fibre $H^2_+(X;\mathbb{R})$ and that $S_{H^+} \to \mathcal{J}$ denotes the unit sphere bundle in $H^+ \oplus \mathbb{R}$. In the case that $b_1(X) = 0$, the Jacobian torus $J(X)$ is just a point and H^+ is a bundle over B.

Proposition 4.11. *Let $E \to B$ be a 4-manifold family with fibre X such that $b_1(X) = 0$ and assume a spin^c structure on $T(E/B)$ is given. Suppose there exists a family of metrics $\{g_b\}_{b\in B}$ on E with positive scalar curvature and that E admits a family of flat spin^c connections $\{A_b\}_{b\in B}$. Then the class $[\mu_E]$ is stably homotopic to the inclusion*

$$\iota : B \times S^0 \to S_{H^+}.$$

Proof. Let n be the number of connected components of E. For $t \in [0,1]$ define a homotopy

$$\mu_t : L^2_k(E, W^+ \oplus T^*(E/B)) \oplus \mathbb{R}^n \to L^2_{k-1}(E, W^- \oplus \Lambda^2_+ T^*(E/B) \oplus \mathbb{R})$$

by the formula

$$\mu_t(\psi, a, f) = (D_{A+ta}\psi, d^+a - t\sigma(\psi), d^*a + f).$$

Since $b_1(X) = 0$ and $F_A = 0$, we have $\mu_1 = \mu_E$. Further, μ_0 is the linearised monopole map $l = D_A \oplus d^+ \oplus d^*$. We will use the techniques from the standard argument to show that μ_t is a homotopy through compact perturbations of l. Suppose that $\mu_t(\psi, a, f) = 0$ for some $t \in [0,1]$. This implies that

$$\begin{aligned} D_{A+ta}\psi &= 0 \\ d^+a &= t\sigma(\psi) \\ d^*a &= 0 \\ f &= 0. \end{aligned}$$

Applying the Weitzenböck formula gives

$$\begin{aligned}
\Delta_g|\psi|^2 &\leq 2\left\langle \nabla^*_{A+ta}\nabla_{A+ta}\psi, \psi\right\rangle \\
&= \left\langle 2D^*_{A+ta}D_{A+ta}\psi - \frac{s}{2}\psi - F^+_{A+ta}\psi, \psi\right\rangle \\
&= -\frac{s}{2}|\psi|^2 - t\left\langle d^+a\psi, \psi\right\rangle \\
&= -\frac{s}{2}|\psi|^2 - t^2\left\langle \sigma(\psi)\psi, \psi\right\rangle \\
&= -\frac{s}{2}|\psi|^2 - \frac{t^2}{2}|\psi|^4.
\end{aligned}$$

It follows that

$$\Delta_g|\psi|^2 + \frac{s}{2}|\psi|^2 + \frac{t^2}{2}|\psi|^4 \leq 0.$$

At a maximum of $|\psi|$ we obtain

$$\frac{s}{2}\|\psi\|^2_{L^\infty} + \frac{t^2}{2}\|\psi\|^4_{L^\infty} \leq 0.$$

Since $s > 0$ we have $\psi = 0$. This in turn implies that $d^+a = 0$. Since $d^*a = 0$ and $b_1(X) = 0$, a is harmonic and therefore $a = 0$. Thus $\mu^{-1}(0)$ contains only one point and certainly is bounded. That is, μ is a compact homotopy.

Recall that $\operatorname{ind} l = \operatorname{ind} D_A - b^+(X)$. The positive scalar curvature and the fact that $F_A = 0$ implies that both $\ker D_A = 0$ and $\operatorname{coker} D_A = 0$. Thus D_A is an isomorphism and therefore the Bauer-Furuta finite dimensional approximation of l is stably homotopic to the inclusion ι. □

Let $V \to B$ be an $SO(4)$-vector bundle with a spin^c structure $\mathfrak{s}$ on the vertical tangent space $T(V/B)$. This induces a spin^c structure on $S_V = S(\mathbb{R} \oplus V)$ in the following way. Let $\mathrm{Fr}(V)$ denote the vertical oriented frame bundle of V. The spin^c structure on V determines a principle $\mathrm{Spin}^c(4)$-bundle $\mathcal{P}_V \to \mathrm{Fr}(V)$ which pulls back to a principle $\mathrm{Spin}^c(5)$-bundle $\mathcal{P}_{\mathbb{R}\oplus V} \to \mathrm{Fr}(\mathbb{R} \oplus V)$. Let $i : \mathrm{Fr}(S(V)) \to \mathrm{Fr}(\mathbb{R} \oplus V)$ be the inclusion map of frames defined by the outward normal first convention. Then $i^*(\mathcal{P}_{\mathbb{R}\oplus V}) \to \mathrm{Fr}(S(V))$ is the spin^c structure on S_V induced by $\mathfrak{s}$.

Corollary 4.12. *Let $V \to B$ be an $SO(4)$-bundle with a spin^c structure and give $\pi : S_V \to B$ the induced spin^c structure on the vertical tangent bundle $T(S_V/B)$. Then the class $[\mu_{S_V}]$ is stably homotopic to the identity $id : B \times S^0 \to B \times S^0$.*

Proof. Since $b_1(S^4) = b_2(S^4) = 0$, the pullback map $\pi^* : H^2(B;\mathbb{Z}) \to \pi^*(S_V;\mathbb{Z})$ is an isomorphism by the Serre spectral sequence. Let $\mathcal{L} \to S_V$ be the canonical line bundle of the induced spin^c structure on $T(S_V/B)$. Then the first chern class $c_1(\mathcal{L}) \in H^2(S_V;\mathbb{Z})$ is in the image of π^*. Thus there exists a connection A on $\mathcal{L}$ with curvature $F_A = \pi^*(\omega)$ for some 2-form $\omega \in \Omega^2(B)$. Let $i_b : \pi^{-1}(b) \to S_V$ be the inclusion of the fibre over $b \in B$. Then the restriction $A_b = i_b^* A$ is flat since $F_{A_b} = i_b^*\pi^*\omega = 0$.

Since the structure group of V is $SO(4)$, the fibres of S_V can be equipped with the standard round metric which has positive scalar curvature. By Proposition 4.11, $[\mu_{S_V}] = [\mathrm{id}]$. $\square$

Finally, we have all the necessary tools to derive Bauer-Furuta connected sum formula. We begin with the unparameterised case, which was first formulated by Bauer in [9]. Afterwards, we prove the families formula which is a new result.

Theorem 4.13 ([9] Theorem 1.1)**.** *Let $X = \#_i X_i$ be a connected sum of n closed, oriented, 4-manifolds. The Bauer-Furuta invariant $[\mu_X]$ is given by the formula*

$$[\mu_X] = \bigwedge_{i=1}^{n} [\mu_{X_i}]. \tag{4.16}$$

Proof. It is enough to prove the result for a connected sum of two 4-manifolds. Define

$$\begin{aligned} Y_1 &= X_1 \# S^4 \\ Y_2 &= S^4 \# X_2 \\ Y_3 &= S^4 \# S^4. \end{aligned} \tag{4.17}$$

Set $Y = \coprod_i Y_i$. By the connected sum construction outlined in 4.4, we can choose a metric that gives Y the structure of a separating neck. The negative components of Y are given by the left summands of (4.17) and the positive components by the right summands. Further, any choice of spin^c structure on X_1 and X_2 extends uniquely to a spin^c structure on Y. Now $[\mu_{Y_1}] = [\mu_{X_1}]$, $[\mu_{Y_2}] = [\mu_{X_2}]$ and Proposition 4.11 implies that $[\mu_{Y_3}] = [\mathrm{id}]$. By Proposition 4.10 we have

$$[\mu_Y] = [\mu_{X_1}] \wedge [\mu_{X_2}].$$

Let τ be the even permutation $\tau = (123)$ so that

$$Y^\tau = (X_1 \# X_2) \amalg (S^4 \# S^4) \amalg (S^4 \# S^4).$$

Applying Propositions 4.10 and 4.11 again yields

$$[\mu_{Y^\tau}] = [\mu_{X_1 \# X_2}].$$

Thus Theorem 3.8 implies that $[\mu_X] = [\mu_{X_1}] \wedge [\mu_{X_2}]$. $\square$

Remark 4.14: In the construction of X^τ it is assumed that τ is an even permutation, however this assumption is unnecessary for Theorem 3.8. If τ happens to be odd, then replace X with the disjoint union

$$X' = X \amalg (S^4 \# S^4) \amalg (S^4 \# S^4).$$

Now include an extra transposition in τ that swaps the last two S^4 components. As shown in the argument above, $[\mu_X] = [\mu_{X'}]$.

Theorem 4.15 (Families Bauer-Furuta Connected Sum Formula)**.** *For $j \in \{1, 2\}$, let $E_j \to B$ be a 4-manifold family equipped with a spin^c structure $\mathfrak{s}_j$ on the vertical tangent bundle. Let $i_j : B \to E_j$ be a section with normal bundle V_j and assume that $\varphi : V_1 \to V_2$ is an orientation reversing isomorphism satisfying $\varphi(i_1^*(\mathfrak{s}_{E_1})) \cong i_2^*(\mathfrak{s}_{E_2})$. Then the families Bauer-Furuta class of the fiberwise connected sum $E = E_1 \#_B E_2$ is*

$$[\mu_E] = [\mu_{E_1}] \wedge_{\mathcal{J}} [\mu_{E_2}]. \tag{4.18}$$

Proof. By Corollary 4.9, there is a unique spin^c structure on the vertical tangent space of E that extends $\mathfrak{s}_1$ and $\mathfrak{s}_2$. Let $U_j = \overline{E_j - D(V_j)}$ as in (4.14) so that

$$E_1 = U_1 \cup_{S(-V_1)} D(V_1)$$
$$E_2 = D(V_2) \cup_{S(V_2)} U_2.$$

Recall that $S(V) \subset E$ denotes $S(-V_1) \subset E_1$ and $S(V_2) = \varphi(S(-V_1)) \subset E_2$. For any $L > 0$, we can choose a metric on E_1 and E_2 that gives both of them a separating neck of length $2L$. Let $F = E_1 \amalg E_2$ so that $[\mu_F] = [\mu_{E_1}] \wedge_{\mathcal{J}} [\mu_{E_2}]$ by Proposition 4.10. Let τ be the transposition (12) so that

$$F^\tau = \left(U_1 \cup_{S(V)} U_2\right) \amalg \left(D(V_2) \cup_{S(V)} D(V_1)\right).$$

That is, $F^\tau = E \amalg S_{V_2}$. The spin^c structure on S_{V_2} is induced by $\mathfrak{s}_2$ and therefore $[\mu_{S_{V_2}}] = [\mathrm{id}]$ by Corollary 4.12. Thus $[\mu_{F^\tau}] = [\mu_E]$ by Proposition 4.10. Theorem 4.3 implies that $[\mu_F] = [\mu_{F^\tau}]$ and therefore

$$[\mu_E] = [\mu_{E_1}] \wedge_{\mathcal{J}} [\mu_{E_2}].$$

Note that the fact that τ is an odd permutation is not an issue by Remark 4.14. $\square$

Of course, this formula extends to a connected sum of arbitrarily many families. Note that the diffeomorphism type of the connected sum $E = E_1 \#_B E_2$ depends on the sections i_1, i_2 and the isomorphism φ, however the class $[\mu_{E_1}] \wedge_{\mathcal{J}} [\mu_{E_2}]$ does not. That is, if E' is obtained as a connected sum of E_1 and E_2 for different i_1, i_2 and φ, then $[\mu_E] = [\mu_{E'}]$.

Chapter 5

Families Seiberg-Witten invariants

In [6], Baraglia-Konno derived a connected sum formula for the families Seiberg-Witten invariant. Their formula required various simplifying assumptions on the fibres of the families which were necessary to perform the delicate gluing arguments. In this section, we improve on their work by deriving a completely general connected sum formula (Theorem 5.27). This is achieved by using the connected sum formula for the families Bauer-Furuta invariant from Theorem 4.15.

First we define the families Seiberg-Witten invariant in terms of the moduli space the monopole map. Next, we discuss Baraglia-Konno's reformulation of the families Seiberg-Witten invariant in equivariant cohomology [7]. This reformulation gives an alternate definition of the families Seiberg-Witten invariant which pairs nicely with the families Bauer-Furuta invariant. In Chapter 5.3 we derive a localisation formula for smash product of two monopole maps and in Chapter 5.4 we perform the necessary cohomology calculations that leads a new result (Theorem 5.26), which implies the final formula (Theorem 5.27).

5.1 Generalised monopole maps

Let B be a compact, smooth, oriented manifold and let $H_{\mathbb{C}}, H'_{\mathbb{C}} \to B$ be complex, separable Hilbert bundles as in Definition 2.22. Let S^1 act on $H_{\mathbb{C}}$ and $H'_{\mathbb{C}}$ by fibrewise multiplication. Similarly, let $H_{\mathbb{R}}, H'_{\mathbb{R}} \to B$ be real Hilbert bundles on which S^1 acts trivially and set

$$H' = H'_{\mathbb{C}} \oplus H'_{\mathbb{R}}$$
$$H = H_{\mathbb{C}} \oplus H_{\mathbb{R}}.$$

Following the definitions made in [7], we say that a smooth equivariant bundle map $f : H' \to H$ is a (generalised) families monopole map if it satisfies the following axioms.

1. (Fredholm) The map f decomposes as $f = l + c$ where $l : H' \to H$ is a smooth, S^1-equivariant, fibrewise linear Fredholm map and $c : H' \to H$ is an S^1-equivariant family of smooth compact maps, not necessarily linear.
2. (Boundedness) The pre-image under f of any disk bundle in H is contained in a disk bundle in H'.
3. (Injectivity) Since l is linear and equivariant, we can write $l = l_{\mathbb{C}} \oplus l_{\mathbb{R}} : H'_{\mathbb{C}} \oplus H'_{\mathbb{R}} \to H_{\mathbb{C}} \oplus H_{\mathbb{R}}$. We require that $l_{\mathbb{R}} : H'_{\mathbb{R}} \to H_{\mathbb{R}}$ is injective.
4. (Vanishing) For any $v \in H'_{\mathbb{R}}$, we have $c(v) = 0$.
5. (Quadratic) There exists a smooth bilinear map $q : H' \times_B H' \to H$ such that $c(v) = q(v, v)$.

Example 5.1: Let $E \to B$ be a 4-manifold family with fibre X. Choose a spin^c structure on the vertical tangent bundle $T(E/b)$ and fix an integer $k \geq 4$. Then as in Section 4.2, the families Seiberg-Witten monopole map $\mu : \mathcal{A} \to \mathcal{C}$ is a map of bundles over the Jacobian bundle $\mathcal{J}$ with

$$\mathcal{A}_{\mathbb{C}} = L^2_k(E, W^+), \qquad \mathcal{A}_{\mathbb{R}} = L^2_k(E, T^*(E/B)) \oplus \mathbb{R}^n$$
$$\mathcal{C}_{\mathbb{C}} = L^2_k(E, W^-), \qquad \mathcal{C}_{\mathbb{R}} = L^2_k(E, \Lambda^2_+ T^*(E/B) \oplus \mathbb{R}) \oplus \mathcal{H}^1(\mathbb{R}).$$

For a reference connection A_0, recall that an element $\theta \in \mathcal{J}$ defines another connection $A_\theta = A_0 + i\theta$. For each $\theta \in \mathcal{J}$, the map μ^θ is defined by

$$\mu^\theta(\psi, a, f) = (D_{A_\theta + ia}\psi, -iF^+_{A_0+ia} + d^+a + iF^+_{A_0}, d^*a, \mathrm{pr}(a)). \tag{5.1}$$

Note that this formula differs slightly from the definition of μ in (4.3) in that we have added a constant $+iF^+_{A_0}$ term. This only shifts the level sets of μ, but the change is made to ensure that μ satisfies the vanishing condition above. There is a decomposition $\mu^\theta = l^\theta + c^\theta$ with

$$l^\theta(\psi, a, f) = (D_{A_\theta}\psi, d^+a, d^*a + f, \mathrm{pr}(a))$$
$$c^\theta(\psi, a, f) = (ia \cdot \psi, i\sigma(\psi), 0, 0).$$

In particular $l^\theta_{\mathbb{C}}(\psi) = D_{A_\theta}\psi$ and $l^\theta_{\mathbb{R}}(a,f) = (d^+a, d^*a + f, \mathrm{pr}(a))$. It is clear that $l_{\mathbb{R}}$ is injective since if $d^+a = 0$ and $d^*a = 0$, then a is harmonic and $\mathrm{pr}(a) = a$. The fact that c satisfies the quadratic condition was shown in the proof of Lemma 3.2.

Since $l_{\mathbb{R}} : H'_{\mathbb{R}} \to H_{\mathbb{R}}$ is injective, we can identify $l_{\mathbb{R}}(H'_{\mathbb{R}}) \subset H_{\mathbb{R}}$ with $H'_{\mathbb{R}}$ and assume that $l_{\mathbb{R}}$ is the inclusion. Let $H^+ = H'_{\mathbb{R}}/H_{\mathbb{R}}$, which is a vector bundle that can be identified with the cokernel of $l_{\mathbb{R}}$. Since $l_{\mathbb{R}}$ is injective and Fredholm, we have that the virtual index bundle $\mathrm{ind}\, l_{\mathbb{R}}$ is equal to H^+. Let b^+ denote the index of $l_{\mathbb{R}}$ so that $\dim(H^+) = b^+$.

Definition 5.2. *A chamber for a families monopole map $f : H' \to H$ is a homotopy class of section $\eta : B \to (H_{\mathbb{R}} - H'_{\mathbb{R}})$. Denote the set of chambers of f by $\mathcal{CH}(f)$.*

Remark 5.3: In order for a map $f : H' \to H$ to be a generalised families monopole map, there is a further technical transversality assumption that needs to be made about the chambers of f (see (M6) in [7]). In ordinary Seiberg-Witten theory, this condition is fulfilled by the existence of regular perturbations.

Note that chambers do not always exist so $\mathcal{CH}(f)$ may be empty. The set of chambers $\mathcal{CH}(f)$ is equivalent to the set of non-vanishing homotopy classes of sections of H^+, which is also equivalent to the homotopy classes of sections of $S(H^+)$. If $b^+ = 0$, then $H^+ = 0$ which is analogous to the case that $H^2_+(X;\mathbb{R}) = 0$ in ordinary Seiberg-Witten theory. If $b^+ \geq 1 + \dim B$, then a chamber is guaranteed to exist and if $b^+ > 1 + \dim B$, there is a unique choice of chamber.

Suppose that $\eta : B \to (H_{\mathbb{R}} - H'_{\mathbb{R}})$ is a representative of a chamber and assume that η is transverse to f. Let $\widetilde{\mathcal{M}}_\eta = f^{-1}(\eta)$, which is a smooth manifold equipped with a free S^1-action. It can be shown using the assumptions above that $\widetilde{\mathcal{M}}_\eta$ is compact and finite dimensional with dimension $\mathrm{ind}(l) + \dim B$. The quotient $\mathcal{M}_\eta = \widetilde{\mathcal{M}}_\eta / S^1$ is thus a compact $\mathrm{ind}(l) + \dim B - 1$ dimension manifold. It is also a fibre bundle over B, with projection map $\pi_{\mathcal{M}_\eta} : \mathcal{M}_\eta \to B$ defined by $\pi_{\mathcal{M}_\eta} = p \circ f$. We will assume that the determinant line bundle $\det(\mathrm{ind}\, l_R)$ is orientable, although this assumption is not strictly necessary. Fix an orientation of $\det(\mathrm{ind}\, l_R)$, which uniquely determines an orientation of $\mathcal{M}_\eta$ [7, Lemma 2.6]. Let $\mathcal{L} \to \mathcal{M}_\eta$ be the associated complex line bundle of the principle S^1-bundle $\widetilde{\mathcal{M}}_\eta \to \mathcal{M}_\eta$ and denote by $c_1(\mathcal{L})$ its first chern class.

Definition 5.4. *For any integer $m \geq 0$ and $[\eta] \in \mathcal{CH}(f)$, the m-th families Seiberg-*

Witten invariant of a monopole map $f : H' \to H$ is

$$SW_m^{f,\eta} = (\pi_{\mathcal{M}_\eta})_*(c_1(\mathcal{L})^m) \in H^{2m-(\operatorname{ind} l-1)}(B;\mathbb{Z}) \tag{5.2}$$

Here the 'wrong-way' map $(\pi_{\mathcal{M}_\eta})_* : H^*(\mathcal{M}_\eta;\mathbb{Z}) \to H^{*-(\operatorname{ind} l-1)}(B;\mathbb{Z})$ is defined using Poincaré duality and the pushforward map in homology. In de Rahm cohomology, this map is given by integration along the fibres. Lemma 2.9 in [7] illustrates that $\mathrm{SW}_m^{f,\eta}$ depends only on the homotopy class of η. In the case that $\det(\operatorname{ind} l_{\mathbb{R}})$ is not orientable, a local coefficient system can be used instead of integer coefficients.

5.1.1 Finite dimensional monopole maps

Let $V, V' \to B$ be finite dimensional complex vector bundles of rank a and a' respectively on which S^1 acts by scalar multiplication. Similarly, let $U, U' \to B$ be real vector bundles of rank b and b' on which S^1 acts trivially. Let $S_{V,U} = S(\mathbb{R} \oplus V \oplus U)$ denote the unit sphere in $\mathbb{R} \oplus V \oplus U$ and note that $S_{V,U} = S_V \wedge S_U$. Identify $B \subset S_{V,U}$ with the image of the section at infinity. Let $f : S_{V',U'} \to S_{V,U}$ be an equivariant map which fixes B.

For any finite dimensional complex vector bundle $A \to B$, define the stabilisation of V by A to be the whitney sum $V \oplus A$. Similarly, for $C \to B$ a finite dimensional real vector bundle, the stabilisation of U by C is $U \oplus C$. It is immediate that $S_{V'\oplus A, U'\oplus C} = S_{V',U'} \wedge S_{A,C}$, hence the stabilisation of f by $A \oplus C$ is

$$f \wedge \mathrm{id}_{A\oplus C} : S_{V',U'} \wedge S_{A,C} \to S_{V,U} \wedge S_{A,C}.$$

One way to obtain a map $f : S_{V',U'} \to S_{V,U}$ is to restrict a monopole map $g : H' \to H$ to a finite dimensional subspace $V' \oplus U' \subset H'$. In this case, the stable homotopy class of $f = g|_{S_{V',U'}}$ is independent of the choice of subspace $V' \oplus U'$ as outlined in Definition 2.33. That is, for any other subspace $(V' \oplus A) \oplus (U' \oplus C)$, it is the case that $g|_{S_{V'\oplus A,U'\oplus C}}$ is stably homotopic to $g|_{S_{V',U'}} \wedge \mathrm{id}_{A\oplus C}$.

For $f : S_{V',U'} \to S_{V,U}$, define virtual bundles $D = V' - V \in K^0(B)$ and $H^+ = U - U' \in KO^0(B)$ with $d = a' - a$ and $b^+ = b - b'$. In the case that f arises from the finite dimensional approximation of a Seiberg-Witten monopole map, D is the virtual index bundle of the Dirac operator and H^+ is trivial with fibre $H^2_+(X;\mathbb{R})$. In this setting, $f|_{U'}$ is injective and U' is identified as a subbundle of U with $f|_{U'}$ the inclusion.

Definition 5.5. *An S^1-equivariant map $f : (S_{V',U'}, B_{V',U'}) \to (S_{V,U}, B_{V,U})$ is a finite dimensional monopole map if $f|_{U'}$ is injective.*

Let $f : S_{V',U'} \to S_{V,U}$ be a finite dimensional monopole map. After stabilisation, we can assume that $U = U' \oplus H^+$ and that H^+ is a genuine vector bundle, with $f|_{U'}$ the inclusion.

Definition 5.6. *A chamber for a finite dimensional monopole map f is a homotopy class of section $\phi : B \to (U - U')$.*

By the discussion above, a chamber $\phi : B \to (U - U')$ is equivalent to the homotopy class of a non-vanishing section of H^+, or a homotopy class of section of the unit sphere bundle $S(H^+) \to B$. Thus this notion of chamber is equivalent to Definition 5.2 and we can denote set of chambers for f by $\mathcal{CH}(f)$.

Fix a chamber $\phi \in \mathcal{CH}(f)$. In the same manner as before, we will assume that f is transverse to ϕ so that $\widetilde{\mathcal{M}}_\phi = f^{-1}(\phi(B))$ is a compact smooth submanifold of $V' \oplus U'$. Since f is equivariant and $\widetilde{\mathcal{M}}_\phi$ is disjoint from U', there is an induced free S^1-action on $\widetilde{\mathcal{M}}_\phi$. This means that $\mathcal{M}_\phi = \widetilde{\mathcal{M}}_\phi / S^1$ is a compact smooth manifold of dimension $(2d - b^+) + \dim(B) - 1$. The space $\mathcal{M}_\phi$ is fibred over B with $\pi_{\mathcal{M}_\phi} : \mathcal{M}_\phi \to B$ the projection and $\widetilde{\mathcal{M}}_\phi \to \mathcal{M}_\phi$ is a principle S^1-bundle with associated complex line bundle $\mathcal{L} = \widetilde{\mathcal{M}}_\phi \times_{S^1} \mathbb{C} \to \mathcal{M}$. Assume that $\det(\operatorname{ind} l|_{U'})$ is oriented, which determines an orientation on $\mathcal{M}_\phi$.

Definition 5.7. *For any integer $m \geq 0$ and $[\phi] \in \mathcal{CH}(f)$, the m-th families Seiberg-Witten invariant of a finite dimensional monopole map f is*

$$SW_m^{f,\phi} = (\pi_{\mathcal{M}_\phi})_*(c_1(\mathcal{L})^m) \in H^{2m-(2d-b^+-1)}(B; \mathbb{Z}). \tag{5.3}$$

As in Chapter 2, $\mathrm{SW}_m^{f,\phi}$ should only depend on the stable homotopy class of f.

Proposition 5.8 ([7] Proposition 2.18)**.** *The invariant $SW_m^{f,\phi}$ depends only on the homotopy class of the chamber ϕ and the stable homotopy class of the finite dimensional monopole map f.*

Further, it was noted above that a finite dimensional monopole map can be obtained by restricting a monopole map to a finite dimensional subspace. To do this, let $f = l + c : H' \to H$ be a monopole map and let $V \oplus U \subset H$ be an admissible S^1-invariant subbundle as in Definition 2.31. Set $V' = l_{\mathbb{C}}^{-1}(V)$ and $U' = l_{\mathbb{R}}^{-1}(U)$. Then the restriction $f|_{V'\oplus U'}$ extended to a map of sphere bundles $f_{V',U'} : S_{V',U'} \to S_{V,U}$ is a finite dimensional monopole map. Let $\eta : B \to (H - H')$ be a chamber for f which uniquely determines a chamber $\phi : B \to (U - U')$ for $f_{V',U'}$.

Theorem 5.9 ([7] Theorem 2.24). *The Seiberg-Witten invariants $SW_m^{f,\eta}$ and $SW_m^{f_{V',U'},\phi}$ are equal.*

5.2 Formulation in equivariant cohomology

In [7], Baraglia-Konno reformulate the Seiberg-Witten invariant $SW^{f,\phi}$ as a map in S^1-equivariant cohomology. One advantage of doing so is that this formulation describes the Seiberg-Witten invariant without explicitly referring to the moduli space $\mathcal{M}$. Instead, it focuses attention on the homotopy class of f. Consequently, this definition pairs well with the families Bauer-Furuta invariant defined in Chapter 4.

5.2.1 Equivariant cohomology

Let G be a continuous Lie group and X a topological space with a continuous left G-action. For ease of notation, we will assume $\mathbb{Z}$ coefficients for all cohomology groups following, although some orientablity issues can be dealt with by using local coefficient systems. Recall that the Borel model of the equivariant cohomology group $H^*_G(X)$ is given by

$$H^*_G(X) = H^*(EG \times_G X).$$

Here EG is a contractible space on which G acts freely, with $BG = EG/G$ the classifying space of G. The spaces EG and BG are defined uniquely up to homotopy. The homotopy quotient $EG \times_G X$ is the quotient $(EG \times X)/G$ where G acts on $(e, x) \in EG \times X$ by $g \cdot (e, x) = (eg, g^{-1}x)$. In the case that G acts freely on X, $H^*_G(X) = H^*(X/G)$ and if G acts trivially on X, then $H^*_G(X) = H^*(BG \times X)$. For more detail, see [12].

For our purposes, we will often set $G = S^1$ so that $BG = \mathbb{CP}^\infty$ and $EG = S^\infty$. Recall that the cohomology ring of $\mathbb{CP}^\infty$ with coefficients in $\mathbb{Z}$ is $\mathbb{Z}[x]$ with x having degree 2 [33]. Suppose that $\pi : E \to B$ is a fibre-bundle with a fibre-preserving left G-action on E. We treat B as a G-space with the trivial action. Then $H^*_{S^1}(B) = H^*(\mathbb{CP}^\infty \times B) = H^*(B)[x]$ and $H^*_{S^1}(E)$ is a module over $H^*(B)[x]$. The module structure is defined by the pullback π^*. For $\alpha \in H^*(B)[x]$ and $\theta \in H^*_{S^1}(E)$, we will often abuse notation by writing $\alpha \cdot \theta$ to denote $\pi^*(\alpha) \wedge \theta$. The pullback of an equivariant bundle map $f : E' \to E$ is a $H^*(B)[x]$-module morphism $f^* : H^*_{S^1}(E) \to H^*_{S^1}(E')$.

Now let $\pi : W \to B$ be a rank k oriented real vector bundle. Recall that S_W denotes the unit sphere bundle of $\mathbb{R} \oplus W$ with $B \subset S$ the image of the section at infinity. In ordinary cohomology, the Thom isomorphism $H^*(B) \to H^{*+k}(S_W, B)$ is given by $\alpha \mapsto \pi^*(\alpha) \smile \Phi$, where $\Phi \in H^k(S_W, B)$ is the Thom class of W [13, §6]. The Thom class is defined by the property that the restriction of Φ to each fibre $(S_W)_b$ is the positive generator of $H^k((S_W)_b) = \mathbb{Z}$. The given orientation of W determines Φ uniquely. The Euler class $e(W) \in H^k(B)$ is the pullback of Φ by the zero section $\zeta : B \to S_W$. The 'wrong-way' map $\pi_* : H^{*+k}(S_W, B) \to H^*(B)$ is the inverse of the Thom isomorphism, which is equivalent to integration over the fibre when using real coefficients.

The same results translate directly to equivariant cohomology. Let $W_G = EG \times W$ and $B_G = EG \times_G B$ so that $W_G \to B_G$ is an oriented equivariant vector bundle with fibre W. Now $H^*_G(S_W, B) = H^*(S_{W_G}, B_{W_G})$ and the equivariant Thom class $\tau_W \in H^k_G(S_W, B)$ is defined to be the ordinary Thom class of $W_G \to B_G$. As before, the equivariant Thom isomorphism $H^*_G(B) \to H^{*+k}_G(S_W, B)$ is induced by taking the cup product with τ_W. Similarly, the equivariant Euler class $e(W) \in H^k_G(B)$ is just the pullback of τ_W by the zero section. Once again the 'wrong-way' map $\pi_* : H^{*+k}_G(S_W, B) \to H^*_G(B)$ is the inverse of the Thom isomorphism. More generally, if $f : M \to N$ is an equivariant map of oriented G-manifolds with $k = \dim M - \dim N$, then $f_* : H^*_G(M) \to H^{*-k}_G(N)$ can be defined by factoring f as an inclusion followed by a fibre-bundle projection [3].

5.2.2 Cohomology of sphere bundles

Let $V \to B$ be a complex vector bundle of rank a and $U \to B$ be a real oriented vector bundle of rank b. Recall that $S_{V,U} \to B$ denotes the unit sphere bundle in $\mathbb{R} \oplus V \oplus U$ and $B \subset S_{V,U}$ is the image of the section at infinity. The inclusions $V \subset V \oplus U$ and $U \subset V \oplus U$ extend to inclusions $S_V \subset S_{V,U}$ and $S_U \subset S_{V,U}$. The orientation of U induces an orientation of $S_{V,U}$ since V has a natural orientation. As discussed above, the Thom isomorphism gives $H^*_{S^1}(S_{V,U}, B)$ the structure of a free rank one $H^*_{S^1}(B)$-module generated by the equivariant Thom class $\tau_{V,U} \in H^*_{S^1}(S_{V,U}, B)$.

Let $e(V \oplus U) \in H^{2a+b}_{S^1}(B)$ denote the equivariant Euler class, which is the pullback of $\tau_{V,U} \in H^{2a+b}_{S^1}(S_{V,U}, B)$ by the zero section. In $H^*_{S^1}(B)$, $e(V \oplus U)$ factors as $e(V \oplus U) = e(V) \cdot e(U)$. Here $e(V)$ is the equivariant Euler class of V and $e(U)$ is just the ordinary Euler class of U since S^1 acts trivially on U. Let $\alpha_i(V) \in H^2(B)$ be the ordinary chern roots of V. Since S^1 acts by scalar multiplication on V, the equivariant Chern roots of V are $\alpha_1(V)+x, ..., \alpha_a(V)+x$ where x is the generator of $H^*_{S^1}(B) = H^*(B)[x]$. By the splitting principle [44], it follows that the equivariant Euler class is given by

$$e(V) = x^a + c_1(V)x^{a-1} + c_2(V)x^{a-2} + ... + c_a(V). \tag{5.4}$$

Notice that this is a monic polynomial and hence is not a zero divisor in $H^*_{S^1}(B)$ so long as $a \geq 1$. If $a = 0$, we of course have $e(V) = 0$.

Let $\pi_U : S_U \to B$ denote the projection onto B. The pullback bundle $\pi_U^*(V) \to S_U$ is a subbundle of $S_{V,U}$ which can be identified with the normal bundle of $S_U \subset S_{V,U}$.

Lemma 5.10. *Let $i_V : S_V \to S_{V,U}$ and $i_U : S_U \to S_{V,U}$ denote inclusion maps with pullbacks $i_V^* : H^*_{S^1}(S_{V,U}, B) \to H^*_{S^1}(S_V, B)$ and $i_U^* : H^*_{S^1}(S_{V,U}, B) \to H^*_{S^1}(S_U, B)$. Then*

$$i_V^*(\tau_{V,U}) = \tau_V \cdot e(U)$$
$$i_U^*(\tau_{V,U}) = e(V) \cdot \tau_U.$$

Proof. Let $p_1 : (S_V \times_B S_U, B \times_B S_U) \to (S_V, B)$ and $p_2 : (S_V \times_B S_U, S_V \times_B B) \to (S_U, B)$ denote projection maps onto the first and second factor respectively. There is an external cup product defined by

$$\smile : H^*_{S^1}(S_V, B) \otimes_{H^*_{S^1}(B)} H^*_{S^1}(S_U, B) \to H^*_{S^1}(S_{V,U}, B)$$
$$\alpha \smile \beta = p_1^*(\alpha) \cdot p_2^*(\beta).$$

The multiplication on the right hand side is the ordinary cup product in $H^*_{S^1}(S_{V,U}, B)$. Since $H^*_{S^1}(S_V, B)$ and $H^*_{S^1}(S_U, B)$ are both one-dimensional free $H^*_{S^1}(B)$-modules, this external cup product is an isomorphism [33, Theorem 3.18]. Under this isomorphism, we can identify $\tau_{V,U} = \tau_V \smile \tau_U$.

We prove the second formula. The map i_U is homotopy equivalent to the zero section of the bundle $\pi_U^*(V) \to S_U$, hence it follows that

$$\begin{aligned} i_U^*(\tau_{V,U}) &= i_U^*(\tau_V \smile \tau_U) \\ &= i_U^*(p_1^*(\tau_V)) \cdot i_U^*(p_2^*\tau_U) \\ &= e(V) \cdot \tau_U. \end{aligned}$$

□

Let $\delta : H^{*-1}_{S^1}(S_U, B) \to H^*_{S^1}(S_{V,U}, S_U)$ be the connecting morphism in the long exact sequence of the triple $(S_{V,U}, S_U, B)$. Recall that the ordinary cohomology of the projective bundle $\mathbb{P}(V) \to B$ is given by $H^*(\mathbb{P}(V)) = H^*(B)[x]/\langle e(V)\rangle$ where $e(V)$ is the equivariant Euler class given by equation (5.4).

Lemma 5.11 ([7] Proposition 3.3)**.** *Suppose that $a = rank(V) \geq 1$. Then the cohomology ring $H^*_{S^1}(S_{V,U}, S_U)$ is a free rank one $H^*(\mathbb{P}(V))$-module generated by $\delta\tau_U \in H^{b+1}_{S^1}(S_{V,U}, S_U)$.*

Proof. The long exact sequence of the triple $(S_{V,U}, S_U, B)$ is given by

$$\begin{tikzcd}
\ldots \arrow[r] & H^*_{S^1}(S_{V,U}, B) \arrow[r, "i_U^*"] \arrow[d, equal] & H^*_{S^1}(S_U, B) \arrow[r, "\delta"] \arrow[d, equal] & H^{*+1}_{S^1}(S_{V,U}, S_U) \arrow[r] & \ldots \\
& H^*_{S^1}(B) \cdot \tau_{V,U} & H^*_{S^1}(B) \cdot \tau_U & &
\end{tikzcd}$$

From Lemma 5.10, $i_U^*(\tau_{V,U}) = e(V)\tau_U$. Since $e(V)$ is not a zero divisor in $H^*_{S^1}(B)$, the map $i_U^* : H^*_{S^1}(B) \cdot \tau_{V,U} \to H^*_{S^1}(B) \cdot \tau_U$ is injective with image $H^*_{S^1}(B) \cdot (e(V)\tau_U)$. Exactness implies that δ is surjective with kernel $H^*_{S^1}(B) \cdot (e(V)\tau_U)$. Thus δ induces an isomorphism

$$\begin{aligned}
\delta : (H^*_{S^1}(B)/\langle e(V)\rangle) \cdot \tau_U &\to H^{*+1}(S_{V,U}, S_U) \\
\alpha\tau_U &\mapsto \alpha\delta\tau_U.
\end{aligned}$$

Recall that $H^*_{S^1}(B) = H^*(B)[x]$, hence $H^*_{S^1}(B)/\langle e(V)\rangle = H^*(\mathbb{P}(V))$ as required. □

Now let $W \subset U$ be a proper subbundle of U with dimension b'. A crucial ingredient in the above calculation of $H^*_{S^1}(S_{V,U}, S_U)$ is the fact that $e(V)$ is not a zero divisor, hence i_U^* is injective. The group $H^*_{S^1}(S_{V,U}, S_W)$ is less straightforward to calculate since $e(V \oplus U/W)$ might be a zero divisor.

Suppose there exists a section $\phi : B \to U$ which is disjoint from W. Then one can define a refined Thom class $\widetilde{\tau}^{\phi}_{V,U}$ in the following manner. Let N be an open tubular neighbourhood of the image $\phi(B) \subset U$, which is diffeomorphic to the total space of U. Since $\phi(B)$ and W are disjoint closed spaces, it can be assumed that N is disjoint from W. Define a homeomorphism $p^{\phi}_U : N \to U$ and extend it to a continuous map

$p_U^\phi : S_U \to S_U$ by collapsing $S_U - N$ to infinity fiberwise. Since $S_W \subset (S_U - N)$, p_U^ϕ defines a map of pairs

$$p_U^\phi : (S_U, S_W) \to (S_U, B).$$

Definition 5.12. *Given a chamber $\phi : B \to (U - W)$, the refined Thom class $\widetilde{\tau}^\phi_{V,U} \in H^*_{S^1}(S_{V,U}, S_W)$ is defined by the formula*

$$\widetilde{\tau}^\phi_{V,U} = (p_U^\phi)^* \tau_{V,U}.$$

This definition is of course independent of the homotopy class of $[\phi]$. Further, under the restriction map $i_W^* : H^{2a+b}_{S^1}(S_{V,U}, S_W) \to H^{2a+b}_{S^1}(S_{V,U}, B)$ the refined Thom class $\widetilde{\tau}^\phi_{V,U}$ is sent to the ordinary Thom class $\tau_{V,U}$. This is due to the fact that $(p_U^\phi)^*$ is a degree one map, hence $i_W^*(\widetilde{\tau}^\phi_{V,U})$ restricts to the positive generator of $H^*_{S^1}((S_{V,U})_b, \infty_b)$ for each $b \in B$.

Lemma 5.13 ([7] Proposition 3.2)**.** *Suppose that $W \subset U$ is a proper subbundle and $\phi : B \to U$ is a section which is disjoint from W. Let $\delta : H^*_{S^1}(S_W, B) \to H^*_{S^1}(S_{V,U}, S_W)$ be the connecting morphism in the long exact sequence of the triple $(S_{V,U}, S_W, B)$. Then $H^*_{S^1}(S_{V,U}, S_W)$ is a free rank two $H^*_{S^1}(B)$-module generated by $\widetilde{\tau}^\phi_{V,U}$ and $\delta\tau_W$.*

Proof. This follows immediately from the following long exact sequence.

$$\dots \longrightarrow H^{*-1}_{S^1}(S_W, B) \xrightarrow{\delta} H^*_{S^1}(S_{V,U}, S_W) \xrightarrow{i_W^*} H^*_{S^1}(S_{V,U}, B) \longrightarrow \dots$$

Since $i_W^* \widetilde{\tau}^\phi_{V,U} = \tau_{V,U}$, the map i_W^* is surjective in all degrees and therefore the above long exact sequence splits into short exact sequences.

$$0 \longrightarrow H^{*-1}_{S^1}(S_W, B) \xhookrightarrow{\delta} H^*_{S^1}(S_{V,U}, S_W) \xrightarrow{i_W^*} H^*_{S^1}(S_{V,U}, B) \longrightarrow 0$$

The choice of lift $\widetilde{\tau}^\phi_{V,U}$ of $\tau_{V,U}$ determines a splitting the result follows. $\square$

Let $\widetilde{U}$ denote the open unit disk bundle in $\pi_U^*(V)$, which is an equivariant tubular neighbourhood of S_U in $S_{V,U}$. Note that S^1 acts freely on $\pi_U^*(V)$. Define $\widetilde{Y} = S_{V,U} - \widetilde{U}$, which is a compact manifold with boundary $\partial\widetilde{Y} = S(\pi_U^*(V))$, the unit sphere bundle of $\pi_U^*(V)$. The S^1-action on $\widetilde{Y}$ is free, hence the quotient $Y = \widetilde{Y}/S^1$

is a compact smooth manifold with boundary $\partial Y = S(\pi_U^*(V))/S^1 = \mathbb{P}(\pi_U^*(V))$, the projective bundle of $\pi_U^*(V)$. The Chern class of the principle S^1-bundle $\widetilde{Y} \to Y$ is the image of $x \in H^*(B)[x]$ under the pullback $H^*_{S^1}(B) \to H^*_{S^1}(\widetilde{Y}) \cong H^*(Y)$. The restriction map $H^*(Y) \to H^*(\partial Y)$ sends x to the Chern class of $\mathcal{O}_{\pi_U^*(V)}(1) \to \mathbb{P}(\pi_U^*(V))$.

It is clear by excision that, as $H^*_{S^1}(B)$-modules,

$$\begin{aligned} H^*_{S^1}(S_{V,U}, S_U) &\cong H^*_{S^1}(\widetilde{Y}, \partial\widetilde{Y}) \\ &= H^*(Y, \partial Y). \end{aligned}$$

The last equality follows since S^1 acts freely on Y. To make an explicit identification of $H^*(Y, \partial Y)$ and $H^*(S_{V,U}, S_U)$, first notice that

$$\begin{aligned} \partial\widetilde{Y} &= S(\pi_U^*(V)) \\ &= \pi_U^*(S(V)) \\ &= S_U \times_B S(V). \end{aligned}$$

Thus the below diagram is a pullback.

$$\begin{tikzcd} \partial\widetilde{Y} \arrow[r, "\tilde{p}_2"] \arrow[d, "\tilde{p}_1"] & S(V) \arrow[d, "\pi_V"] \\ S_U \arrow[r, "\pi_U"] & B \end{tikzcd}$$

Let $\tilde{\delta} : H^{*-1}_{S^1}(\partial\widetilde{Y}) \to H^*_{S^1}(\widetilde{Y}, \partial\widetilde{Y})$ denote the connecting map in the long exact sequence of the pair $(\widetilde{Y}, \partial\widetilde{Y})$. The following result is adapted from [7, Proposition 3.5]

Lemma 5.14. *There is an isomorphism $\gamma : H^*_{S^1}(S_{V,U}, S_U) \to H^*(Y, \partial Y)$ of $H^*_{S^1}(B)$-modules which makes the following diagram commute.*

$$\begin{tikzcd} H^{*-1}_{S^1}(S_U, B) \arrow[r, "\delta"] \arrow[d, "\tilde{p}_1^*"] & H^*_{S^1}(S_{V,U}, S_U) \arrow[d, "\gamma"] \\ H^{*-1}(\partial Y) \arrow[r, "\tilde{\delta}"] & H^*(Y, \partial Y) \end{tikzcd} \tag{5.5}$$

Proof. Let $i : (\widetilde{Y}, \partial\widetilde{Y}, \emptyset) \to (S_{V,U}, \widetilde{U}, B)$ be the inclusion of triples, which induces the following commutative diagram between long exact sequences

$$\begin{tikzcd}
\dots \arrow[r] & H^{*-1}_{S^1}(\widetilde{U}, B) \arrow[r, "\delta"] \arrow[d, "i^*"] & H^*_{S^1}(S_{V,U}, \widetilde{U}) \arrow[r] \arrow[d, "\cong"', "i^*"] & H^*_{S^1}(S_{V,U}, B) \arrow[r] \arrow[d, "i^*"] & \dots \\
\dots \arrow[r] & H^{*-1}_{S^1}(\partial\widetilde{Y}) \arrow[r, "\tilde{\delta}"] & H^*_{S^1}(\widetilde{Y}, \partial\widetilde{Y}) \arrow[r] & H^*_{S^1}(\widetilde{Y}) \arrow[r] & \dots
\end{tikzcd}$$

It follows from excision that $i^* : H_{S^1}(S_{V,U}, \widetilde{U}) \to H_{S^1}(\widetilde{Y}, \partial\widetilde{Y})$ is an isomorphism. Similarly, let $j : (S_{V,U}, S_U, B) \to (S_{V,U}, \widetilde{U}, B)$ be the inclusion of triples, which induces the following commutative diagram between long exact sequences

$$\begin{tikzcd}
\dots \arrow[r] & H^{*-1}_{S^1}(S_U, B) \arrow[r, "\delta"] & H^*_{S^1}(S_{V,U}, S_U) \arrow[r] & H^*_{S^1}(S_{V,U}, B) \arrow[r] & \dots \\
\dots \arrow[r] & H^{*-1}_{S^1}(\widetilde{U}, B) \arrow[r, "\delta"] \arrow[u, "j^*", "\cong"'] & H^*_{S^1}(S_{V,U}, \widetilde{U}) \arrow[r] \arrow[u, "j^*", "\cong"'] & H^*_{S^1}(S_{V,U}, B) \arrow[r] \arrow[u, "j^*", "\cong"'] & \dots
\end{tikzcd}$$

Since j is a homotopy equivalence, all the vertical maps are isomorphisms. Since $\widetilde{U} = S_U \times_B D(V)$, the projection map $p : \widetilde{U} \to S_U$ is a homotopy inverse for j. Combining the two diagram gives

$$\begin{tikzcd}
H^{*-1}_{S^1}(S_U, B) \arrow[r, "\delta"] \arrow[d, bend right, "p^*"'] & H^*_{S^1}(S_{V,U}, S_U) \arrow[dd, bend left, "\gamma"] \\
H^{*-1}_{S^1}(\widetilde{U}, B) \arrow[r, "\delta"] \arrow[u, "j^*", "\cong"'] \arrow[d, "i^*"] & H^*_{S^1}(S_{V,U}, \widetilde{U}) \arrow[u, "j^*", "\cong"'] \arrow[d, "\cong"', "i^*"] \\
H^{*-1}_{S^1}(\partial\widetilde{Y}) \arrow[r, "\tilde{\delta}"] & H^*_{S^1}(\widetilde{Y}, \partial\widetilde{Y})
\end{tikzcd}$$

Let $\gamma = i^* \circ (j^*)^{-1}$, which is an isomorphism. Further $i^* \circ p^* = (p \circ i)^* = \tilde{p}_1^*$, hence diagram (5.5) commutes. □

The isomorphism γ naturally identifies $H^*(Y, \partial Y)$ with $H_{S^1}(S_{V,U}, S_U)$ and thus identifies $H^*(Y, \partial Y)$ as a free rank one $H^*(\mathbb{P}(V))$-module generate by $\gamma(\delta\tau_U)$. Diagram (5.5) shows that calculations on $H^*(Y, \partial Y)$ simplify to calculations in $H^*(\mathbb{P}(V))$ in the sense of the following result.

Proposition 5.15. *Let $\alpha \in H^*(Y, \partial Y)$ and write $\alpha = \eta \cdot \gamma(\delta\tau_U)$ for some $\eta \in H^*(\mathbb{P}(V))$. Then*

$$(\pi_Y)_*(\alpha) = (\pi_{\mathbb{P}(V)})_*(\eta).$$

Proof. From Lemma 5.14 we have

$$\begin{aligned}(\pi_Y)_*(\alpha) &= (\pi_Y)_*(\gamma(\eta \cdot \delta\tau_U)) \\ &= (\pi_Y)_*(\gamma(\delta(\eta \cdot \tau_U))) \\ &= (\pi_Y)_*(\tilde{\delta}(\tilde{p}_1^*(\eta \cdot \tau_U))).\end{aligned}$$

Let $\iota : \partial Y \to Y$ be the inclusion map. It was shown in [7, Lemma 3.4] by direct calculation that $\tilde{\delta} = \iota_*$, hence

$$\begin{aligned}(\pi_Y)_*(\alpha) &= (\pi_Y)_* \left(\iota_*(\tilde{p}_1^*(\eta \cdot \tau_U))\right) \\ &= (\pi_{\partial Y})_*(\eta \cdot \tilde{p}_1^*\tau_U).\end{aligned}$$

Note that $(\pi_{\partial Y})_* = (\pi_Y)_* \circ \iota_*$. Recall that $\partial Y = S_U \times_B \mathbb{P}(V)$ and consider the following diagram.

$$\begin{tikzcd} \partial Y \arrow[r, "p_1"] \arrow[d, "p_2"] & S_U \arrow[d, "\pi_U"] \\ \mathbb{P}(V) \arrow[r, "\pi_{\mathbb{P}(V)}"] & B \end{tikzcd}$$

The bundle $p_2 : S_U \times_B \mathbb{P}(V) \to \mathbb{P}(V)$ is a sphere bundle over $\mathbb{P}(V)$ which is the pullback of S_U by $\pi_{\mathbb{P}(V)}$. We can identify $\mathbb{P}(V) \subset \partial Y$ as the image of the section at infinity. Viewing $\tilde{p}_1^*(\tau_U)$ as in element of $H^*(\partial Y, \mathbb{P}(V)) = H^*_{S^1}(\widetilde{\partial Y}, S(V))$ we see that $\tilde{p}_1^*(\tau_U)$ is the Thom class of $S_U \times_B \mathbb{P}(V) \to \mathbb{P}(V)$. Hence $(p_2)_*(\tilde{p}_1^*(\tau_U)) = 1$. Writing $\pi_{\partial Y} = \pi_{\mathbb{P}(V)} \circ p_2$ and applying the projection formula we have

$$\begin{aligned}(\pi_Y)_*(\alpha) &= (\pi_{\mathbb{P}(V)})_* \left((p_2)_*(\eta \cdot \tilde{p}_1^*\tau_U)\right) \\ &= (\pi_{\mathbb{P}(V)})_*(\eta).\end{aligned}$$

□

The Segre classes $s_j(V) \in H^{2j}(B)$ of $V \to B$ are defined by the relation $s(V)c(V) = 1$ where $s(V) = 1 + s_1(V) + ... + s_a(V)$ is the total Segre class of V and $c(V) = 1 + c_1(V) + ... + c_a(V)$ is the total Chern class of V. It can be shown by direct calculation that

$$(\pi_{\mathbb{P}(V)})_*(x^j) = \begin{cases} 0 & \text{if } j < a-1 \\ s_{j-(a-1)}(V) & \text{if } j \geq a-1 \end{cases}$$

By Proposition 5.15, this data can be used to calculate $(\pi_Y)_*(\alpha)$ for any $\alpha \in H^*(Y, \partial Y)$.

5.2.3 Cohomological formula for $SW_m^{f,\phi}$

Let $f : S_{V',U'} \to S_{V,U}$ be a finite dimensional monopole map with $D = V' - V \in K^0(B)$ and $H^+ = U - U' \in KO^0(B)$. As in Definition 5.5, we can assume that $U = U' \oplus H^+$ with $f|_{U'}$ the inclusion and H^+ a genuine vector bundle. Let $\phi : B \to (U - U')$ be a representative of a chamber. Let $\widetilde{\tau}^{\phi}_{V,U} \in H^{2a+b}(S_{V,U}, S_{U'})$ be the refined Thom class described in Definition 5.12. Since $f|_{U'}$ is an inclusion, we have $f^* : H^*(S_{V,U}, S_{U'}) \to H^*(S_{V',U'}, S_{U'})$. One of the primary results of [7] is the following theorem

Theorem 5.16. *[7, Theorem 3.6] For each $m \geq 0$, we have*

$$SW_m^{f,\phi} = (\pi_Y)_*(x^m \gamma(f^* \phi_*(1))). \tag{5.6}$$

Notice the differences in this formula compared to Definition 5.7. The above formula makes no reference to the moduli space $\mathcal{M}_\phi$ or the line bundle $\mathcal{L}$. Furthermore, this definition of $SW_m^{f,\phi}$ clearly only depends on f and ϕ up to homotopy. Moreover, since $f^*\phi_*(1) \in H^*(S_{V',U'}, S_{U'})$, we can write $f^*\phi_*(1) = \eta^\phi \delta \tau_U$ for some $\eta^\phi \in H^*(\mathbb{P}(V))$. Then by Proposition 5.15 we have, for any $\alpha \in H^*(\mathbb{P}(V))$,

$$SW_m^{f,\phi}(\alpha) = (\pi_{\mathbb{P}(V)})_*(x^m \eta^\phi). \tag{5.7}$$

We extend this notation to a map $SW^{f,\phi} : H^*(\mathbb{P}(V)) \to H^*(B)$ defined for any $\alpha \in H^*(\mathbb{P}(V))$ by

$$SW^{f,\phi}(\alpha) = (\pi_{\mathbb{P}(V)})_*(\alpha \eta^\phi). \tag{5.8}$$

Of course, the m-th Seiberg-Witten invariant is given by $SW_m^{f,\phi} = SW^{f,\phi}(x^m)$.

5.3 Localised formula

To derive a connected sum formula for $SW_m^{f,\phi}$, equation (5.7) illustrates that we can focus our attention on the map $(\pi_{\mathbb{P}(V)})_*$. This will be accomplished using the Atiyah-Bott localisation theorem [3] in equivariant cohomology.

Given a ring R and a multiplicative subset $S \subset R$, the localisation of R with respect to S is

$$S^{-1}R = \left\{ \frac{r}{s} \mid r \in R, s \in S \right\} / \sim$$

where $\frac{r_1}{s_1} \sim \frac{r_2}{s_2}$ if and only if there exists some $t \in S$ such that

$$t(r_1 s_2 - r_2 s_1) = 0.$$

The localisation $S^{-1}R$ is a ring by adding and multiplying fractions in the usual way. There is a natural map $l_S : R \to S^{-1}R$ defined by

$$l_S(r) = \frac{r}{1}. \tag{5.9}$$

This map has the property that $l_S(S)$ is contained in the group of units of $S^{-1}R$. If S contains no zero divisors, then l_S is injective and R can be identified as a subring of $S^{-1}R$. Alternatively, $S^{-1}R$ can be defined by the universal property that for any map $f : R \to T$ of rings with $f(S)$ contained in the group of units of T, there is a unique extension $\tilde{f} : S^{-1}R \to T$ such that the following diagram commutes.

$$\begin{tikzcd} S^{-1}R \arrow[rr, dashed, "\exists ! \tilde{f}"] & & T \\ & R \arrow[ul, "l_S"] \arrow[ur, "f"'] & \end{tikzcd}$$

Let $f : R_1 \to R_2$ be a morphism of rings with multiplicative subsets $S_1 \subset R_1, S_2 \subset R_2$. The universal property of localisation implies that $l_{S_2} \circ f$ extends uniquely to a morphism $\tilde{f} : S_1^{-1}R_1 \to S_2^{-1}R_2$ so long as $f(S_1) \subset S_2$. The extension makes the following diagram commute.

$$\begin{tikzcd} S_1^{-1}R_1 \arrow[r, dashed, "\exists ! \tilde{f}"] & S_2^{-1}R_2 \\ R_1 \arrow[u, "l_{S_1}"] \arrow[r, "f"] & R_2 \arrow[u, "l_{S_2}"'] \end{tikzcd} \tag{5.10}$$

We will often abuse notation by just writing $f : S_1^{-1}R_1 \to S_2^{-1}R_2$ to denote the extension $\tilde{f}$. For more detail on localised rings, see [21].

As usual let V_1, V_2 be complex vector bundles of complex rank a_1 and a_2 respectively. Set $V = V_1 \oplus V_2$ with $a = a_1 + a_2$. Define an abnormal action on V by letting S^1 act by scalar multiplication on the V_1 component and trivially on the V_2 component. Notice that this is different to the usual action of scalar multiplication on both components of V. Let $\pi : \mathbb{P}(V) \to B$ be the complex projective bundle of V and extend this S^1-action to $\mathbb{P}(V)$. Let $\iota_i : \mathbb{P}(V_i) \to \mathbb{P}(V)$ be inclusions for $i \in \{1, 2\}$ and note that the subset of fixed points of $\mathbb{P}(V)$ is exactly $\mathbb{P}(V_1) \cup \mathbb{P}(V_2)$.

Recall that the S^1-equivariant cohomology of a point is $H_{S^1}(*) = H^*(\mathbb{CP}^\infty)$ and pullback by the terminal map $\mathbb{P}(V) \to \{*\}$ endows $H^{*+2a}_{S^1}(\mathbb{P}(V))$ with the structure of a $\mathbb{Z}[y]$-module. Here y is the pullback of the generator of $H^*(\mathbb{CP}^\infty) = \mathbb{Z}[y]$, which has degree 2. Let $S \subset H^*_{S^1}(\mathbb{P}(V))$ be the subset $S = \{1, y, y^2, ...\}$ containing all non-negative powers of y. The map

$$\iota_1^* \oplus \iota_2^* : H^*_{S^1}(\mathbb{P}(V)) \to H^*_{S^1}(\mathbb{P}(V_1)) \oplus H^*_{S^1}(\mathbb{P}(V_2))$$

extends to a map

$$\iota_1^* \oplus \iota_2^* : S^{-1}H^*_{S^1}(\mathbb{P}(V)) \to S^{-1}H^*_{S^1}(\mathbb{P}(V_1)) \oplus S^{-1}H^*_{S^1}(\mathbb{P}(V_2))$$

which satisfies the commutativity relation outlined in Diagram (5.10). The localisation theorem as described in [3, Proposition 4.3] states that $\iota_1^* \oplus \iota_2^*$ induces an isomorphism between the localised cohomology groups. We will derive an explicit formula for its inverse.

Let N_i be the normal bundle of $\mathbb{P}(V_i)$ in $\mathbb{P}(V)$. Denote by $\mathcal{O}(-1) \to \mathbb{P}(V)$ the tautological line bundle with fibre at a line $l_b \in \mathbb{P}(V)_b$ the one-dimensional subspace $l_b \subset \pi^*(V)_b$. We will write $\mathcal{O}_i(-1) \to \mathbb{P}(V_i)$ to denote the tautological line over $\mathbb{P}(V_i)$, noting that $\mathcal{O}_i(-1) = \iota_i^*\mathcal{O}(-1)$. Let $\mathcal{O}(1) = \mathrm{Hom}(\mathcal{O}(-1), \mathbb{C})$ denote the dual of $\mathcal{O}(-1)$. We view V_j as a vector bundle over $\mathbb{P}(V_i)$ through pull back by $\pi_i : \mathbb{P}(V_i) \to B$.

Lemma 5.17. *There are equivariant isomorphisms $N_1 \cong (V_2 \otimes \mathcal{O}_1(1))_{-1}$ and $N_2 \cong (V_1 \otimes \mathcal{O}_2(1))_1$, where the subscripts denote the weight of the S^1-action.*

Proof. Let $T(\mathbb{P}(V)/B)$ denote the vertical tangent bundle of $\pi : \mathbb{P}(V) \to B$. Consider the following short exact sequence of vector bundles over $\mathbb{P}(V)$, defined by inclusion and projection maps.

$$0 \to \mathcal{O}(-1) \to V \to V/\mathcal{O}(-1) \to 0$$

Tensoring with $\mathcal{O}(1)$ produces the following exact sequence of vector bundles over $\mathbb{P}(V)$, known as the Euler sequence.

$$0 \to \mathbb{C} \to V \otimes \mathcal{O}(1) \to T(\mathbb{P}(V)/B) \to 0$$

Pulling back by $\iota_i : \mathbb{P}(V_i) \to \mathbb{P}(V)$ gives an exact sequence of vector bundles over $\mathbb{P}(V_i)$. Combining this with the Euler sequence for V_i, we obtain the following commutative diagram of vector bundles over $\mathbb{P}(V_i)$.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbb{C} \arrow[r] \arrow[d] & V_i \otimes \mathcal{O}_i(1) \arrow[r] \arrow[d] & T(\mathbb{P}(V_i)/B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{C} \arrow[r] \arrow[d] & V \otimes \mathcal{O}_i(1) \arrow[r] \arrow[d] & \iota_i^* T(\mathbb{P}(V)/B) \arrow[r] \arrow[d] & 0 \\
 & 0 & V/V_i \otimes \mathcal{O}_i(1) \arrow[r, dashed, "f"] \arrow[d] & N_i \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

The columns and first two rows of this diagram are exact. An elementary diagram chase shows that there is a well defined map $f : V/V_i \otimes \mathcal{O}_i(1) \to N_i$ represented by the dashed arrow. The nine lemma proves that f is an isomorphism. Identifying $V/V_1 \cong V_2$ and $V/V_2 \cong V_1$ shows that $N_1 \cong (V_2 \otimes \mathcal{O}_1(1))$ and $N_2 \cong (V_1 \otimes \mathcal{O}_2(1))$ on the level of ordinary vector bundles.

To show that the weight of the S^1-action on N_1 is -1, fix $l_b \in \mathbb{P}(V_1)$ for some $b \in B$ and local trivialisations $V_1|_U \cong \mathbb{C}^{a_1}$, $V_2|_U \cong \mathbb{C}^{a_2}$ for some open $U \subset \mathbb{P}(V_1)$ containing l_b. We can assume that $l_b = [1, 0, ..., 0]$ in the homogenous coordinates associated to this trivialisation. We can also assume that $V|_U \cong \mathbb{C}^{a_1} \oplus \mathbb{C}^{a_2}$ is a trivialisation with $V_1|_U = \mathbb{C}^{a_1} \oplus 0$ and $V_2|_U = 0 \oplus \mathbb{C}^{a_2}$. A vector $v \in (N_1)_{l_b}$ in a fibre of N_1 can be represented as the tangent vector to a curve γ_t transverse to $\mathbb{P}(V_1)|_U \subset \mathbb{P}(V)|_U$, given by.

$$\begin{aligned}
\gamma_t &= [1, 0, ..., 0] + t[0, ..., 0, v_1, ..., v_n] \\
&= [1, 0, ..., 0, tv_1, ..., tv_n].
\end{aligned}$$

Recall that $e^{i\theta} \cdot [w, z] = [e^{i\theta} w, z]$ for $w \in V_1|_U$, $z \in V_2|_U$ with at least one of them non-zero. Acting on γ_t gives

$$\begin{aligned}
e^{i\theta} \cdot \gamma_t &= e^{i\theta} \cdot [1, 0, ..., 0, tv_1, ..., tv_n] \\
&= [e^{i\theta}, 0, ..., 0, tv_1, ..., tv_n] \\
&= [1, 0, ..., 0, te^{-i\theta} v_1, ..., te^{-i\theta} v_n] \\
&= [1, 0, ..., 0] + t[0, ..., 0, e^{-i\theta} v_1, ..., e^{-i\theta} v_n].
\end{aligned}$$

Hence S^1 acts on N_1 with weight -1. A similar argument shows that S^1 acts on N_2 with weight $+1$. $\square$

Since S^1 acts trivially on $\mathbb{P}(V_i)$, we have that $H^*_{S^1}(\mathbb{P}(V_i)) = H^*(\mathbb{P}(V_i))[y]$ and $S^{-1}H^*_{S^1}(\mathbb{P}(V_i)) = H^*(\mathbb{P}(V_i))[y, y^{-1}]$. That is, we can identify $H^*_{S^1}(\mathbb{P}(V_i))$ as a subring of $S^{-1}H^*_{S^1}(\mathbb{P}(V_i))$ using the injective map defined in (5.9). Let $e(N_i) \in H^{a_i}_{S^1}(\mathbb{P}(V_i))$ denote the S^1 equivariant Euler class of N_i.

Lemma 5.18. *The equivariant Euler class $e(N_i)$ is invertible with in the localised ring $S^{-1}H^*_{S^1}(\mathbb{P}(V_i))$.*

Proof. We will prove the result for N_2. The N_1 case follows the same argument with y replaced with $-y$. Let α_i for $0 \leq i =\leq a_1$ be the Chern roots of V_1. Then by Lemma 5.17, the Chern roots of $N_2 = (V_1 \otimes \mathcal{O}_2(1))_1$ are $\alpha_i(V_1) + y$. Thus by the splitting principle, the total chern class $c(N_2)$ of N_2 is

$$c(N_2) = \sum_{i=0}^{a_1} \sum_{j=0}^{i} c_j(V_1) y^{i-j}.$$

Thus the equivariant Euler class $e(N_2)$ is given by

$$e(N_2) = \sum_{j=0}^{a_1} c_j(V_1) y^{a_1-j}.$$

We claim that the inverse of $e(N_2)$ in $S^{-1}H^*_{S^1}(\mathbb{P}(V_2))$ is given by

$$e(N_2)^{-1} = \sum_{k=0}^{a_1} s_k(V_1) y^{-a_1-k}$$

Here $s_j(V_1) \in H^{2j}(B)$ denotes the j-th Segre class and is defined by the equation $c(V_1)s(V_1) = 1$ for $s(V_1) = 1 + s_1(V_1) + s_2(V_1) + ...$ the total Segre class. Note that this sum is finite since B is a finite dimensional manifold. It follows that

$$\begin{aligned} e(N_2)e(N_2)^{-1} &= \sum_{j=0}^{a_1} \sum_{k=0}^{a_1} c_j(V_1) s_k(V_1) y^{-(j+k)} \\ &= \sum_{m=0}^{2a_1} \left(\sum_{j=0}^{m} c_j(V_1) s_{m-j}(V_1) \right) y^{-m}. \end{aligned}$$

Since $c(V_1)s(V_1) = 1$, we must have that $\sum_{j=0}^{m} c_j(V_1)s_{m-j}(V_1)$ is zero for $m > 0$ by the grading of $H^*(B)$. Therefore $e(N_2)e(N_2)^{-1} = 1$. □

Lemma 5.19. *For any $\alpha \in H^*_{S^1}(\mathbb{P}(V_j))$ with $j, j' \in \{1, 2\}$, $j \neq j'$, we have*

$$\begin{aligned} \iota_j^*((\iota_j)_*\alpha) &= e(N_j)\alpha \\ \iota_j^*((\iota_{j'})_*\alpha) &= 0. \end{aligned}$$

Proof. Let $\pi_j : N_j \to \mathbb{P}(V_j)$ denote the projection with Thom class τ_j. First note that since $(\pi_j)_*$ is an isomorphism, $(\iota_j)_*\alpha = \pi_j^*\alpha \cdot \tau_j$. This follows because $(\pi_j)_*(\iota_j)_*$ is the identity and $(\pi_j)_*(\pi_j^*\alpha \cdot \tau_j) = \alpha$ by the projection formula. Applying this produces the following calculation.

$$\begin{aligned}\iota_j^*((\iota_j)_*\alpha) &= \iota_j^*(\pi_j^*\alpha \cdot \tau_i)\\ &= \alpha\iota_j^*(\tau_j)\\ &= \alpha e(N_j).\end{aligned}$$

Note that the inclusion map ι_j is just the zero section of the normal bundle N_j, hence pulling back τ_j by ι_j gives $e(N_j)$. For $j, j' \in \{1, 2\}$ with $j \neq j'$, we have

$$\begin{aligned}\iota_j^*(\iota_{j'})_*\alpha &= \iota_j^*(\pi_{j'}^*\alpha \cdot \tau_{j'})\\ &= (\pi_{j'} \circ \iota_j)^*\alpha \cdot \iota_j^*\tau_j\\ &= 0.\end{aligned}$$

The last equality follows since $\pi_{j'} \circ \iota_j$ is a constant map. □

The above relations also hold for the extended maps $\iota_j^* : S^{-1}H^*_{S^1}(\mathbb{P}(V)) \to S^{-1}H^*_{S^1}(\mathbb{P}(V_j))$ and $(\tilde{\iota}_j)_* : S^{-1}H^*_{S^1}(\mathbb{P}(V_j)) \to S^{-1}H^*_{S^1}(\mathbb{P}(V))$ by the uniqueness outlined in diagram (5.10).

Lemma 5.20. *For any $\gamma \in S^{-1}H^*_{S^1}(\mathbb{P}(V))$, we have*

$$\gamma = (\iota_1)_*(e(N_1)^{-1}\iota_1^*\gamma) + (\iota_2)_*(e(N_2)^{-1}\iota_2^*\gamma)$$

Proof. Let $\beta = (\iota_1)_*(e(N_1)^{-1}\iota_1^*\gamma) + (\iota_2)_*(e(N_2)^{-1}\iota_2^*\gamma)$. Calculating using Lemma 5.19 shows that $(\iota_1^* + \iota_2^*)\beta = (\iota_1^* + \iota_2^*)\gamma$, hence $\gamma = \beta$ since $\iota_1^* + \iota_2^*$ is an isomorphism between the localised cohomology rings. □

Let $l_{\mathbb{P}(V)} : H^*_{S^1}(\mathbb{P}(V)) \to S^{-1}H^*_{S^1}(\mathbb{P}(V))$ denote the map sending $\alpha \in H^*_{S^1}(\mathbb{P}(V))$ to $l_{\mathbb{P}(V)}(\alpha) = \frac{\alpha}{1} \in S^{-1}H^*_{S^1}(\mathbb{P}(V))$. Note that we cannot necessarily identify $H^*_{S^1}(\mathbb{P}(V))$ as a subring of $S^{-1}H_{S^1}(\mathbb{P}(V))$ since $y \in S$ may be a zero divisor of $H^*_{S^1}(\mathbb{P}(V))$, in which case the map $\alpha \mapsto \frac{\alpha}{1}$ is not injective. However $H^*_{S^1}(B) = H^*(B)[y]$, hence y is not a zero-divisor in $H^*_{S^1}(B)$ and the associated map $l_B : H^*_{S^1}(B) \to S^{-1}H^*_{S^1}(B)$ is the inclusion $H^*(B)[y] \subset H^*(B)[y, y^{-1}]$. Let $\pi_1 = \pi \circ \iota_i$ and extend $(\pi_i)_* : S^{-1}H^*_{S^1}(\mathbb{P}(V_i)) \to S^{-1}H^*_{S^1}(B)$ to a map between the localised cohomology rings.

Theorem 5.21 (Localised Formula)**.** *For any $\alpha \in H^*_{S^1}(\mathbb{P}(V))$, we have*

$$\pi_*(\alpha) = (\pi_1)_*(e(N_1)^{-1}\iota_1^*\gamma) + (\pi_2)_*(e(N_2)^{-1}\iota_2^*\gamma) \tag{5.11}$$

where $\gamma = l_{\mathbb{P}(V)}(\alpha) \in S^{-1}H^*_{S^1}(\mathbb{P}(V))$.

Proof. From Lemma 5.20, we have

$$\begin{aligned}
&(\pi_1)_*(e(N_1)^{-1}\iota_1^*\gamma) + (\pi_2)_*(e(N_2)^{-1}\iota_2^*\gamma)\\
&= \pi_*\big((\iota_1)_*(e(N_1)^{-1}\iota_1^*\gamma) + (\iota_2)_*(e(N_2)^{-1}\iota_2^*\gamma)\big)\\
&= \pi_*\gamma\\
&= \pi_* l_{\mathbb{P}(V)}(\alpha).
\end{aligned}$$

From (5.10), the following diagram is commutative.

$$\begin{tikzcd}
S^{-1}H^*_{S^1}(\mathbb{P}(V)) \arrow[r, "\pi_*"] & S^{-1}H^*_{S^1}(B) \\
H^*_{S^1}(\mathbb{P}(V)) \arrow[u, "l_{\mathbb{P}(V)}"] \arrow[r, "\pi_*"] & H^*_{S^1}(B) \arrow[u, hook, "l_B"']
\end{tikzcd} \tag{5.12}$$

Therefore $\pi_* l_{\mathbb{P}(V)}(\alpha) = l_B(\pi_*(\alpha))$. In particular, the right-hand side of (5.11) is valued in $H^*_{S^1}(B)$ and the formula follows. □

5.4 Seiberg-Witten connected sum formula

For $i \in \{1,2\}$, let $f_i : S_{V_i',U_i'} \to S_{V_i,U_i}$ be two finite dimensional monopole maps where $U_i \to B$ is a real vector bundle of rank b_i and $V_i \to B$ is a complex vector bundle of rank a_i. We assume that $U_i' \subset U_i$ with $f_i|_{U_i'}$ the inclusion. Set $U = U_1 \oplus U_2$, $V = V_1 \oplus V_2$ and define U' and V' similarly. Let $f = f_1 \wedge_B f_2 : S_{V',U'} \to S_{V,U}$ denote the fibrewise smash product of f_1 and f_2. Then $f|_{U'}$ is the inclusion and f is a finite dimensional monopole map.

For the purpose of applying the localised formula 5.21, define a $\mathbb{T} = S^1 \times S^1$ action on $V = V_1 \oplus V_2$ in the following manner. The first factor $\mathbb{T}_1 = S^1$ acts in the usual fashion, scalar multiplication on both V_1 and V_2. The second factor $\mathbb{T}_2 = S^1$ acts on V_1 by scalar multiplication but on V_2 trivially. The action of $\mathbb{T}_2$ on V is the same as the S^1-action defined in Chapter 5.3. Let $\mathbb{T}$ act on V' in the same way and extend this action to $S_{V,U}$ and $S_{V',U'}$. Once again let $\mathbb{T}$ act trivially on B so that $H^*_{\mathbb{T}}(B) = H^*(B)[x,y]$, where x generates the action of the first factor $\mathbb{T}_1$ and y generates the action of the second factor $\mathbb{T}_2$. By pullback, $H_{\mathbb{T}}(S_{V,U})$ and $H_{\mathbb{T}}(S_{V',U'})$ are modules over $H^*(B)[x,y]$.

Let $S(V) \to B$ be the unit sphere bundle in V. Then $\mathbb{T}_1$ acts freely on $S(V)$ and $\mathbb{P}(V) = S(V)/\mathbb{T}_1$. As shown in the previous section, there is an induced action of $\mathbb{T}_2$ on $\mathbb{P}(V)$ with fixed point set $\mathbb{P}(V_1) \cup \mathbb{P}(V_2)$. Moreover $H_{\mathbb{T}}(S(V)) = H_{\mathbb{T}_2}(\mathbb{P}(V))$, which is a module over $H^*(B)[y]$.

Let $\phi_2 : B \to (U_2 - U_2')$ represent a chamber for f_2 with $\phi = (0, \phi_2) : B \to (U - U')$ representing a chamber for f. Note that not every chamber of f can be represented by a map of this form, however we make the simplifying assumption that such a chamber exists. Let $\widetilde{\tau}^{\phi}_{V,U} \in H^*_{\mathbb{T}}(S_{V,U}, S_U)$ be the refined Thom class induced by ϕ. The long exact sequence of the triple $(S_{V,U}, S_U, B)$ in $\mathbb{T}$-equivariant cohomology is given by

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & H^*_{\mathbb{T}}(S_{V,U}, B) & \xrightarrow{i^*} & H^*_{\mathbb{T}}(S_U, B) & \xrightarrow{\delta} & H^{*+1}_{\mathbb{T}}(S_{V,U}, S_U) & \longrightarrow & \ldots \\ & & \| & & \| & & & & \\ & & H^*_{\mathbb{T}}(B)\cdot \tau_{V,U} & & H^*_{\mathbb{T}}(B)\cdot \tau_U & & & & \end{array}$$

Thus the proof of Lemma 5.11 shows that $H^*_{\mathbb{T}}(S_{V,U}, S_U)$ is a free rank one $H^*_{\mathbb{T}_2}(\mathbb{P}(V))$-module generated by $\delta\tau_U$. Recall that $\widetilde{Y}$ is the complement of a $\mathbb{T}$-invariant tubular neighbourhood of $S_U \subset S_{V,U}$ with $Y = \widetilde{Y}/\mathbb{T}_1$. As before, $\pi_Y : Y \to B$ is an oriented fibre bundle with each fibre a smooth manifold with boundary. The push forward map $(\pi_Y)_* : H^*_{\mathbb{T}_2}(Y, \partial Y) \to H^*_{\mathbb{T}_2}(B)$ defines a map in $\mathbb{T}_2$-equivariant cohomology. Lemma 5.14 and Proposition 5.15 extend to this setting and show that there is an isomorphism $\gamma : H^*_{\mathbb{T}}(S_{V,U}, S_U) \to H^*_{\mathbb{T}_2}(Y, \partial Y)$ which gives $H^*_{\mathbb{T}_2}(Y, \partial Y)$ the structure of a free $H^*_{\mathbb{T}_2}(\mathbb{P}(V))$-module generated by $\gamma(\delta\tau_U)$. Further, for $\alpha \in H^*_{\mathbb{T}_2}(Y, \partial Y)$ written as $\alpha = \eta \cdot \gamma(\delta\tau_U)$ with $\eta \in H^*_{\mathbb{T}_2}(\mathbb{P}(V))$, we have

$$(\pi_Y)_*(\alpha) = (\pi_{\mathbb{P}(V)})_*(\eta).$$

Note that $(\pi_{\mathbb{P}(V)})_* : H^*_{\mathbb{T}_2}(\mathbb{P}(V)) \to H^*_{\mathbb{T}_2}(B)$ is a map in $\mathbb{T}_2$-equivariant cohomology and so $(\pi_{\mathbb{P}(V)})_*(\eta)$ is valued in $H^*(B)[y]$. Moreover, $(\pi_{\mathbb{P}(V)})_*$ is $H^*(B)[y]$-linear.

Definition 5.22. *The m-th $\mathbb{T}_2$-generalised Seiberg-Witten invariant $\widehat{SW}^{f,\phi}_m$ of f with respect to the chamber ϕ is given by*

$$\widehat{SW}^{f,\phi}_m = (\pi_Y)_*(x^m \gamma(f^* \widetilde{\tau}^{\phi}_{V,U})) \tag{5.13}$$

which is valued in $H^*_{\mathbb{T}_2}(B) = H^*(B)[y]$.

We call $\widehat{SW}_m^{f,\phi}$ the $\mathbb{T}_2$-generalised Seiberg-Witten since it is a polynomial in $H^*(B)[y]$ and evaluating at $y=0$ gives the ordinary Seiberg-Witten invariant. That is,

$$\widehat{SW}_m^{f,\phi}|_{y=0} = SW_m^{f,\phi}.$$

Also note by the discussion above that we can write $\gamma(f^*\widetilde{\tau}_{V,U}^{\phi}) = \eta^{\phi}\gamma(\delta\tau_U)$ for some $\eta^{\phi} \in H^*_{\mathbb{T}_2}(\mathbb{P}(V))$ such that

$$\widehat{SW}_m^{f,\phi} = (\pi_{\mathbb{P}(V)})_*(x^m\eta^{\phi}).$$

In this fashion we have a map $\widehat{SW}^{f,\phi} : H^*_{\mathbb{T}_2}(\mathbb{P}(V)) \to H^*_{\mathbb{T}_2}(B)$ defined by

$$\widehat{SW}^{f,\phi}(\alpha) = (\pi_{\mathbb{P}(V)})_*(\alpha\eta^{\phi}).$$

We will use this description and the localisation formula applied to $(\pi_{\mathbb{P}(V)})_*$ to derive a connected sum formula for $\widehat{SW}^{f,\phi}$. Let $\iota_i' : \mathbb{P}(V_i') \to \mathbb{P}(V')$ denote the inclusion for $i \in \{1,2\}$ with projection $\pi_i' : \mathbb{P}(V_i') \to B$. Also let N_i' denote the normal bundle of $\mathbb{P}(V_i')$ in $\mathbb{P}(V')$, which is a $\mathbb{T}_2$-equivariant vector bundle. As in Theorem 5.21, let $l_{\mathbb{P}(V')} : H_{\mathbb{T}_2}(\mathbb{P}(V')) \to S^{-1}H_{\mathbb{T}_2}(\mathbb{P}(V'))$ be the canonical map sending $\alpha \mapsto \frac{\alpha}{1}$ and note that $l_{\mathbb{P}(V')}$ might not be injective. Denoting $l_{\mathbb{P}(V')}(\eta^{\phi})$ by $\tilde{\eta}^{\phi}$, we have

$$\widehat{SW}_m^{f,\phi} = (\pi_1')_*(x^m e(N_1')^{-1}(\iota_1')^*\tilde{\eta}^{\phi}) + (\pi_2')_*(x^m e(N_2')^{-1}(\iota_2')^*\tilde{\eta}^{\phi}). \tag{5.14}$$

In order to apply the above formula, we first need to perform some cohomology calculations. The first calculation invokes the external cup product, which is described in [33, Theorem 3.18]

Lemma 5.23. *Let $\phi = (0, \phi_2) : B \to (U - U')$ be a chamber with refined Thom class $\widetilde{\tau}_{V,U}^{\phi} \in H_{\mathbb{T}_2}(S_{V,U}, S_{U'})$. Then*

$$\widetilde{\tau}_{V,U}^{\phi} = \tau_{V_1,U_1} \smile \widetilde{\tau}_{V_2,U_2}^{\phi_2} \tag{5.15}$$

where $\tau_{V_1,U_1} \in H^*_{\mathbb{T}_2}(S_{V_1,U_1}, B)$ is the equivariant Thom class of S_{V_1,U_1} and $\widetilde{\tau}_{V_2,U_2}^{\phi_2} \in H^*_{\mathbb{T}_2}(S_{V_2,U_2}, S_{U_2'})$ is the refined Thom class induced by ϕ_2.

Proof. We define the external cup product $\tau_{V_1,U_1} \smile \widetilde{\tau}_{V_2,U_2}^{\phi_2} \in H_{\mathbb{T}_2}(S_{V,U}, S_{U'})$ for $\tau_{V_1,U_1} \in H^*_{\mathbb{T}_2}(S_{V_1,U_1}, B)$ and $\widetilde{\tau}_{V_2,U_2}^{\phi_2} \in H^*_{\mathbb{T}_2}(S_{V_2,U_2}, S_{U_2'})$, then prove that (5.15) holds. For ease of notation, we assume that $V = 0$, but the general argument is identical.

Define two projection maps

$$\begin{aligned} p_1 &: (S_{U_1} \times_B S_{U_2}, B \times_B S_{U_2}) \to (S_{U_1}, B) \\ p_2 &: (S_{U_1} \times_B S_{U_2}, S_{U_1} \times S_{U_2'}) \to (S_{U_2}, S_{U_2'}) \end{aligned}$$

with corresponding pullbacks

$$\begin{aligned} p_1^* &: H^*_{\mathbb{T}_2}(S_{U_1}, B) \to H^*_{\mathbb{T}_2}(S_{U_1} \times_B S_{U_2}, B \times_B S_{U_2}) \\ p_2^* &: H^*_{\mathbb{T}_2}(S_{U_2}, S_{U_2'}) \to H^*_{\mathbb{T}_2}(S_{U_1} \times_B S_{U_2}, S_{U_1} \times_B S_{U_2'}). \end{aligned}$$

Note that $B \times_B S_{U_2} \cup S_{U_1} \times_B S_{U_2'} = S_{U_1} \vee_B S_{U_2} \cup S_{U_1} \times_B S_{U_2'}$ so we have

$$p_1^* \smile p_2^* : H^*_{\mathbb{T}_2}(S_{U_1}, B) \otimes_{H^*_{\mathbb{T}_2}(B)} H^*_{\mathbb{T}_2}(S_{U_2}, S_{U_2'}) \to H^*_{\mathbb{T}_2}(S_{U_1} \times_B S_{U_2}, S_{U_1} \times_B S_{U_2'} \cup S_{U_1} \vee_B S_{U_2}).$$

Observe that

$$(S_{U_1} \times_B S_{U_2'} \cup S_{U_1} \vee_B S_{U_2})/(S_{U_1} \vee_B S_{U_2}) \cong (S_{U_1} \wedge_B S_{U_2'})/B.$$

Applying [33, Proposition 2.22] it follows that

$$H^*_{\mathbb{T}_2}(S_{U_1} \times_B S_{U_2}, S_{U_1} \times_B S_{U_2'} \cup S_{U_1} \vee_B S_{U_2}) \cong H^*_{\mathbb{T}_2}(S_{U_1} \wedge_B S_{U_2}, S_{U_1} \wedge_B S_{U_2'}).$$

Composing with the restriction $H^*_{\mathbb{T}_2}(S_{U_1} \wedge_B S_{U_2}, S_{U_1} \wedge_B S_{U_2'}) \to H^*_{\mathbb{T}_2}(S_{U_1} \wedge_B S_{U_2}, S_{U'})$, we obtain a map

$$\begin{aligned} \smile : H^*_{\mathbb{T}_2}(S_{U_1}, B) \otimes_{H^*_{\mathbb{T}_2}(B)} H^*_{\mathbb{T}_2}(S_{U_2}, S_{U_2'}) &\to H^*_{\mathbb{T}_2}(S_U, S_{U'}) \\ \alpha \smile \beta &:= p_1^*(\alpha) \cdot p_2^*(\beta). \end{aligned}$$

Note the slight abuse of notation; the multiplication on the right is the ordinary cup product in $H^*_{\mathbb{T}_2}(S_U, S_{U_1} \wedge S_{U_2'})$, which is then viewed under restriction as an element of $H^*_{\mathbb{T}_2}(S_U, S_{U'})$.

Let $f_1 : (S_{W_1}, S_{W_1'}) \to (S_{U_1}, S_{U_1'})$ and $f_2 : (S_{W_2}, S_{W_2'}) \to (S_{U_2}, S_{U_2'})$ be maps of pairs and write $f = f_1 \wedge f_2 : (S_W, S_{W'}) \to (S_U, S_{U'})$. Naturality of the ordinary cup product implies that the following diagram commutes.

$$\begin{tikzcd} H^*_{\mathbb{T}_2}(S_{W_1}, B) \otimes_{H^*_{\mathbb{T}_2}(B)} H^*_{\mathbb{T}_2}(S_{W_2}, S_{W_2'}) \arrow[r, "\smile"] & H^*_{\mathbb{T}_2}(S_W, S_{W'}) \\ H^*_{\mathbb{T}_2}(S_{U_1}, B) \otimes_{H^*_{\mathbb{T}_2}(B)} H^*_{\mathbb{T}_2}(S_{U_2}, S_{U_2'}) \arrow[r, "\smile"] \arrow[u, "f_1^* \otimes f_2^*"] & H^*_{\mathbb{T}_2}(S_U, S_{U'}) \arrow[u, "f^*"'] \end{tikzcd} \tag{5.16}$$

Hence we obtain the formula

$$(f_1 \wedge f_2)^*(\alpha \smile \beta) = f_1^*(\alpha) \smile f_2^*(\beta).$$

Finally, let $p^{\phi}_U : (S_U, S_{U'}) \to (S_U, B)$ denote the degree one map which collapses the complement of a neighbourhood of $\phi(B)$ to infinity, as described in Definition 5.12. Up to homotopy, $p^{\phi}_U = \mathrm{id}_{U_1} \wedge p^{\phi_2}_{U_2}$ hence

$$\begin{aligned}\widetilde{\tau}^{\phi}_U &= (p^{\phi}_U)^*\tau_U \\ &= (\mathrm{id}_{U_1} \wedge p^{\phi_2}_{U_2})^*(\tau_{U_1} \smile \tau_{U_2}) \\ &= \tau_{U_1} \smile (p^{\phi_2}_{U_2})^*\tau_{U_2} \\ &= \tau_{U_1} \smile \widetilde{\tau}^{\phi_2}_{U_2}.\end{aligned}$$

This proves (5.15). □

As in Lemma 5.11, let $\delta : H^*_{\mathbb{T}}(S_{U'}, B) \to H^*_{\mathbb{T}}(S_{V',U'}, S_{U'})$ be the connecting morphism in the long exact sequence of the triple $(S_{V',U'}, S_{U'}, B)$ and define δ_2 similarly for the triple $(S_{V'_2,U'_2}, S_{U'_2}, B)$. By the definition of the external cup product above, we have

$$\begin{aligned}\delta\tau_{U'} &= \delta(p_1^*\tau_{U'_1} \cdot p_2^*\tau_{U'_2}) \\ &= \delta p_1^*\tau_{U'_1} \cdot p_2^*\tau_{U'_2} + (-1)^{a'_1} p_1^*\tau_{U'_1} \cdot \delta p_2^*\tau_{U'_2}.\end{aligned}$$

Further, $\delta p_1^*\tau_{U'_1} = 0$. This can be seen from the following commutative diagram, where the bottom row is the long exact sequence of the triple $(S_{U'_1}, S_{U'_1}, B)$.

$$\begin{tikzcd} \ldots \arrow[r] & H^{*-1}_{\mathbb{T}}(S_{U'}, B) \arrow[r, "\delta"] & H^*_{\mathbb{T}}(S_{V',U'}, S_{U'}) \arrow[r] & H^*_{\mathbb{T}}(S_{V',U'}, B) \arrow[r] & \ldots \\ \ldots \arrow[r] & H^{*-1}_{\mathbb{T}}(S_{U'_1}, B) \arrow[r, "\delta_1"] \arrow[u, "p_1^*"] & H^*_{\mathbb{T}}(S_{U'_1}, S_{U'_1}) \arrow[r] \arrow[u, "p_1^*"] & H^*_{\mathbb{T}}(S_{U'_1}, B) \arrow[r] \arrow[u, "p_1^*"] & \ldots \end{tikzcd}$$

It follows that

$$\begin{aligned}\delta\tau_{U'} &= (-1)^{a'_1} p_1^*\tau_{U'_1} \cdot p_2^*\delta_2\tau_{U'_2} \\ &= (-1)^{a'_1}\tau_{U'_1} \smile \delta_2\tau_{U'_2}. \end{aligned} \tag{5.17}$$

Recall that $\phi : B \to (U - U')$ is a chamber satisfying $\phi = (0, \phi_2)$ where $\phi_2 : B \to (U_2 - U'_2)$ is a chamber for f_2. Let $f_2^*(\widetilde{\tau}^{\phi_2}_{V_2,U_2}) = \eta^{\phi_2}\gamma(\delta\tau_{U'_2})$ for some uniquely determined $\eta^{\phi_2} \in H^*_{\mathbb{T}_2}(\mathbb{P}(V'_2))$.

Lemma 5.24. *For $\phi : B \to (U - U')$ a chamber satisfying $\phi = (0, \phi_2)$, we have*

$$\begin{aligned}(\iota'_1)^*(\eta^{\phi}) &= 0 \\ (\iota'_2)^*(\eta^{\phi}) &= e(N_2)e(H_1^+)\eta_2^{\phi_2}\end{aligned}$$

where $N_2 \to \mathbb{P}(V_2')$ *is the* $\mathbb{T}_2$*-equivariant vector bundle* $N_2 = V_1 \otimes \mathcal{O}_{\mathbb{P}(V_2')}(1)$*.*

Proof. We start with the first equality. The inclusion map $\iota_1 : S_{V_1} \to S_V$ extends to an inclusion $\iota_1 : S_{V_1,U} = S_{V_1} \wedge_B S_U \to S_V \wedge_B S_U = S_{V,U}$ by taking the smash product of ι_1 and the identity on S_U. Let ι_1' denote the same extension $\iota_1' : S_{V_1',U'} \to S_{V',U'}$. Let $i_{U_j'} : U_j' \to U_j$ denote inclusions for $j \in \{1,2\}$. There is a map $f_1 \wedge i_{U_2'} : S_{V_1',U'} \to S_{V_1,U}$ which is defined below.

$$S_{V_1',U'} = S_{V_1',U_1'} \wedge_B S_{U_2'} \xrightarrow{f_1 \wedge i_{U_2'}} S_{V_1,U_1} \wedge_B S_{U_2} = S_{V_1,U} \tag{5.18}$$

Since $f_2|_{U_2'} = i_{U_2'}$, the following diagram commutes.

$$\begin{tikzcd} S_{V',U'} \arrow[r, "f_1 \wedge f_2"] & S_{V,U} \\ S_{V_1',U'} \arrow[u, "\iota_1'"] \arrow[r, "f_1 \wedge i_{U_2'}"] & S_{V_1,U} \arrow[u, "\iota_1"'] \end{tikzcd}$$

Note that each map fixes $S_{U'}$, hence we obtain the below diagram in equivariant cohomology.

$$\begin{tikzcd} H^*_{\mathbb{T}}(S_{V',U'}, S_{U'}) \arrow[d, "(\iota_1')^*"'] & H^*_{\mathbb{T}}(S_{V,U}, S_{U'}) \arrow[l, "(f_1 \wedge f_2)^*"'] \arrow[d, "\iota_1^*"] \\ H^*_{\mathbb{T}}(S_{V_1',U'}, S_{U'}) & H^*_{\mathbb{T}}(S_{V_1,U}, S_{U'}) \arrow[l, "(f_1 \wedge i_{U_2'})^*"] \end{tikzcd} \tag{5.19}$$

Let $\widetilde{\tau}^{\phi}_{V_1,U} \in H^*_{\mathbb{T}}(S_{V_1,U}, S_{U'})$ be the refinement of $\tau_{V_1,U} \in H^*_{\mathbb{T}}(S_{V_1,U}, B)$ induced by ϕ. By Lemma 5.23 we have

$$\begin{aligned} \iota_1^*(\widetilde{\tau}^{\phi}_{V,U}) &= \iota_1^*(\tau_{V_2} \smile \widetilde{\tau}^{\phi}_{V_1,U}) \\ &= e(V_2) \cdot \widetilde{\tau}^{\phi}_{V_1,U} \end{aligned}$$

Note that $\tau_{V_2} \in H^{2a_2}_{\mathbb{T}_2}(S_{V_2}, B)$ has even degree, hence the sign is correct. The map $(f_1 \wedge i_{U_2'})^*$ factors as follows

$$\begin{tikzcd} H^*_{\mathbb{T}}(S_{V_1',U'}, S_{U'}) & H^*_{\mathbb{T}}(S_{V_1,U_1} \wedge_B S_{U_2'}, S_{U'}) \arrow[l, "(f_1 \wedge \mathrm{id})^*"'] & H^*_{\mathbb{T}}(S_{V_1,U}, S_{U'}) \arrow[l, "(\mathrm{id} \wedge i_{U_2'})^*"'] \arrow[ll, bend right, "(f_1 \wedge i_{U_2'})^*"'] \end{tikzcd} \tag{5.20}$$

Again $\widetilde{\tau}^{\phi}_{V_1,U} = \tau_{V_1,U_1} \smile \widetilde{\tau}^{\phi_2}_{U_2}$ where $\widetilde{\tau}^{\phi_2}_{U_2} \in H^*_{\mathbb{T}}(S_{U_2}, S_{U_2'})$ is the refined Thom class induced by ϕ_2. As in (5.16), the naturality of the external cup product gives the following commutative diagram.

$$\begin{tikzcd} H^*_{\mathbb{T}}(S_{V_1,U_1} \wedge_B S_{U_2'}, S_{U'}) & H^*_{\mathbb{T}}(S_{V_1,U}, S_{U'}) \arrow[l, "(\mathrm{id}\wedge i_{U_2'})^*"'] \\ H^*_{\mathbb{T}}(S_{V_1,U_1}, B) \otimes_{H^*_{\mathbb{T}}(B)} H^*_{\mathbb{T}}(S_{U_2'}, S_{U_2'}) \arrow[u, "\smile"] & H^*_{\mathbb{T}}(S_{V_1,U_1}, B) \otimes_{H^*_{\mathbb{T}}(B)} H^*_{\mathbb{T}}(S_U, S_{U'}) \arrow[l, "\mathrm{id}\otimes i^*_{U_2'}"'] \arrow[u, "\smile"'] \end{tikzcd}$$

Since $H^*_{\mathbb{T}}(S_{U_2'}, S_{U_2'}) = 0$, it follows that

$$\begin{aligned}(\mathrm{id} \wedge i_{U_2'})^*(\iota_1^* \widetilde{\tau}^{\phi}_{V,U}) &= (\mathrm{id} \wedge i_{U_2'})^*(e(V_2)\widetilde{\tau}^{\phi}_{V_1,U}) \\ &= e(V_2)(\mathrm{id} \wedge i_{U_2'})^*(\tau_{V_1,U_1} \smile \widetilde{\tau}^{\phi_2}_{U_2}) \\ &= 0.\end{aligned}$$

It follows from the commutativity of (5.19) that

$$\begin{aligned}((\iota_1')^*\eta^{\phi})\delta\tau_{U'} &= (\iota_1')^*(f^*\widetilde{\tau}^{\phi}_{V,U}) \\ &= (f_1 \wedge i_{U_2'})^*(\iota_1^*\tau^{\phi}_{V,U}) \\ &= 0.\end{aligned}$$

Consequently $(\iota_1')^*\eta^{\phi} = 0$.

For the second equality, there is a similar commutative diagram.

$$\begin{tikzcd} H^*_{\mathbb{T}}(S_{V',U'}, S_{U'}) \arrow[d, "(\iota_2')^*"'] & H^*_{\mathbb{T}}(S_{V,U}, S_{U'}) \arrow[l, "(f_1\wedge f_2)^*"'] \arrow[d, "\iota_2^*"] \\ H^*_{\mathbb{T}}(S_{V_2',U'}, S_{U'}) & H^*_{\mathbb{T}}(S_{V_2,U}, S_{U'}) \arrow[l, "(i_{U_1'}\wedge f_2)^*"] \end{tikzcd}$$

We have that $\iota_2^*\widetilde{\tau}^{\phi}_{V,U} = e(V_1)\widetilde{\tau}^{\phi}_{V_2,U}$ for $\widetilde{\tau}^{\phi}_{V_2,U} \in H^*_{\mathbb{T}}(S_{V_2,U}, S_{U'})$. Lemma 5.23 ensures that $\widetilde{\tau}^{\phi}_{V_2,U}$ factors as $\widetilde{\tau}^{\phi}_{V_2,U} = \tau_{U_1} \smile \widetilde{\tau}^{\phi_2}_{V_2,U_2}$ under the external cup product

$$\smile : H^*_{\mathbb{T}}(S_{U_1}, B) \otimes_{H^*_{\mathbb{T}}(B)} H^*_{\mathbb{T}}(S_{V_2,U}, S_{U'}) \to H^*_{\mathbb{T}}(S_{V_2,U}, S_{U'}).$$

The calculation continues as follows.

$$\begin{aligned}((\iota_2')^*\eta^{\phi})\delta\tau_{U'} &= (\iota_2')^*(f^*\widetilde{\tau}^{\phi}_{V,U}) \\ &= (i_{U_1'} \wedge f_2)^*(\iota_2^*\widetilde{\tau}^{\phi}_{V,U}) \\ &= (i_{U_1'} \wedge f_2)^*(e(V_1)\widetilde{\tau}^{\phi}_{V_2,U}) \\ &= e(V_1)(i_{U_1'} \wedge f_2)^*(\tau_{U_1} \smile \widetilde{\tau}^{\phi_2}_{V_2,U_2}) \\ &= e(V_1)i^*_{U_1'}(\tau_{U_1}) \smile f_2^*(\widetilde{\tau}^{\phi_2}_{V_2,U_2})\end{aligned}$$

Recall that H_1^+ is the fibre of the bundle $U_1 \to U_1'$. Applying the formula (5.17) we obtain

$$\begin{aligned}((\iota_2')^*\eta^\phi)\delta\tau_{U'} &= e(V_1)e(H_1^+)\tau_{U_1'} \smile \eta_2^{\phi_2}\delta\tau_{U_2'} \\ &= e(V_1)e(H_1^+)\eta_2^{\phi_2}\delta\tau_{U'}\end{aligned}$$

Thus $(\iota_2')^*\eta^\phi = e(V_1)e(H_1^+)\eta_2^{\phi_2}$. Interpreting V_1 as a vector bundle over $\mathbb{P}(V_2')$, the product $\mathbb{P}(V_2') \times_B V_1 \to \mathbb{P}(V_2')$ can be identified as the normal bundle $N_2 \to \mathbb{P}(V_2')$ of $\mathbb{P}(V_2') \subset \mathbb{P}(V_2' \oplus V_1)$. Applying the same argument as Lemma 5.17 shows that N_2 is equivariantly isomorphic to $(V_1 \otimes \mathcal{O}_{\mathbb{P}(V_2')}(1))_1$. Since $e(V_1)$ is the $\mathbb{T}$-equivariant Euler class of V_1, it follows that $(\iota_2')^*\eta^\phi = e(N_2)e(H_1^+)\eta_2^{\phi_2}$ □

Let M_2 denote the virtual vector bundle $M_2 = N_2' - N_2$ over $\mathbb{P}(V_2)$. Notice that

$$\begin{aligned}M_2 &= V_1' \otimes \mathcal{O}_{\mathbb{P}(V_2')}(1) - V_1 \otimes \mathcal{O}_{\mathbb{P}(V_2')}(1) \\ &= D_1 \otimes \mathcal{O}_{\mathbb{P}(V_2')}(1). \end{aligned} \tag{5.21}$$

Here $D_1 = V_1' - V_1$ is a virtual vector bundle over $\mathbb{P}(V_2)$ of rank $d_1 = a_1' - a_1$. By Lemma 5.18 we have

$$e(M_2)^{-1} = \sum_{j\geq 0} s_j(D_1 \otimes \mathcal{O}_{\mathbb{P}(V_2')}(1))y^{-d_1-j}.$$

Note that this sum is finite since the j-th Segre class is an element of $H^*(B)$ and B is a finite dimensional manifold.

Lemma 5.25. *The j-th Segre class of the virtual bundle $D_1 \otimes \mathcal{O}_{\mathbb{P}(V_2)}(1)$ is given by the formula*

$$s_j(D_1 \otimes \mathcal{O}_{\mathbb{P}(V_2)}(1)) = \sum_{l=0}^{j} \binom{-d_1 - l}{j-l} s_l(D_1)x^{j-1}$$

where $x = c_1(\mathcal{O}_{\mathbb{P}(V_2)})$.

Proof. Suppose that $V \to B$ is an n-dimensional vector bundle with Chern roots $\alpha_1, ..., \alpha_n$. Then by the splitting principle, the Chern roots of $V \otimes \mathcal{O}_{\mathbb{P}(V_2)}(1)$ are $\alpha_1 + x, ..., \alpha_n + x$ and we have

$$c_j(V \otimes \mathcal{O}_{\mathbb{P}(V_2)}(1)) = \sum_{l=0}^{j} c_l(V)x^{j-l}\binom{n-l}{j-l}.$$

This formula also holds if $V = V_1 - V_2$ is a rank n virtual bundle. In this case, $s(V) = c(V)^{-1} = c(-V)$ and $s_j(V) = c_j(-V)$. Applying this formula to the virtual vector bundle D_1 of rank d_1, we obtain

$$\begin{aligned} s_j(D_1 \otimes \mathcal{O}_{\mathbb{P}(V_2)}(1)) &= c_j((-D_1) \otimes \mathcal{O}_{\mathbb{P}(V_2)}(1)) \\ &= \sum_{l=0}^{j} \binom{-d_1 - l}{j-l} c_l(-D_1) x^{j-1} \\ &= \sum_{l=0}^{j} \binom{-d_1 - l}{j-l} s_l(D_1) x^{j-1}. \end{aligned}$$

□

The map $f_1 : S_{V_1',U_1'} \to S_{V_1,U_1}$ induces a map $f_1^* : H_{\mathbb{T}_2}(S_{V_1,U_1}, B) \to H_{\mathbb{T}_2}(S_{V_1',U_1'}, B)$ which sends τ_{V_1,U_1} to some $H^*_{\mathbb{T}_2}(B)$-multiple of $\tau_{V_1',U_1'}$ in $H_{\mathbb{T}_2}(S_{V_1',U_1'}, B)$. Define the degree of f_1 to be the class $\deg_{S^1}(f_1) \in H_{\mathbb{T}_2}(S_{V_1',U_1'}, B)$ such that

$$f_1^*(\tau_{V_1,U_1}) = \deg_{S^1}(f_1)\tau_{V_1',U_1'}. \tag{5.22}$$

To calculate $\deg_{S^1}(f_1)$, consider the following commutative diagram.

$$\begin{tikzcd} S_{V_1',U_1'} \arrow[r, "f_1"] & S_{V_1,U_1} \\ S_{U_1'} \arrow[u, "i_{U_1'}"] \arrow[r, "f_1|_{U_1'}"] & S_{U_1} \arrow[u, "i_{U_1}"'] \end{tikzcd}$$

Recall that $f|_{U'}$ is the inclusion and that $U_1 = U_1' \oplus H_1^+$, hence

$$\begin{aligned} \deg_{S^1}(f_1)e(V_1')\tau_{U_1'} &= i_{U'}^*(\deg_{S^1}(f_1)\tau_{V_1',U_1'}) \\ &= i_{U'}^*(f_1^*\tau_{V_1,U_1}) \\ &= f_1|_{U_1'}^*(i_{U_1}^*\tau_{V_1,U_1}) \\ &= f_1|_{U_1'}^*(e(V_1)\tau_{U_1}) \\ &= e(H_1^+)e(V_1)\tau_{U_1'}. \end{aligned}$$

It follows that

$$\deg_{S^1}(f_1) = e(H_1^+)e(-D_1)$$

where D_1 is the virtual bundle $D_1 = V_1' - V_1$. The calculation in the proof of Lemma 5.17 extends to virtual bundles and shows that

$$\deg_{S^1}(f_1) = e(H_1^+) \sum_{l=0}^{-d_1} s_l(D_1) x^{-d_1 - l}. \tag{5.23}$$

This formula for the degree of f_1 can be applied to derive cohomological proof of Donaldson's theorem [5, Theorem 3.1]. We will use it to calculate the Seiberg-Witten invariant of a smash product, from which the Seiberg-Witten connected sum formula follows immediately. Theorem 5.26 and Theorem 5.27 are new results.

Theorem 5.26. *Suppose $f_i : S_{V_i',U_i'} \to S_{V_i,U_i}$ are finite dimensional monopole maps for $i \in \{1,2\}$ and that $\phi_2 : B \to (U_2 - U_2')$ is a chamber for f_2. Let $\phi = (0, \phi_2)$ be a chamber for the monopole map $f = f_1 \wedge_B f_2 : S_{V',U'} \to S_{V,U}$ and write $U_1 = U_1' \oplus H_1^+$. Then*

$$SW_m^{f,\phi} = SW^{f_2,\phi_2}(x^m \deg_{S^1}(f_1)). \tag{5.24}$$

Proof. From (5.14) we have

$$\widehat{SW}_m^{f,\phi} = (\pi_1')_*(x^m e(N_1')^{-1}(\iota_1')^*\tilde{\eta}^\phi) + (\pi_2')_*(x^m e(N_2')^{-1}(\iota_2')^*\tilde{\eta}^\phi).$$

Applying Lemma 5.24 gives

$$\begin{aligned}\widehat{SW}_m^{f,\phi} &= (\pi_2')_*(x^m e(N_2')^{-1}e(N_2)e(H_1^+)\eta^{\phi_2}) \\ &= \widehat{SW}^{f_2,\phi_2}(x^m e(H_1^+)e(M_2)^{-1})\end{aligned}$$

Note that $M_2 = N_2 - N_2'$ is a complex virtual bundle and therefore has even rank. Apply (5.22) to obtain

$$\begin{aligned}\widehat{SW}_m^{f,\phi} &= \widehat{SW}^{f_2,\phi_2}\left(x^m e(H_1^+)\sum_{j\geq 0} s_j(D_1 \otimes \mathcal{O}_{\mathbb{P}(V_2')}(1))y^{-d_1-j}\right) \\ &= \sum_{j\geq 0} \widehat{SW}^{f_2,\phi_2}\left(x^m e(H_1^+)s_j(D_1 \otimes \mathcal{O}_{\mathbb{P}(V_2')}(1))\right) y^{-d_1-j}.\end{aligned}$$

Recall that the ordinary Seiberg Witten invariant $SW_m^{f,\phi}$ is given by evaluating the $\mathbb{T}_2$-generalised Seiberg-Witten invariant at $y = 0$. It follows that

$$SW_m^{f,\phi} = SW^{f_2,\phi_2}\left(x^m e(H_1^+)s_{-d_1}(D_1 \otimes \mathcal{O}_{\mathbb{P}(V_2')}(1))\right).$$

Apply Lemma 5.25 and equation (5.23) to obtain

$$\begin{aligned}SW_m^{f,\phi} &= SW^{f_2,\phi_2}\left(x^m e(H_1^+)\left(\sum_{l=0}^{-d_1} s_l(D_1)x^{-d_1-1}\right)\right) \\ &= SW^{f_2,\phi_2}(x^m \deg_{S^1}(f_1)).\end{aligned}$$

$\square$

Substituting in $\deg_{S^1}(f_1)$ gives an alternative presentation of the formula

$$SW_m^{f,\phi} = e(H_1^+) \sum_{l=0}^{-d_1} s_l(D_1) SW^{f_2,\phi_2}(x^{m-d_1-1}).$$

Theorem 5.27 (Families Seiberg-Witten Connected Sum Formula). *For $j \in \{1,2\}$, let $E_j \to B$ be a 4-manifold family equipped with a spin^c structure $\mathfrak{s}_j$ on the vertical tangent bundle. Let $i_j : B \to E_j$ be a section with normal bundle V_j and assume that $\varphi : V_1 \to V_2$ is an orientation reversing isomorphism satisfying $\varphi(i_1^*(\mathfrak{s}_{E_1})) \cong i_2^*(\mathfrak{s}_{E_2})$. Set $E = E_1 \#_B E_2$ and let ϕ_2 be a chamber for μ_2 and with $\phi = (0, \phi_2)$ defining a chamber for μ. Then*

$$SW_m^{\mu,\phi} = SW^{\mu_2,\phi_2}(x^m \deg_{S^1}(\mu_1)). \tag{5.25}$$

Proof. Theorem 4.15 gives $[\mu] = [\mu_1] \wedge_{\mathcal{J}} [\mu_2]$ and thus the result follows immediately from Theorem 5.26. □

Let $E_1, E_2 \to B$ be families as in Theorem 5.27 with fibre X_1 and X_2 respectively. Assume that the virtual dimension of the moduli space for X_1 is given by $d(X_1, \mathfrak{s}_{X_1}) = 2m \geq 0$. In [6, Theorem 4.4], Baraglia-Konno derived a Seiberg-Witten connected sum formula under the assumptions that

1. $b_1(X_2) = 0$
2. $c_1(\mathfrak{s}_{X_2})^2 = \sigma(X_2)$
3. $b^+(X_1) + b^+(X_2) > \dim B + 1$

The first assumption implies $\mathcal{J}(X_2) = 0$ and simplifies the structure of the reducible monopoles on X_2. The second assumption ensures that the index of the Dirac operator on X_2 is zero, which implies that $\deg_{S^1}\mu_1 = e(H_1^+)$ by (5.23). Assumption (3) ensures that there is a unique choice of chamber for μ. Given these three assumptions, the formula (5.25) reduces to

$$SW_m^{\mu,\phi}(\alpha) = SW(X_1, \mathfrak{s}_{X_1}) \cdot \left\langle \alpha \cup e(H_1^+), [B] \right\rangle.$$

Here $\alpha \in H^s(B)$ where $s = \dim B - b^+_{X_2}$. Thus Theorem (5.27) is a significant generalisation of Baraglia-Konno's formula. It is able to accommodate chambers and handle cases where $\mathcal{J}(X_2)$ and $\deg_{S^1}(\mu_1)$ are non-trivial.

Theorem 5.27 has many potential applications, one such instance being the ability to detect diffeomorphisms which are topologically isotopic to the identity, but not smoothly isotopic. Let $f : X \to X$ be a diffeomorphism and let $E_f \to S^1$ denote the mapping torus of f, which is a 4-manifold family over the circle. Assume that X is simply connected. By work of Freedman and Quinn, f is topologically isotopic to the identity if and only if it acts trivially on $H^2(X;\mathbb{Z})$ [50]. This is an easily verifiable condition, however detecting smooth isotopy is more difficult.

One obstruction to f being smoothly isotopic to the identity is the smoothability of E_f. This can be detected by the families Seiberg-Witten invariant. This approach has been taken by both Ruberman [51] and Baraglia-Konno [6]. Since $b_1(X) = 0$, it is necessary that $b^+(X)$ be even and greater than 2 to have a non-zero families Seiberg-Witten invariant of E_f. Baraglia-Konno constructed examples using their formula with $b^+(X) \geq 4$, which was necessary to avoid chambers. Theorem 5.27 can accommodate chambers, hence could be used to construct examples with $b^+(X) = 2$.

References

[1] M.F. Atiyah. Algebraic topology and operators in Hilbert space. In *Lectures in Modern Analysis and Applications, I*, volume Vol. 103 of *Lecture Notes in Math.*, pages 101–121. Springer, Berlin-New York, 1969.

[2] M.F. Atiyah and D.W. Anderson. *K-Theory; Lectures by M.F. Atiyah. Notes by D.W. Anderson, Fall, 1964*. 1967.

[3] M.F. Atiyah and R. Bott. The moment map and equivariant cohomology. *Topology*, 23(1):1–28, 1984.

[4] D. Baraglia. Obstructions to smooth group actions on 4-manifolds from families Seiberg-Witten theory. *Adv. Math.*, 354:106730, 32, 2019.

[5] D. Baraglia. Constraints on families of smooth 4-manifolds from Bauer-Furuta invariants. *Algebr. Geom. Topol.*, 21(1):317–349, 2021.

[6] D. Baraglia and H. Konno. A gluing formula for families Seiberg-Witten invariants. *Geom. Topol.*, 24(3):1381–1456, 2020.

[7] D. Baraglia and H. Konno. On the Bauer-Furuta and Seiberg-Witten invariants of families of 4-manifolds. *J. Topol.*, 15(2):505–586, 2022.

[8] S. Bauer. Refined Seiberg-Witten invariants. 3:1–46, 2004.

[9] S. Bauer. A stable cohomotopy refinement of Seiberg-Witten invariants. II. *Invent. Math.*, 155(1):21–40, 2004.

[10] S. Bauer and M. Furuta. A stable cohomotopy refinement of Seiberg-Witten invariants. I. *Invent. Math.*, 155(1):1–19, 2004.

[11] M.S. Berger. *Nonlinearity and Functional Analysis: Lectures on Nonlinear Problems in Mathematical Analysis.* Pure and Applied Mathematics. Elsevier Science, 1977.

[12] R. Bott. *An Introduction to Equivariant Cohomology*, pages 35–57. Springer Netherlands, Dordrecht, 1999.

[13] R. Bott and L.W. Tu. *Differential Forms in Algebraic Topology*. Graduate Texts in Mathematics. Springer New York, 2013.

[14] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.

[15] E.H Brown-Jr. Cohomology theories. *Ann. of Math. (2)*, 75:467–484, 1962.

[16] N. Doll, H. Schulz-Baldes, and N. Waterstraat. *Spectral flow—a functional analytic and index-theoretic approach*, volume 94 of *De Gruyter Studies in Mathematics*. De Gruyter, Berlin, [2023] ©2023.

[17] S.K. Donaldson. An application of gauge theory to four-dimensional topology. *J. Differential Geom.*, 18(2):279–315, 1983.

[18] S.K. Donaldson. The orientation of yang-mills moduli spaces and 4-manifold topology. *Journal of Differential Geometry*, 26(3):397–428, 1987.

[19] S.K. Donaldson. The Seiberg-Witten equations and 4-manifold topology. *Bull. Amer. Math. Soc. (N.S.)*, 33(1):45–70, 1996.

[20] S.K. Donaldson. *Floer homology groups in Yang-Mills theory*, volume 147 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2002. With the assistance of M. Furuta and D. Kotschick.

[21] D. Eisenbud. *Localization*, pages 57–86. Springer New York, New York, NY, 1995.

[22] D.S. Freed and K.K. Uhlenbeck. *Instantons and Four-Manifolds*. Mathematical Sciences Research Institute Publications. Springer New York, 2012.

[23] M.H. Freedman and F. Quinn. *Topology of 4-Manifolds*. Princeton Mathematical Series. Princeton University Press, 2014.

[24] H. Freudenthal. Über die Klassen der Sphärenabbildungen I. Große Dimensionen. *Compositio Math.*, 5:299–314, 1938.

[25] K.A. Froyshov. Monopoles over 4-manifolds containing long necks. I. *Geom. Topol.*, 9:1–93, 2005.

[26] K.A. Froyshov. A generalized blow-up formula for seiberg–witten invariants, 2006.

[27] K.A. Froyshov. Monopoles over 4-manifolds containing long necks, ii, 2006.

[28] K.A. Froyshov. *Compactness and gluing theory for monopoles*, volume 15 of *Geometry & Topology Monographs*. Geometry & Topology Publications, Coventry, 2008.

[29] M. Furuta. Monopole equation and the $\frac{11}{8}$-conjecture. *Math. Res. Lett.*, 8(3):279–291, 2001.

[30] O. Garcia-Prada and R.O. Wells. *Differential Analysis on Complex Manifolds*. Graduate Texts in Mathematics. Springer New York, 2007.

[31] P.B. Gilkey. *Invariance theory: the heat equation and the Atiyah-Singer index theorem*, volume 16. CRC press, 2018.

[32] J.P.C. Greenlees and J.P. May. Equivariant stable homotopy theory. In *Handbook of algebraic topology*, pages 277–323. North-Holland, Amsterdam, 1995.

[33] A. Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.

[34] M. Ishida and H. Sasahira. Stable cohomotopy Seiberg-Witten invariants of connected sums of four-manifolds with positive first Betti number, I: non-vanishing theorem. *Internat. J. Math.*, 26(6):1541004, 23, 2015.

[35] M. Ishida and H. Sasahira. Stable cohomotopy Seiberg-Witten invariants of connected sums of four-manifolds with positive first Betti number II: Applications. *Comm. Anal. Geom.*, 25(2):373–393, 2017.

[36] T. Kato, N. Nakamura, and K. Yasui. The simple type conjecture for mod 2 Seiberg-Witten invariants. *J. Eur. Math. Soc. (JEMS)*, 25(12):4869–4877, 2023.

[37] P.B. Kronheimer and T.S. Mrowka. The genus of embedded surfaces in the projective plane. *Math. Res. Lett.*, 1(6):797–808, 1994.

[38] P.B. Kronheimer and T.S. Mrowka. *Monopoles and Three-Manifolds*. 2008.

[39] N.H. Kuiper. The homotopy type of the unitary group of Hilbert space. *Topology*, 3:19–30, 1965.

[40] H.B. Lawson-Jr. and M.L. Michelsohn. *Spin geometry*, volume 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1989.

[41] T-J Li and A-K Liu. Family Seiberg-Witten invariants and wall crossing formulas. *Comm. Anal. Geom.*, 9(4):777–823, 2001.

[42] F. Lin. Lectures on families of Dirac operators and applications. In *Frontiers in geometry and topology*, volume 109 of *Proc. Sympos. Pure Math.*, pages 1–16. Amer. Math. Soc., Providence, RI, [2024] ©2024.

[43] A-K Liu. Family blowup formula, admissible graphs and the enumeration of singular curves. I. *J. Differential Geom.*, 56(3):381–579, 2000.

[44] J.W. Milnor and J.D. Stasheff. *Characteristic Classes*. Annals of mathematics studies. Princeton University Press, 1974.

[45] J.D. Moore. *Lectures on Seiberg-Witten invariants*, volume 1629 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, second edition, 2001.

[46] J.W. Morgan, Z. Szabó, and C.H. Taubes. A product formula for the Seiberg-Witten invariants and the generalized Thom conjecture. *J. Differential Geom.*, 44(4):706–788, 1996.

[47] N. Nakamura. The Seiberg-Witten equations for families and diffeomorphisms of 4-manifolds. *Asian J. Math.*, 7(1):133–138, 2003.

[48] L.I. Nicolaescu. *Notes on Seiberg-Witten Theory*. Graduate studies in mathematics. American Mathematical Soc.

[49] L.I. Nicolaescu. *Lectures on the Geometry of Manifolds*. G - Reference,Information and Interdisciplinary Subjects Series. World Scientific, 2007.

[50] F. Quinn. Isotopy of 4-manifolds. *J. Differential Geom.*, 24(3):343–372, 1986.

[51] D. Ruberman. An obstruction to smooth isotopy in dimension 4. *Math. Res. Lett.*, 5(6):743–758, 1998.

[52] D. Ruberman. An obstruction to smooth isotopy in dimension 4. *Math. Res. Lett.*, 5(6):743–758, 1998.

[53] D. Ruberman. Positive scalar curvature, diffeomorphisms and the Seiberg-Witten invariants. *Geom. Topol.*, 5:895–924, 2001.

[54] D. Salamon. Removable singularities and a vanishing theorem for Seiberg-Witten invariants. *Turkish J. Math.*, 20(1):61–73, 1996.

[55] D. Salamon. *Spin Geometry and Sieberg-Witten Invariants*. ETH, 2000.

[56] G. Segal. Fredholm complexes. *Quart. J. Math. Oxford Ser. (2)*, 21:385–402, 1970.

[57] N. Seiberg and E. Witten. Electric-magnetic duality, monopole condensation, and confinement in n=2 supersymmetric yang-mills theory. *Nuclear Physics B*, 426(1):19–52, September 1994.

[58] N. Seiberg and E. Witten. Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD. *Nuclear Phys. B*, 431(3):484–550, 1994.

[59] M. Szymik. Characteristic cohomotopy classes for families of 4-manifolds. *Forum Math.*, 22(3):509–523, 2010.

[60] A.S. Švarc. On the homotopic topology of Banach spaces. *Dokl. Akad. Nauk SSSR*, 154:61–63, 1964.

[61] K. Wehrheim. *Uhlenbeck compactness*. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2004.

[62] E. Witten. Monopoles and four-manifolds. *Math. Res. Lett.*, 1(6):769–796, 1994.